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# Singular LQ Suboptimal Control Problem with Disturbance Rejection for Descriptor Systems

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*Abstract*— **This paper deals with singular linear quadratic (LQ) suboptimal control problem with disturbance rejection for descriptor systems. Under some conditions, a suboptimal control-state pair can be found such that the performance of the closed-loop system is within some range; the suboptimal control can be synthesized as state feedback and the state trajectory of the closed-loop system is uniquely determined by disturbance and initial state.**

*Index Terms* — **Descriptor system; Singular linear quadratic cost; Disturbance rejection; Suboptimal control; State feedback.**

## I. INTRODUCTION

Descriptor systems,which are also called singular systems,generalized systems or differential-algebraic systems, are more general than normal state space systems. Descriptor systems have comprehensive practical background. Great progress has been made in the theory and its applications since 1970s [1][3][4]. In this paper, we deals with singular linear quadratic (LQ) suboptimal control problem with disturbance rejection for descriptor systems. Using the method in [2], we transform the singular linear quadratic (LQ) suboptimal control problem with disturbance rejection for descriptor systems equivalently to the nonsingular LQ suboptimal control problem with disturbance rejection for linear systems which can be solved by solving the nonsingular LQ optimal control problems for two other linear systems. Under some conditions, a suboptimal control-state pair can be found such that the performance of the closed-loop system is within some range; the suboptimal control can be synthesized as state feedback and the state trajectory of the closed-loop system is uniquely determined by disturbance and initial state.

This paper is organized as follows. Section II is a statement and transformation of the problem. Section III is the solution of the problem. Section IV is an example. Section V is a brief conclusion.

*Notation*. Throughout the paper, the superscript "$\tau$" stands for matrix transposition; $\bar{C}^+$ denotes the closed right-half complex plane; $R^n$ denotes the $n$-dimensional Euclidean space; $R^{n\times m}$ is the set of $n\times m$ real matrices; $I_n$ is the $n\times n$ identity matrix; $A>0$ means that $A$ is positive definite; $A\geq 0$ means that $A$ is positive semi-definite.

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## II. STATEMENT AND TRANSFORMATION OF THE PROBLEM

Consider a descriptor system

$$\Sigma\begin{cases} E\dot{x}(t)=Ax(t)+B_1w(t)+B_2u(t), \; Ex(0)=x_0 \\ y(t)=Cx(t)+D_1w(t)+D_2u(t) \end{cases} \tag{1}$$

with a performance index

$$J(u,x,w)=\int_0^{+\infty} y^\tau(t)y(t)\mathrm{d}t \tag{2}$$

where $E,A\in R^{n\times n}$, $B_1\in R^{n\times l}$, $B_2\in R^{n\times r}$, $C\in R^{m\times n}$, $D_1\in R^{m\times l}$, $D_2\in R^{m\times r}$; $x,y,u,w$ are state, output, input and unknown disturbance respectively; $x_0$ is a given initial condition. $rankE=p$, $0<p<n$, $r+l\leq m\leq p$, $\|w\|_{\mathcal{L}^2}\leq\rho$. $\rho>0$ is given. Here $\mathcal{L}^2$ denotes the vector space of measurable functions, $w:\ R^+\to R^l$, such that

$$\|w\|_{\mathcal{L}^2}=\left(\int_0^{+\infty} w^\tau(t)w(t)\mathrm{d}t\right)^{\frac{1}{2}}<\infty$$

Let the admissible control-state pair set be

$$\mathcal{J}=\{(u,x)|(u,x) \text{ satisfies (1) and } u,x\in\mathcal{L}^2\}$$

The aim of this paper is to find a suboptimal control-state pair $(u^*,x^*)\in\mathcal{J}$ such that $J(u^*,x^*,w)$ is within some range for all $w$ with $\|w\|_{\mathcal{L}^2}\leq\rho$.

We assume the following:

$(A1)$ $rank\begin{bmatrix} 0 & E & 0 \\ E & A & B_2 \end{bmatrix}=n+p$, i.e. the system $\Sigma$ is impulse controllable;

$(A2)$ $rank\begin{bmatrix} sE-A & B_2 \end{bmatrix}=n, \forall s\in\bar{C}^+$, i.e. the system $\Sigma$ is $R-$stabilizable;

$(A3)$ $rank\begin{bmatrix} 0 & E & 0 \\ E & A & B_2 \\ 0 & C & D_2 \end{bmatrix}=n+p+r$;

$(A4)$ $rank\begin{bmatrix} sE-A & B_2 & B_1 \\ C & -D_2 & -D_1 \end{bmatrix}=n+r+l$, $\forall s\in\bar{C}^+$.

*Definition 1*. If there is another descriptor system $\hat{\Sigma}$: $\hat{E}\dot{\hat{x}}=\hat{A}\hat{x}+\hat{B}_1w+\hat{B}_2u$, $y=\hat{C}\hat{x}+D_1w+D_2u$ and two nonsingular matrices $M,N\in R^{n\times n}$ such that $x=N\hat{x}$, $MEN=\hat{E}$, $MAN=\hat{A}$, $MB_1=\hat{B}_1$, $MB_2=\hat{B}_2$ and $CN=\hat{C}$, then the systems $\Sigma$ and $\hat{\Sigma}$ are called restricted system equivalent (r.s.e.).

Since $rankE=p$, $0<p<n$, there exist nonsingular matrices $M,N\in R^{n\times n}$ such that

$$MEN=\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

Accordingly, let

$$MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, MB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},$$

$$MB_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, CN = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, N^{-1}x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

where

$A_{11} \in R^{p\times p}, B_{11} \in R^{p\times l}, B_{21} \in R^{p\times r}, C_1 \in R^{m\times p}$, $x_1 \in R^p, x_2 \in R^{n-p}$

From $Ex(0) = x_0$ and (3)(4), we have

$$x_1(0) = \begin{bmatrix} I_p & 0 \end{bmatrix} Mx_0$$

Denote

$$x_{10} = \begin{bmatrix} I_p & 0 \end{bmatrix} Mx_0$$

Then $\Sigma$ is r.s.e. to the system

$$\Sigma_1 \begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}w + B_{21}u, x_1(0) = x_{10} \\ 0 = A_{21}x_1 + A_{22}x_2 + B_{12}w + B_{22}u \\ y = C_1x_1 + C_2x_2 + D_1w + D_2u \end{cases} \tag{5}$$

with

$$J_1(u,(x_1,x_2),w) = \int_0^{+\infty} y^\tau(t)y(t)\mathrm{d}t \tag{6}$$

and

$$\mathcal{J}_1 = \{(u,(x_1,x_2))|(u,(x_1,x_2)) \text{ satisfies (5) and } u, x_1, x_2 \in \mathcal{L}^2\}$$

Denote by $\mathcal{P}$ the LQ suboptimal control problem with disturbance rejection for $\Sigma$ with $J$ and $\mathcal{J}$, and by $\mathcal{P}_1$ the problem for $\Sigma_1$ with $J_1$ and $\mathcal{J}_1$. Then $\mathcal{P}$ is equivalent to $\mathcal{P}_1$.

*Lemma 1*. The system $\Sigma$ is impulse controllable, i.e. $\Sigma$ satisfies $(A1)$, if and only if

$$rank \begin{bmatrix} A_{22} & B_{22} \end{bmatrix} = n - p$$

From lemma 1 we know that $\begin{bmatrix} A_{22} & B_{22} \end{bmatrix}$ has full row rank if $\Sigma$ satisfies $(A1)$. Then there exists $K_2 \in R^{r\times(n-p)}$ such that $A_{22} + B_{22}K_2$ is nonsingular. Let

$$\mathcal{K} = \{K_2 | K_2 \in R^{r\times(n-p)},\ A_{22} + B_{22}K_2 \text{ is nonsingular}\}$$

and

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} V_0^{-1} & -V_0^{-1}B_{22} \\ K_2V_0^{-1} & I_r - K_2V_0^{-1}B_{22} \end{bmatrix} \tag{7}$$

where $V_0 = A_{22} + B_{22}K_2$. Note that $V$ defined in (7) is nonsingular and it is easy to prove that

$$\begin{bmatrix} A_{22} & B_{22} \end{bmatrix} V = \begin{bmatrix} I_{n-p} & 0 \end{bmatrix} \tag{8}$$

Let

$$\begin{bmatrix} A_{12} & B_{21} \end{bmatrix} V = \begin{bmatrix} \bar{A}_{12} & \bar{B}_{21} \end{bmatrix},$$

$$\begin{bmatrix} C_2 & D_2 \end{bmatrix} V = \begin{bmatrix} \bar{C}_2 & \bar{D}_2 \end{bmatrix}, V^{-1}\begin{bmatrix} x_2 \\ u \end{bmatrix} = \begin{bmatrix} \bar{x}_2 \\ \bar{u} \end{bmatrix}$$

where

$\bar{A}_{12} = A_{12}V_{11} + B_{21}V_{21},\ \bar{B}_{21} = A_{12}V_{12} + B_{21}V_{22}$, $\bar{C}_2 = C_2V_{11} + D_2V_{21}, \bar{D}_2 = C_2V_{12} + D_2V_{22}$, $\bar{x}_2 \in R^{n-p},\ \bar{u} \in R^r$

Then the system $\Sigma_1$ is converted to the system

$$\tilde{\Sigma}_2 \begin{cases} \dot{x}_1 = \tilde{A}_{11}x_1 + \tilde{B}_{11}w + \bar{B}_{21}\bar{u}, x_1(0) = x_{10} \\ \bar{x}_2 = -A_{21}x_1 - B_{12}w \\ y = \tilde{C}_1x_1 + \tilde{D}_1w + \bar{D}_2\bar{u} \end{cases} \tag{9}$$

where

$\tilde{A}_{11} = A_{11} - \bar{A}_{12}A_{21}, \tilde{B}_{11} = B_{11} - \bar{A}_{12}B_{12}$, $\tilde{C}_1 = C_1 - \bar{C}_2A_{21}, \tilde{D}_1 = D_1 - \bar{C}_2B_{12}$.

*Lemma 2*. Assume $T_1 \in R^{n_1\times n_1}$ is nonsingular. $T_2 \in R^{n_1\times n_2}$, $T_3 \in R^{n_3\times n_1}$, $T_4 \in R^{n_3\times n_2}$. Then

$$T_4 - T_3T_1^{-1}T_2 = 0$$

if and only if

$$rank \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = n_1$$

*Lemma 3*. $B_{11} - \bar{A}_{12}B_{12} = 0$ if and only if

$$rank \begin{bmatrix} A_{22} & B_{22} & B_{12} \\ -K_2 & I_r & 0 \\ A_{12} & B_{21} & B_{11} \end{bmatrix} = n + r - p \tag{10}$$

*Proof*: Note that
$B_{11} - \bar{A}_{12}B_{12} = B_{11} - (A_{12} + B_{21}K_2)(A_{22} + B_{22}K_2)^{-1}B_{12}$
Then from lemma 2 we know that $B_{11} - \bar{A}_{12}B_{12} = 0$ if and only if

$$rank \begin{bmatrix} A_{22} + B_{22}K_2 & B_{12} \\ A_{12} + B_{21}K_2 & B_{11} \end{bmatrix} = n - p \tag{11}$$

It is easy to prove (11) holds if and only if (10) holds.Proof is over.

From lemma 3 we know that if there exists $K_2 \in \mathcal{K}$ such that (10) holds, then $\tilde{\Sigma}_2$ is the system

$$\Sigma_2 \begin{cases} \dot{x}_1 = \tilde{A}_{11}x_1 + \bar{B}_{21}\bar{u}, \quad x_1(0) = x_{10} \\ \bar{x}_2 = -A_{21}x_1 - B_{12}w \\ y = \tilde{C}_1x_1 + \tilde{D}_1w + \bar{D}_2\bar{u} \end{cases} \tag{12}$$

Note that

$$\begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} = \begin{bmatrix} I_p & 0 & 0 \\ -V_{11}A_{21} & V_{12} & -V_{11}B_{12} \\ -V_{21}A_{21} & V_{22} & -V_{21}B_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ \bar{u} \\ w \end{bmatrix} \tag{13}$$

then we have

$$J_2(\bar{u}, x_1, w) = \int_0^{+\infty} y^\tau(t)y(t)\mathrm{d}t$$

$$= \int_0^{+\infty} \begin{bmatrix} x_1^\tau & \bar{u}^\tau & w^\tau \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^\tau & Q_{22} & Q_{23} \\ Q_{13}^\tau & Q_{23}^\tau & Q_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ \bar{u} \\ w \end{bmatrix} \mathrm{d}t \tag{14}$$

where

$$\begin{array}{l} Q_{11} = (C_1 - \bar{C}_2 A_{21})^\tau (C_1 - \bar{C}_2 A_{21}), \\ Q_{12} = (C_1 - \bar{C}_2 A_{21})^\tau \bar{D}_2 \\ Q_{13} = (C_1 - \bar{C}_2 A_{21})^\tau (D_1 - \bar{C}_2 B_{12}),\ Q_{22} = \bar{D}_2^\tau \bar{D}_2 \\ Q_{23} = \bar{D}_2^\tau (D_1 - \bar{C}_2 B_{12}), \\ Q_{33} = (D_1 - \bar{C}_2 B_{12})^\tau (D_1 - \bar{C}_2 B_{12}) \end{array}$$

The admissible control-state pair set of $\Sigma_2$ is

$$\mathcal{J}_2 = \{(\bar{u}, x_1) | (\bar{u}, x_1) \text{ satisfies (12)},\ \bar{u}, x_1 \in \mathcal{L}^2\}$$

Denote by $\mathcal{P}_2$ the LQ suboptimal control problem with disturbance rejection for $\Sigma_2$ with $J_2$ and $\mathcal{J}_2$. Then $\mathcal{P}_1$ is equivalent to $\mathcal{P}_2$.

Now we give the sufficient and necessary conditions of $Q_{22} > 0$.

*Theorem 1*. The following statements are equivalent:

(i) $Q_{22} > 0$

(ii) $rank \begin{bmatrix} A_{22} & B_{22} \\ C_2 & D_2 \end{bmatrix} = n - p + r$

(iii) $rank \begin{bmatrix} 0 & E & 0 \\ E & A & B_2 \\ 0 & C & D_2 \end{bmatrix} = n + p + r$

*Proof*:((i)$\leftrightarrow$(ii)).Note that

$$Q_{22} > 0 \longleftrightarrow rank(C_2 V_{12} + D_2 V_{22}) = r$$

Then we only need to prove

$$rank(C_2 V_{12} + D_2 V_{22}) = r \longleftrightarrow \text{(ii)}$$

(Sufficiency).We assume $rank(C_2 V_{12} + D_2 V_{22}) < r$. Then there exists $\alpha \in R^r$ and $\alpha \neq 0$ such that $\begin{bmatrix} C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \alpha = 0$. Let $\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \alpha = \beta$. Then we have $\beta \neq 0$ for $\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$ has full column rank and $\alpha \neq 0$. From (8), we have $\begin{bmatrix} A_{22} & B_{22} \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = 0$. Then we obtain $\begin{bmatrix} A_{22} & B_{22} \end{bmatrix} \beta = 0$. So $\begin{bmatrix} A_{22} & B_{22} \\ C_2 & D_2 \end{bmatrix} \beta = 0$. This is contradictory to (ii) for $\beta \neq 0$.

(Necessity).The proof of necessity is similar to that of sufficiency.

It is easy to prove that (ii)$\leftrightarrow$(iii).Proof is over.

Now we will find a performance $\tilde{J}_2(\bar{u}, x_1)$ such that

$$J_2(\bar{u}, x_1, w) \leq \tilde{J}_2(\bar{u}, x_1)$$

for all disturbance $w$ with $\| w \|_{\mathcal{L}^2} \leq \rho$. Note that

$$\begin{array}{l} \begin{bmatrix} x_1^\tau & \bar{u}^\tau & w^\tau \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^\tau & Q_{22} & Q_{23} \\ Q_{13}^\tau & Q_{23}^\tau & Q_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ \bar{u} \\ w \end{bmatrix} \\ = x_1^\tau Q_{11} x_1 + \bar{u}^\tau Q_{12}^\tau x_1 + x_1^\tau Q_{12} \bar{u} + \bar{u}^\tau Q_{22} \bar{u} \\ +w^\tau Q_{33} w + 2(w^\tau Q_{13}^\tau x_1 + w^\tau Q_{23}^\tau \bar{u}) \end{array}$$

It is obvious that

$$(w - Q_{13}^\tau x_1)^\tau (w - Q_{13}^\tau x_1) \geq 0$$

Then we have

$$w^\tau w + x_1^\tau Q_{13} Q_{13}^\tau x_1 \geq 2 w^\tau Q_{13}^\tau x_1$$

Similarly, we have

$$w^\tau w + \bar{u}^\tau Q_{23} Q_{23}^\tau \bar{u} \geq 2 w^\tau Q_{23}^\tau \bar{u}$$

Then we obtain

$$\begin{array}{l} \begin{bmatrix} x_1^\tau & \bar{u}^\tau & w^\tau \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^\tau & Q_{22} & Q_{23} \\ Q_{13}^\tau & Q_{23}^\tau & Q_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ \bar{u} \\ w \end{bmatrix} \\ \leq x_1^\tau Q_{11} x_1 + \bar{u}^\tau Q_{12}^\tau x_1 + x_1^\tau Q_{12} \bar{u} + \bar{u}^\tau Q_{22} \bar{u} \\ +w^\tau Q_{33} w + x_1^\tau Q_{13} Q_{13}^\tau x_1 + \bar{u}^\tau Q_{23} Q_{23}^\tau \bar{u} + 2 w^\tau w \\ \leq \begin{bmatrix} x_1^\tau & \bar{u}^\tau \end{bmatrix} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^\tau & Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ \bar{u} \end{bmatrix} + [\lambda_{max} Q_3]\, w^\tau w \end{array}$$

where

$$\begin{array}{l} Q_1 = Q_{11} + Q_{13} Q_{13}^\tau, Q_2 = Q_{22} + Q_{23} Q_{23}^\tau, \\ Q_3 = Q_{33} + 2I \end{array}$$

and $\lambda_{max} Q_3$ is the maximal eigenvalue of $Q_3$. So

$$J_2(\bar{u}, x_1, w) \leq \tilde{J}_2(\bar{u}, x_1)$$

for all $w$ with $\| w \|_{\mathcal{L}^2} \leq \rho$ where

$$\begin{array}{l} \tilde{J}_2(\bar{u}, x_1) = \int_0^{+\infty} \begin{bmatrix} x_1^\tau & \bar{u}^\tau \end{bmatrix} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^\tau & Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ \bar{u} \end{bmatrix} \mathrm{d}t \\ \qquad + [\lambda_{max} Q_3]\, \rho^2 \end{array}$$

Note that $Q_{33} \geq 0$, so $Q_3 > 0$, then $\lambda_{max} Q_3 > 0$. From theorem 1 we know that $Q_{22} > 0$. So $Q_2 > 0$.

By the transformation

$$\begin{bmatrix} x_1 \\ \bar{u} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ -Q_2^{-1} Q_{12}^\tau & Q_2^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ \hat{u} \end{bmatrix} \tag{15}$$

$\Sigma_2$ is transformed to the system

$$\Sigma_3 \begin{cases} \dot{x}_1 = \hat{A} x_1 + \hat{B}_2 \hat{u}, \quad x_1(0) = x_{10} \\ \bar{x}_2 = -A_{21} x_1 - B_{12} w \\ y = \hat{C} x_1 + \hat{D}_1 w + \hat{D}_2 \hat{u} \end{cases} \tag{16}$$

where

$$\begin{array}{l} \hat{A} = A_{11} - \bar{A}_{12} A_{21} - \bar{B}_{21} Q_2^{-1} Q_{12}^\tau,\ \hat{B}_2 = \bar{B}_{21} Q_2^{-\frac{1}{2}}, \\ \hat{C} = C_1 - \bar{C}_2 A_{21} - \bar{D}_2 Q_2^{-1} Q_{12}^\tau,\ \hat{D}_1 = D_1 - \bar{C}_2 B_{12}, \\ \hat{D}_2 = \bar{D}_2 Q_2^{-\frac{1}{2}} \end{array}$$

And $\tilde{J}_2(\bar{u}, x_1)$ is equivalently transformed to

$$\begin{array}{l} \tilde{J}_3(\hat{u}, x_1) = \int_0^{+\infty} (x_1^\tau (Q_1 - Q_{12} Q_2^{-1} Q_{12}^\tau) x_1 + \hat{u}^\tau \hat{u}) \mathrm{d}t \\ \qquad + [\lambda_{max} Q_3]\, \rho^2 \end{array} \tag{17}$$

Without considering output, we regard $\tilde{J}_3(\hat{u}, x_1)$ as the performance of the linear system

$$\dot{x}_1 = \hat{A} x_1 + \hat{B}_2 \hat{u} \tag{18}$$

The admissible control-state pair set of the system (18) is

$$\mathcal{J}_3 = \{(\hat{u}, x_1) | (\hat{u}, x_1) \text{ satisfies (18)},\ \hat{u}, x_1 \in \mathcal{L}^2\}$$

If we find an optimal control-state pair $(\hat{u}^*, x_1^*) \in \mathcal{J}_3$ such that

$$\tilde{J}_3(\hat{u}^*, x_1^*) = \min_{(\hat{u}, x_1) \in \mathcal{J}_3} \tilde{J}_3(\hat{u}, x_1)$$

then we will find a suboptimal control-state pair $(\bar{u}^*, x_1^*) \in \mathcal{J}_2$ such that

$$J_2(\bar{u}^*, x_1^*, w) \leq \tilde{J}_2(\bar{u}^*, x_1^*) = \tilde{J}_3(\hat{u}^*, x_1^*)$$

for all $w$ with $\| w \|_{\mathcal{L}^2} \leq \rho$. Denote by $\mathcal{P}_3$ the LQ optimal control problem for the system (18) with $\tilde{J}_3$ and $\mathcal{J}_3$. Then we can solve $\mathcal{P}_2$ via solving $\mathcal{P}_3$.

## III. SOLUTION OF THE PROBLEM

First we prove that $(\hat{A}, \hat{B}_2)$ is stabilizable, $(\hat{A}, Q_1 - Q_{12}Q_2^{-1}Q_{12}^\tau)$ and $(\hat{A}, \hat{C})$ are detectable.

*Lemma 4.* The system $\Sigma$ is $R$-stabilizable, i.e. $\Sigma$ satisfies $(A2)$, if and only if $(\hat{A}, \hat{B}_2)$ is stabilizable.

*Theorem 2.* Assume the system $\Sigma$ satisfies $(A4)$. If there exists $K_2 \in \mathcal{K}$ such that

$$rank \begin{bmatrix} A_{22} & B_{22} & A_{21} \\ -K_2 & I_r & 0 \\ C_2 & D_2 & C_1 \end{bmatrix} = n - p + m + r \qquad (19)$$

and (10) holds, then $(\hat{A}, Q_1 - Q_{12}Q_2^{-1}Q_{12}^\tau)$ is detectable.

*Proof*:Denote $V_0 = A_{22} + B_{22}K_2$,then

$$\begin{aligned} & rank \begin{bmatrix} A_{22} & B_{22} & A_{21} \\ -K_2 & I_r & 0 \\ C_2 & D_2 & C_1 \end{bmatrix} \\ &= rank \begin{bmatrix} V_0 & A_{21} \\ C_2 + D_2K_2 & C_1 \end{bmatrix} + r \\ &= rank \begin{bmatrix} V_0 & A_{21} \\ 0 & C_1 - (C_2 + D_2K_2)V_0^{-1}A_{21} \end{bmatrix} + r \\ &= rank \left[C_1 - (C_2 + D_2K_2)V_0^{-1}A_{21}\right] + n - p + r \\ &= n - p + m + r \end{aligned}$$

So

$$rank \left[C_1 - (C_2 + D_2K_2)V_0^{-1}A_{21}\right] = m$$

Note that

$C_1 - \bar{C}_2A_{21} = C_1 - (C_2 + D_2K_2)V_0^{-1}A_{21}$

Then we have $rank\left[C_1 - \bar{C}_2A_{21}\right] = m$. From lemma 3 we know that $B_{11} - \bar{A}_{12}B_{12} = 0$. Then

$$\begin{aligned} & rank \begin{bmatrix} sE - A & B_2 & B_1 \\ C & -D_2 & -D_1 \end{bmatrix} \\ &= rank \begin{bmatrix} sI_p - A_{11} & -A_{12} & B_{21} & B_{11} \\ -A_{21} & -A_{22} & B_{22} & B_{12} \\ C_1 & C_2 & -D_2 & -D_1 \end{bmatrix} \\ &= rank \begin{bmatrix} sI_p - A_{11} & -\bar{A}_{12} & -\bar{B}_{21} & -B_{11} \\ -A_{21} & -I_{n-p} & 0 & -B_{12} \\ C_1 & \bar{C}_2 & \bar{D}_2 & D_1 \end{bmatrix} \\ &= rank \begin{bmatrix} sI_p - (A_{11} - \bar{A}_{12}A_{21}) & -\bar{B}_{21} & 0 \\ C_1 - \bar{C}_2A_{21} & \bar{D}_2 & D_1 - \bar{C}_2B_{12} \end{bmatrix} \\ &\quad + n - p \\ &= rank \begin{bmatrix} sI_p - (A_{11} - \bar{A}_{12}A_{21}) & -\bar{B}_{21} & 0 \\ Q_{11} & Q_{12} & Q_{13} \end{bmatrix} + n - p \\ &= rank \begin{bmatrix} sI_p - (A_{11} - \bar{A}_{12}A_{21}) & -\bar{B}_{21} & 0 \\ Q_1 & Q_{12} & Q_{13} \end{bmatrix} + n - p \\ &= rank \begin{bmatrix} sI_p - \hat{A} & -\bar{B}_{21} & 0 \\ Q_1 - Q_{12}Q_2^{-1}Q_{12}^\tau & Q_{12} & Q_{13} \end{bmatrix} + n - p \\ &= n + r + l, \ \forall s \in \bar{C}^+ \end{aligned}$$

So $rank \begin{bmatrix} sI_p - \hat{A} \\ Q_1 - Q_{12}Q_2^{-1}Q_{12}^\tau \end{bmatrix} = p \ , \forall s \in \bar{C}^+.$

Proof is over.

*Theorem 3.* If the system $\Sigma$ satisfies $(A4)$, then $(\hat{A}, \hat{C})$ is detectable.

*Proof*:If $\Sigma$ satisfies $(A4)$, then we have

$$rank \begin{bmatrix} sE - A & B_2 \\ C & -D_2 \end{bmatrix} = n + r, \ \forall s \in \bar{C}^+$$

Note that

$$\begin{aligned} & rank \begin{bmatrix} sE - A & B_2 \\ C & -D_2 \end{bmatrix} \\ &= rank \begin{bmatrix} sI_p - A_{11} & -A_{12} & B_{21} \\ -A_{21} & -A_{22} & B_{22} \\ C_1 & C_2 & -D_2 \end{bmatrix} \\ &= rank \begin{bmatrix} sI_p - A_{11} & -\bar{A}_{12} & -\bar{B}_{21} \\ -A_{21} & -I_{n-p} & 0 \\ C_1 & \bar{C}_2 & \bar{D}_2 \end{bmatrix} \\ &= rank \begin{bmatrix} sI_p - (A_{11} - \bar{A}_{12}A_{21}) & -\bar{B}_{21} \\ C_1 - \bar{C}_2A_{21} & \bar{D}_2 \end{bmatrix} + n - p \\ &= rank \begin{bmatrix} sI_p - \hat{A} & -\bar{B}_{21} \\ \hat{C} & \bar{D}_2 \end{bmatrix} + n - p = n + r, \forall s \in \bar{C}^+ \end{aligned}$$

So $rank \begin{bmatrix} sI_p - \hat{A} \\ \hat{C} \end{bmatrix} = p, \forall s \in \bar{C}^+$. Proof is over.

If there is no disturbance in $\Sigma$, then $\Sigma$ is the system

$$\Sigma' \begin{cases} E\dot{x}(t) = Ax(t) + B_2u(t), & Ex(0) = x_0 \\ y(t) = Cx(t) + D_2u(t) \end{cases} \qquad (20)$$

with the performance

$$J'(u, x) = \int_0^{+\infty} y^\tau(t)y(t)\mathrm{d}t \qquad (21)$$

It is easy to prove that $\Sigma'$ can be equivalently converted to the system

$$\Sigma_3' \begin{cases} \dot{x}_1 = \hat{A}x_1 + \hat{B}_2\hat{u}, & x_1(0) = x_{10} \\ \bar{x}_2 = -A_{21}x_1 \\ \hat{y} = y - \bar{D}_2Q_{22}^{-\frac{1}{2}}\hat{u} = \hat{C}x_1 \end{cases} \qquad (22)$$

with the performance

$$J_3'(\hat{u}, x_1) = \int_0^{+\infty} (x_1^\tau(Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^\tau)x_1 + \hat{u}^\tau\hat{u})\mathrm{d}t \qquad (23)$$

*Lemma 5.* Suppose that $(A, B)$ is stabilizable and $(C, A)$ is detectable. Then the Riccati equation

$$A^\tau P + PA - PBB^\tau P + C^\tau C = 0$$

has a unique positive semi-definite solution. Moreover, the solution is stabilizing (i.e. $A - BB^\tau P$ is stable).

Lemma 5 is corollary 13.8 in [5].

*Theorem 4.* Assume the system $\Sigma$ satisfies $(A1)$ $(A2)$ $(A3)$ $(A4)$. If there exists $K_2 \in \mathcal{K}$ such that (10) and

(19) hold, then there exists a suboptimal control-state pair $(u^*, x^*) \in \mathcal{J}$ such that

$$x_{10}^\tau P_1 x_{10} \leq J(u^*, x^*, w) \leq x_{10}^\tau P_2 x_{10} + [\lambda_{max} Q_3]\, \rho^2 \tag{24}$$

for all $w$ with $\| w \|_{\mathcal{L}^2} \leq \rho$, where $P_1$ is the unique positive semi-definite solution of the Riccati equation

$$\hat{A}^\tau P_1 + P_1 \hat{A} - P_1 \hat{B}_2 \hat{B}_2^\tau P_1 + \hat{C}^\tau \hat{C} = 0 \tag{25}$$

and $P_2$ is the unique positive semi-definite solution of the Riccati equation

$$\hat{A}^\tau P_2 + P_2 \hat{A} - P_2 \hat{B}_2 \hat{B}_2^\tau P_2 + Q_1 - Q_{12} Q_2^{-1} Q_{12}^\tau = 0 \tag{26}$$

The suboptimal control $u^*$ can be synthesized as state feedback. The state trajectory of the closed-loop system is uniquely determined by disturbance and initial state.

*Proof*: If $w = 0$, then we know that the system $\Sigma'$ can be equivalently converted to the system $\Sigma_3'$. From lemma 4 and theorem 3, we know that $(\hat{A}, \hat{B}_2)$ is stabilizable and $(\hat{A}, \hat{C})$ is detectable. Then from linear system theory and lemma 5 we know that the optimal value of the performance (23) is

$$J_{3opt}' = x_{10}^\tau P_1 x_{10}$$

where $P_1$ is the unique positive semi-definite solution of the Riccati equation (25). Note that $J'(u, x) = J_3'(\hat{u}, x_1)$, so

$$J_{opt}' = x_{10}^\tau P_1 x_{10} \leq J(u, x, w) \tag{27}$$

From lemma 4 and theorem 2 we know that $(\hat{A}, \hat{B}_2)$ is stabilizable and $(\hat{A}, Q_1 - Q_{12} Q_2^{-1} Q_{12}^\tau)$ is detectable. Then from linear system theory and lemma 5 we know that the optimal control of the linear system (18) with the performance (17) is

$$\hat{u}^* = -\hat{B}_2^\tau P_2 x_1^* \tag{28}$$

and the optimal value of (17) is

$$\tilde{J}_3(\hat{u}^*, x_1^*) = x_{10}^\tau P_2 x_{10} + [\lambda_{max} Q_3]\, \rho^2 \tag{29}$$

where $P_2$ is the unique positive semi-definite solution of the Riccati equation (26) and $x_1^*$ satisfies

$$\dot{x}_1 = (\hat{A} - \hat{B}_2 \hat{B}_2^\tau P_2) x_1, \;\; x_1(0) = x_{10}$$

Then the suboptimal control-state pair of $\Sigma_1$ is

$$\begin{bmatrix} x_1^* \\ x_2^* \\ u^* \end{bmatrix} = \begin{bmatrix} I_p & 0 & 0 \\ -V_{11} A_{21} & V_{12} & -V_{11} B_{12} \\ -V_{21} A_{21} & V_{22} & -V_{21} B_{12} \end{bmatrix} \begin{bmatrix} x_1^* \\ \bar{u}^* \\ w \end{bmatrix}$$
$$= \begin{bmatrix} I_p & 0 & 0 \\ -V_{11} A_{21} & V_{12} & -V_{11} B_{12} \\ -V_{21} A_{21} & V_{22} & -V_{21} B_{12} \end{bmatrix} \begin{bmatrix} I_p \\ -L \\ 0 \end{bmatrix} x_1^*$$

where $L = Q_2^{-1} Q_{12}^\tau + Q_2^{-\frac{1}{2}} \hat{B}_2^\tau P_2$. Then we obtain

$$x_2^* = -(V_{11} A_{21} + V_{12} L) x_1^* \tag{30}$$

and

$$u^* = -(V_{21} A_{21} + V_{22} L) x_1^* \tag{31}$$

Now we synthesize $u^*$ as the form of $u^* = K_1 x_1^* + \tilde{K}_2 x_2^*$. Let $\tilde{K}_2 = K_2$, then $A_{22} + B_{22} \tilde{K}_2$ is nonsingular. From (30)(31) we know that

$$K_1 = K_2 (V_{11} A_{21} + V_{12} L) - (V_{21} A_{21} + V_{22} L)$$

Then the closed-loop system of $\Sigma_1$ is

$$\Sigma_{1c} \begin{cases} \dot{x}_1 = A_{11c} x_1 + A_{12c} x_2 + B_{11} w, x_1(0) = x_{10} \\ 0 = A_{21c} x_1 + A_{22c} x_2 + B_{12} w \\ y = C_{1c} x_1 + C_{2c} x_2 + D_1 w \end{cases}$$

where

$$\begin{aligned} &A_{11c} = A_{11} + B_{21} K_1, A_{12c} = A_{12} + B_{21} K_2, \\ &A_{21c} = A_{21} + B_{22} K_1, A_{22c} = A_{22} + B_{22} K_2, \\ &C_{1c} = C_1 + D_2 K_1, C_{2c} = C_2 + D_2 K_2. \end{aligned}$$

Note that $A_{22c}$ is nonsingular, so the state trajectory of $\Sigma_{1c}$ is uniquely determined by disturbance $w$ and initial condition $x_{10}$.

The suboptimal control-state pair of $\Sigma$ is

$$\begin{bmatrix} x^* \\ u^* \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ u^* \end{bmatrix}$$

the suboptimal control is

$$u^* = K x^*$$

where

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} N^{-1}$$

Note that

$$\begin{aligned} &J(u, x, w) = J_1(u, (x_1, x_2), w) = J_2(\bar{u}, x_1, w) \\ &\leq \tilde{J}_2(\bar{u}, x_1) = \tilde{J}_3(\hat{u}, x_1) \end{aligned}$$

Then we have

$$J(u^*, x^*, w) \leq \tilde{J}_3(\hat{u}^*, x_1^*) = x_{10}^\tau P_2 x_{10} + [\lambda_{max} Q_3]\, \rho^2 \tag{32}$$

for all $w$ with $\| w \|_{\mathcal{L}^2} \leq \rho$. From (27)(32) we get (24).

## IV. AN EXAMPLE

Consider the following descriptor system

$$\Sigma \begin{cases} E \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_1 w + B_2 u, \\ y = C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_1 w + D_2 u \end{cases}$$

with the initial condition

$$E \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \end{bmatrix}$$

and the performance index

$$J(u, (x_1, x_2), w) = \int_0^{+\infty} y^\tau(t) y(t) \mathrm{d}t$$

where

$$E = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, D_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \| w \|_{\mathcal{L}^2} \leq \tfrac{1}{2}$$

It is easy to prove that $\Sigma$ satisfies $(A1)(A2)(A3)(A4)$. Let $K_2 = 1$, then

$$A_{22} + B_{22} K_2 = 1, \quad V = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

and (10)(19) hold. By calculation we know that

$$\bar{A}_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \bar{B}_{21} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$\bar{D}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, Q_{11} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}, Q_{12} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$
$$Q_{13} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, Q_{22} = 1, Q_{23} = -1, Q_{33} = 2,$$
$$Q_1 = \begin{bmatrix} 8 & -6 \\ -6 & 6 \end{bmatrix}, Q_2 = 2, Q_3 = 4$$

From the results in section II we know that $\Sigma$ can be transformed to the system

$$\Sigma_3 \begin{cases} \dot{x}_1 = \begin{bmatrix} -1 & 0 \\ 1 & -\frac{3}{2} \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \hat{u}, \\ \bar{x}_2 = \begin{bmatrix} -1 & 0 \end{bmatrix} x_1 + w \\ y = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \hat{u} \end{cases}$$

with

$$x_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and we have

$$J(u, (x_1, x_2), w) \leq \tilde{J}_3(\hat{u}, x_1)$$

for all $w$ with $\| w \|_{\mathcal{L}^2} \leq \frac{1}{2}$where

$$\tilde{J}_3(\hat{u}, x_1) = \int_0^{+\infty} (x_1^\tau \begin{bmatrix} 6 & -5 \\ -5 & \frac{11}{2} \end{bmatrix} x_1 + \hat{u}^\tau \hat{u}) \mathrm{d}t + 1$$

By solving the Riccati equations (25) and (26) we have

$$P_1 = \begin{bmatrix} \frac{-1538+242\sqrt{46}}{225} & \frac{74-11\sqrt{46}}{15} \\ \frac{74-11\sqrt{46}}{15} & -3 + \frac{\sqrt{46}}{2} \end{bmatrix},$$

$$P_2 = \begin{bmatrix} \frac{25\sqrt{5}-43}{8} & \frac{9-5\sqrt{5}}{2} \\ \frac{9-5\sqrt{5}}{2} & -3 + 2\sqrt{5} \end{bmatrix}$$

Then

$$L = \begin{bmatrix} \frac{5\sqrt{5}-5}{4} & 1 - \sqrt{5} \end{bmatrix}, \; K_1 = \begin{bmatrix} \frac{5-5\sqrt{5}}{4} & \sqrt{5} - 1 \end{bmatrix}$$

So

$$u^* = \begin{bmatrix} \begin{bmatrix} \frac{5-5\sqrt{5}}{4} & \sqrt{5} - 1 \end{bmatrix} & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$$

where $x_1^*$ and $x_2^*$ satisfy

$$\dot{x}_1^* = \begin{bmatrix} -1 & 0 \\ \frac{5\sqrt{5}-5}{4} & -\sqrt{5} \end{bmatrix} x_1^*, \quad x_1^*(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$x_2^* = \begin{bmatrix} \frac{5\sqrt{5}-9}{4} & 1 - \sqrt{5} \end{bmatrix} x_1^*$$

Note that

$$x_1^\tau(0) P_1 x_1(0) = \frac{-1538 + 242\sqrt{46}}{225} \approx 0.4592$$

and

$$x_1^\tau(0) P_2 x_1(0) + [\lambda_{max} Q_3] \, \rho^2 = \frac{25\sqrt{5} - 43}{8} + 1 \approx 2.6127$$

So

$$0.4592 \leq J(u^*, (x_1^*, x_2^*), w) \leq 2.6127$$

for all $w$ with $\| w \|_{\mathcal{L}^2} \leq \frac{1}{2}$.

## V. CONCLUSIONS

This paper deals with singular linear quadratic (LQ) suboptimal control problem with disturbance rejection for descriptor systems. Under the conditions we give, a suboptimal control-state pair is found such that the performance of the closed-loop system is within some range; the suboptimal control is synthesized as state feedback and the state trajectory of the closed-loop system is uniquely determined by disturbance and initial state.

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