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Stability Analysis for Switched Systems with Continuous-Time and Discrete-Time Subsystems

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***Abstract*— In this paper, we study stability property for a *new type* of switched systems which are composed of a continuous-time LTI subsystem and a discrete-time LTI subsystem. When the two subsystems are Hurwitz and Schur stable, respectively, we show that if the subsystem matrices commute each other, or if they are symmetric, then a common Lyapunov function exists for the two subsystems and that the switched system is exponentially stable under arbitrary switching. Without the assumption of commutation or symmetricity condition, we show that the switched system is exponentailly stable if the average dwell time between the subsystems is larger than a specified constant. When neither of the two subsystems is stable, we propose a sufficient condition in the form of a combination of the two subsystem matrices, under which we propose a stabilizing switching law.**

I. INTRODUCTION

In the last two decades, there has been increasing interest in stability analysis and controller design for switched systems; see the survey papers [1], [2], the recent book [3] and the references cited therein. The motivation for studying switched systems is from many aspect. It is known that many practical systems are inherently multimodal in the sense that several dynamical subsystems are required to describe their behavior which may depend on various environmental factors. Since these systems are essentially switched systems, powerful analysis or design results of switched systems are helpful dealing with real systems. Another important observation is that switching among a set of controllers for a specified system can be regarded as a switched system, and that switching has been used in adaptive control to assure stability in situations where stability can not be proved otherwise [4], [5], or to improve transient response of adaptive control systems [6]. Also, the methods of intelligent control design are based on the idea of switching among different controllers [7], [8]. Therefore, study on switched systems contributes greatly in switching controller and intelligent controller design.

When focusing on stability analysis of switched systems, there are three basic problems in stability and design of switched systems: (i) find conditions for stabilizability under arbitrary switching; (ii) identify the limited but useful class of stabilizing switching laws; and (iii) construct a stabilizing switching law. There are many existing works on these problems in the case where the switched systems are composed of continuous-time subsystems. For Problem (i), Ref. [9] showed that when all subsystems are stable and commutative pairwise, the switched linear system is stable under arbitrary switching. Ref. [10] extended this result from the commutation condition to a Lie-algebraic condition. Ref. [11] showed that a class of symmetric switched systems are asymptotically stable under arbitrary switching since a common Lyapunov function, in the form of $V(x) = x^T x$, exists for all the subsystems. Refs. [12]-[14] considered Problem (ii) using piecewise Lyapunov functions, and Ref. [15] considered Problem (ii) for switched systems with pairwise commutation or Lie-algebraic properties. Ref. [16] considered Problem (iii) by dividing the state space associated with appropriate switching depending on state, and Ref. [17] considered quadratic stabilization, which belongs to Problem (iii), for switched systems composed of a pair of unstable linear subsystems by using a linear stable combination of unstable subsystems. Ref. [18] considered quadratic stabilizability of switched linear systems with polytopic uncertainties, and Ref. [19] dealt with robust quadratic stabilization for switched LTI systems by using piecewise quadratic Lyapunov functions so that the synthesis problem can be formulated as a matrix inequality feasibility problem. Refs. [11], [20], [21], [22] extended the consideration to stability analysis problems for switched systems composed of discrete-time subsystems.

Noticing that all the above references deal with switched systems composed of only continuous-time subsystems or only discrete-time ones, we are motivated to ask the following questions: *Does it make sense to consider switched systems composed of both continuous-time and discrete-time dynamical subsystems? If so, is it possible to obtain similar results for such switched systems concerning the three basic problems?*

The answer to the first question is "YES". It is very easy to find many applications involving such switched systems. For example, in a switched system whose subsystems are all continuous-time, if we use computer to activate some of the subsystems in a discrete-time manner, then the switched system is in fact composed of both continuous-time and discrete-time subsystems. A cascaded system composed of a continuous-time plant, a set of discrete-time controller and switchings among the controllers is also a good example. Another example of a system of this kind is a continuous-time plant controlled either by a physically implemented regulator or by a digitally implemented one (and a switching rule between them). Concerning the second question, we will show that some existing results are still valid for such type of switched systems, though they may take different forms, while analysis and design problems become much more difficult for such switched systems if the approach

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is involved using combination of subsystems. For example, it is impossible to apply the idea of linear stable combination of unstable continuous-time subsystems, which was proposed in [17], to the present case.

The switched system we consider is composed of a continuous-time subsystem

$$\textbf{CS:}\ \ \dot{x}(t) = A_c x(t) \tag{1}$$

and a discrete-time subsystem

$$\textbf{DS:}\ \ x(k+1) = A_d x(k)\,, \tag{2}$$

where $x(t), x(k) \in \mathcal{R}^n$ are the subsystem states, A_c and A_d are constant matrices of appropriate dimension. To discuss stability of the switched system, we assume that the sampling period of **DS** is τ. Since the state of the discrete-time subsystem can be viewed as a piecewise constant vector between sampling points, we can consider the value of the system states in continuous-time domain. Although for notation simplicity we focus our attention on the switched system including only one continuous-time subsystem and one discrete-time subsystem, most of the results in this paper can be extended to the case of more than three subsystems in a natural way, as will be remarked later.

This paper is organized as follows. In Section 2, we consider Problem (i) for the switched system, assuming that **CS** is Hurwitz stable and **DS** is Schur stable. We show that if the two subsystem matrices commute each other, the switched system is exponentially stable, and there is a common Lyapunov function for the two subsystems. Without the commutation condition, we show that if the two subsystem matrices are symmetric, then a common Lyapunov function also exists and the switched system is exponentially stable under arbitrary switching. In Section 3, we consider Problem (ii) for the switched system when the two subsystems are Hurwitz and Schur stable, respectively. We show that if the average dwell time [12], [13] between the two subsystems is larger than a specified contant, then the switched system is exponentially stable. The lower bound of the average dwell time is computed using desired decay rate of the system. Section 4 considers the case where **CS** is not Hurwitz stable and **DS** is not Schur stable. We propose a sufficient condition in the form of a combination of the two subsystem matrices, under which we propose a stabilizing switching law. Finally, Section 5 concludes the paper.

II. ARBITRARY SWITCHING

In this section, we discuss the case where arbitrary switching is possible for the switched system composed of (1) and (2). Since arbitrary switching includes the case of dwelling on **CS** or **DS** for all time, we make the following necessary assumption.

Assumption 1: A_c is Hurwitz stable and A_d is Schur stable.

It is known that Assumption 1 is not enough to guarantee arbitrary switching. That is, a switched system composed of stable subsystems could be unstable if the switching is not done appropriately [1], [23]. Together with Assumption 1, we consider two different conditions under which arbitrary switching is possible.

A. Commutation condition

We first consider the following assumption.

Assumption 2: A_c and A_d commute each other, i.e., $A_c A_d = A_d A_c$.

Under this assumption, we can easily confirm that

$$e^{A_c t} A_d^k = A_d^k e^{A_c t} \tag{3}$$

holds for any scalar t and any positive integer k.

We now state and prove the first result.

Theorem 1: Under Assumptions 1 and 2, the switched system composed of (1) and (2) is exponentially stable under arbitrary switching.

Proof: For any time $t > 0$, we can always divide the time interval $[0, t]$ as $t = t_c + m\tau$ ($m \geq 0$), where t_c is the total duration time on **CS** and $m\tau$ is the total duration time on **DS**. Under Assumption 2, we obtain

$$x(t) = e^{A_c t_c} A_d^m x(0)\,. \tag{4}$$

According to Assumption 1,

$$\|e^{A_c t_c}\| \leq \alpha_c e^{-\lambda_c t_c} \tag{5}$$

holds with two positive scalars α_c, λ_c, and

$$\|A_d^m\| \leq \alpha_d \left(\frac{1}{\lambda_d}\right)^m \tag{6}$$

holds with two positive scalars α_d and $\lambda_d > 1$. Combining these inequalities, we obtain

$$\|x(t)\| \leq \alpha_c \alpha_d e^{-\lambda_c t_c} \left(\frac{1}{\lambda_d}\right)^m \|x(0)\| \leq \alpha_c \alpha_d e^{-\lambda t}\|x(0)\|\,, \tag{7}$$

where $\lambda = \min\left\{\lambda_c, \frac{\ln \lambda_d}{\tau}\right\}$. Since we did not add any restriction on switching laws, the switched system is exponentially stable under arbitrary switching. ∎

Remark 1: Theorem 1 and its proof remains true in the case where there are more than three subsystems who are all Hurwitz/Schur stable and commute pairwise. ∎

In the proof of Theorem 1, we used a direct method of estimating the norm of the system state. It is also known [9], [11], [22] that if we can find a common Lyapunov function for the subsystems, then we can declare immediately that the switched system is exponentially stable under arbitrary switching. The following theorem is an extension to Theorem 1 in [9], and gives a clear answer concerning existence of a common Lyapunov function for **CS** and **DS**.

Theorem 2: Suppose that Assumptions 1 and 2 hold. Let Q be an arbitrary positive definite matrix, and let P_d and

P_c be the unique positive definite solutions respectively to the Lyapunov equations

$$A_d^T P_d A_d - P_d = -Q \tag{8}$$

$$A_c^T P_c + P_c A_c = -P_d \,, \tag{9}$$

then the function

$$V(x) = x^T P_c x \tag{10}$$

is a common Lyapunov function for **CS** and **DS**.

Proof: Since the derivative of $V(x)$ along the trajectories of the subsystem **CS** is

$$\dot{V} = x^T (A_c^T P_c + P_c A_c) x = -x^T P_d x < 0 \,, \tag{11}$$

$V(x)$ is a Lyapunov function for the continuous-time subsystem.

Since the difference of $V(x)$ along the trajectories of the subsystem **DS** is

$$V(x(k+1)) - V(x(k)) = x^T(k)(A_d^T P_c A_d - P_c) x(k) \,, \tag{12}$$

what remains is to show $A_d^T P_c A_d - P_c < 0$. To do this, we substitute P_d in (9) into (8) to obtain

$$Q = A_d^T (A_c^T P_c + P_c A_c) A_d - (A_c^T P_c + P_c A_c) \,. \tag{13}$$

Using the assumption of $A_c A_d = A_d A_c$, we rewrite the above inequality as

$$Q = A_c^T (A_d^T P_c A_d - P_c) + (A_d^T P_c A_d - P_c) A_c \,. \tag{14}$$

Since A_c is Hurwitz stable and $Q > 0$, we obtain $A_d^T P_c A_d - P_c < 0$, which implies that $V(x)$ is a Lyapunov function also for the discrete-time subsystem. ∎

Remark 2: Consider the case where there are more than two continuous-time subsystems described by

$$\textbf{CS-}i\textbf{:} \quad \dot{x}(t) = A_{ci} x(t) \,, \qquad i = 1, \cdots, N_c \tag{15}$$

and there are more than two discrete-time subsystems described by

$$\textbf{DS-}j\textbf{:} \quad x(k+1) = A_{dj} x(k) \,, \qquad j = 1, \cdots, N_d \,, \tag{16}$$

where $N_c \geq 2$ and $N_d \geq 2$ are respectively the number of continuous-time and discrete-time subsystems, A_{ci}'s and A_{dj}'s are respectively constant Hurwitz and Schur stable matrices. Then, using the same proof technique of Theorems 1 and 2, we can show that if all the subsystem matrices (A_{ci}'s and A_{dj}'s) commute pairwise, then

(i) The switched system composed of (15) and (16) is exponentially stable under arbitrary switching;

(ii) There exists a common Lyapunov function for all the subsystems. The procedure of computing it is as follows. First, for any positive definite matrix Q, solve

$$A_{d1}^T P_{d1} A_{d1} - P_{d1} = -Q \tag{17}$$

with respect to $P_{d1} > 0$, and then solve for $j = 2, \cdots, N_d$,

$$A_{dj}^T P_{dj} A_{dj} - P_{dj} = -P_{d,j-1} \tag{18}$$

with respect to $P_{dj} > 0$. Secondly, solve

$$A_{c1}^T P_{c1} + P_{c1} A_{d1} = -P_{d,N_d} \tag{19}$$

with respect to $P_{c1} > 0$, and then solve for $i = 2, \cdots, N_c$,

$$A_{ci}^T P_{ci} + P_{ci} A_{ci} = -P_{c,i-1} \tag{20}$$

with respect to $P_{ci} > 0$. Then, $V(x) = x^T P_{c,N_c} x$ is a common Lyapunov function for all the subsystems in (15) and (16). ∎

B. Symmetricity condition

Here, we consider the case where Assumption 2 does not hold, yet we desire to have exponential stability under arbitrary switching. Hinted by Refs. [11], [22], we make the following assumption.

Assumption 3: Both A_c and A_d are symmetric, i.e., $A_c = A_c^T$, $A_d = A_d^T$.

The next theorem describes another case where arbitrary switching is possible.

Theorem 3: Under Assumptions 1 and 3, the switched system composed of (1) and (2) is exponentially stable under arbitrary switching.

Proof: For any switching law and any time $t > 0$, we assume without loss of generality that the switching points on $[0, t]$ are

$$t_{c1} \,,\; t_{c1} + m_1\tau \,,\; t_{c2} \,,\; t_{c2} + m_2\tau \,,\; \cdots \,,\; t_{cr} + m_r\tau \,, \tag{21}$$

which means that the switched system starts from **CS**, changes to **DS** at t_{c1}, and then changes back to **CS** at $t_{c1} + m_1\tau$, and so on, and that we are now situated on **CS**.

According to Assumptions 1 and 3,

$$A_c < -\mu_c I \tag{22}$$

holds with some positive scalar μ_c, and

$$A_d^2 < \mu_d^{-2} I \tag{23}$$

holds with some positive scalar $\mu_d > 1$.

Now, we consider the Lyapunov function candidate

$$V(x) = x^T x \tag{24}$$

for the system. On the interval $[t_{cr} + m_r\tau, t]$, we have

$$\dot{V} = 2x^T A_c x \leq -2\mu_c V \tag{25}$$

and thus

$$V(x(t)) \leq e^{-2\mu_c(t-(t_{cr}+m_r\tau))} V(x(t_{cr} + m_r\tau)) \,. \tag{26}$$

On the interval $[t_{cr}, t_{cr} + m_r\tau]$, we have

$$V(x(t_{cr} + m_r\tau)) \leq \mu_d^{-2} V(x(t_{cr} + (m_r - 1)\tau)) \tag{27}$$

due to (23), and thus

$$V(x(t_{cr} + m_r\tau)) \leq \mu_d^{-2m_r} V(x(t_{cr})) = e^{-2m_r \ln \mu_d} V(x(t_{cr})) \,. \tag{28}$$

Combining the above two inequalities results in

$$V(x(t)) \le e^{-2\mu(t-t_{cr})} V(x(t_{cr}))\,, \tag{29}$$

where $\mu = \min\{\mu_c, \frac{\ln \mu_d}{\tau}\} > 0$. By induction, we obtain

$$V(x(t)) \le e^{-2\mu t} V(x(0)) \leftrightarrow \|x(t)\| \le e^{-\mu t}\|x(0)\|\,. \tag{30}$$

Therefore, the switched system is exponentially stable under arbitrary switching. ∎

Remark 3: According to (25) and (27), we see that $V(x) = x^T x$ is a common Lyapunov function for the subsystems under Assumptions 1 and 3. ∎

III. Slow Switching by Average Dwell Time

In this section, assuming that Assumption 1 holds, we consider Problem (ii) for the switched system. Specifically, we propose a class of switching law using the average dwell time concept [12], [13] between the two subsystems. There are two different approaches, "direction computation" and "piecewise Lyapunov function", which lead to the same result.

A. Direct computation

As in Section 2, for any time $t > 0$, we assume that the switching points on $[0, t]$ are given by (21). On the interval $[0, t_{c1}]$, we have

$$x(t_{c1}) = e^{A_c t_{c1}} x(0)\,. \tag{31}$$

On the interval $[t_{c1}, t_{c1} + m_1\tau]$,

$$x(t_{c1} + m_1\tau) = A_d^{m_1} x(t_{c1})\,. \tag{32}$$

Thus, by induction, we obtain

$$\begin{aligned} x(t) &= e^{A_c(t-(t_{cr}+m_r\tau))} A_d^{m_r} e^{A_c\left(t_{cr}-(t_{c(r-1)}+m_{r-1}\tau)\right)} \\ &\quad \times A_d^{m_{r-1}} \cdots A_d^{m_1} e^{A_c t_{c1}} x(0)\,. \end{aligned} \tag{33}$$

Using the norm estimates given in (5) and (6), we obtain

$$\begin{aligned} \|x(t)\| &\le \alpha_c^{r+1} e^{-\lambda_c t_c} \alpha_d^r \left(\frac{1}{\lambda_d}\right)^{\sum_{i=1}^r m_i} \\ &= \alpha_c^{r+1}\alpha_d^r e^{-\lambda_c t_c - (\sum_{i=1}^r m_i)\ln\lambda_d} \|x(0)\| \end{aligned} \tag{34}$$

where t_c is the total duration time on **CS** as before. Then, we use $\alpha = \max\{\alpha_c, \alpha_d\}$ and $\lambda = \min\left\{\lambda_c, \frac{\ln\lambda_d}{\tau}\right\}$ to rewrite the above inequality as

$$\|x(t)\| \le \alpha^{N_t+1} e^{-\lambda t}\|x(0)\| \tag{35}$$

where N_t denotes the number of switchings occuring on $[0, t]$, and $N_t = 2r$ in the present case. It is easy to confirm that the above inequality holds for any other case besides (21).

Since $\alpha \le 1$ is a very trivial case, we consider $\alpha > 1$. According to (35), if

$$N_t \le N_0 + \frac{t}{\tau_{ad}}\,, \quad N_0 = \frac{\ln c}{\ln\alpha} - 1\,, \quad \tau_{ad} = \frac{\ln\alpha}{\lambda - \tilde{\lambda}}\,, \tag{36}$$

holds for a positive scalar $\tilde{\lambda} < \lambda$, then

$$\|x(t)\| \le c e^{-\tilde{\lambda} t}\|x(0)\|\,. \tag{37}$$

It is known that the quantity τ_{ad} given in (36) describes a lower bound for the average dwell time between the subsystems.

We summarize the above discussion in the following theorem.

Theorem 4: Under Assumption 1, the switched system composed of (1) and (2) is exponentially stable with decay rate $\tilde{\lambda}$ if the average dwell time is larger than τ_{ad} in (36). ∎

Remark 4: Theorem 4 and its proof remains true in the case where there are more than three subsystems who are all Hurwitz/Schur stable. ∎

B. Piecewise Lyapunov function

Piecewise Lyapunov functions have been used for stability analysis and design of switched systems in many references; for example, [12]-[14]. However, the piecewise Lyapunov functions proposed in these references are for the case where all subsystems are continuous-time dynamical ones. Here, we extend these considerations to the switched system composed of the continuous-time subsystem (1) and the discrete-time subsystem (2).

Since **CS** is Hurwitz stable, we can always find a positive scalar λ_c such that $A_c + \lambda_c I$ is still Hurwitz stable and thus there is a matrix $P_c > 0$ satisfying

$$\begin{aligned} &(A_c + \lambda_c I)^T P_c + P_c(A_c + \lambda_c I) < 0 \\ &\Longleftrightarrow A_c^T P_c + P_c A_c < -2\lambda_c P_c\,. \end{aligned} \tag{38}$$

Since **DS** is Schur stable, we can always find a positive scacar $\lambda_d > 1$ such that $\lambda_d A_d$ is still Schur stable and thus there is a matrix $P_d > 0$ satisfying

$$(\lambda_d A_d)^T P_d(\lambda_d A_d) - P_d < 0 \Longleftrightarrow A_d^T P_d A_d - \lambda_d^{-2} P_d < 0\,. \tag{39}$$

Note that the above inequalities are LMIs with respect to P_c and P_d, and thus are easily solved using any one of several existing softwares, such as the LMI Control Toolbox .

Using the solutions P_c and P_d, we define the following *piecewise Lyapunov function* candidate

$$V(t) = \begin{cases} x^T(t)P_c x(t) & \text{when } \textbf{CS} \text{ active at } t \\ x^T(k)P_d x(k) & \text{when } \textbf{DS} \text{ active on } [k\tau, (k+1)\tau)\,, \end{cases} \tag{40}$$

for the switched system. Then, there exist constant scalars $\alpha_2 \ge \alpha_1 > 0$ such that

$$\alpha_1\|x\|^2 \le V(t) \le \alpha_2\|x\|^2\,, \quad \forall x\,; \tag{41}$$

and there exists a constant scalar $\mu > 1$ such that for any fixed x,

$$x^T P_c x \le \mu x^T P_d x\,, \qquad x^T P_d x \le \mu x^T P_c x\,. \tag{42}$$

One example of choosing α_1, α_2, μ is $\alpha_1 = \min\{\lambda_m(P_c), \lambda_m(P_d)\}$, $\alpha_2 = \max\{\lambda_M(P_c), \lambda_M(P_d)\}$, and $\mu = \frac{\alpha_2}{\alpha_1}$. Here, $\lambda_M(\cdot)$ ($\lambda_m(\cdot)$) denotes the largest (smallest) eigenvalue of a symmetric matrix. Furthermore, using the matrix inequalities (38) and (39), we see that on the time interval where **CS** is active,

$$\dot{V}(t) \le -2\lambda_c V(t)\,, \tag{43}$$

and on the time interval where **DS** is active,

$$V((k+1)\tau) \le \lambda_d^{-2} V(k\tau). \tag{44}$$

We consider the same switching situation as in Sections 2.2 and 3.1. That is, for any time $t > 0$, assume that the switching points on $[0, t]$ are given by (21). Then, according to (43), (44) together with (42), we obtain by induction that

$$\begin{aligned} V(t) &\le e^{-2\lambda_c(t-(t_{cr}+m_r\tau))} V((t_{cr}+m_r\tau)^+) \\ &\le \mu e^{-2\lambda_c(t-(t_{cr}+m_r\tau))} V((t_{cr}+m_r\tau)^-) \\ &\le \mu \lambda_d^{-m_r} e^{-2\lambda_c(t-(t_{cr}+m_r\tau))} V(t_{cr}^+) \\ &\le \mu^2 \lambda_d^{-2m_r} e^{-2\lambda_c(t-(t_{cr}+m_r\tau))} V(t_{cr}^-) \\ &\le \cdots\cdots \\ &\le \mu^{2r} \lambda_d^{-2\sum_{i=1}^r m_i} e^{-2\lambda_c t_c} V(0)\,. \end{aligned} \tag{45}$$

Using (41), we obtain

$$\begin{aligned} \|x(t)\| &\le \sqrt{\frac{\alpha_2}{\alpha_1}} \mu^r \lambda_d^{-\sum_{i=1}^r m_i} e^{-\lambda_c t_c} \|x(0)\| \\ &= \sqrt{\frac{\alpha_2}{\alpha_1}} (\sqrt{\mu})^{2r} e^{-\lambda_c t_c - (\sum_{i=1}^r m_i)\ln \lambda_d} \|x(0)\|\,. \end{aligned} \tag{46}$$

Noting that the above inequality is almost the same as (34), we declare that an average dwell time scheme, similar to (36), can be proposed so that the switched system is exponentially stable with a specified decay rate. In this case, (40) serves as a piecewise Lyapunov function for the switched system under the average dwell time scheme.

IV. Stabilizing Switching Law

In this section, assuming that A_c is not Hurwitz stable and that A_d is not Schur stable, we consider Problem (iii) for the switched system. That is, we aim to derive a switching law stabilizing the switched system.

We assume here that Assumption 2 holds, i.e., $A_cA_d = A_dA_c$. In addition, when there exist a positive real number ξ and a positive integer q satisfying

$$\|e^{A_c\xi} A_d^q\| = \gamma < 1\,, \tag{47}$$

we propose the following switching law.

Switching law: Activate the subsystems **CS** and **DS** alternatively so that their duration time is ξU_0 and $qU_0\tau$, respectively, where U_0 is an arbitrary positive integer.

Theorem 5: Under the above switching law with Assumption 2 and Condition (47), the switched system composed of (1) and (2) is exponentially stable.

Proof: Without loss of generality, assume that we start activating **CS** and then **DS**, and so on. For any positive time t, when the present subsystem is **CS**, we can divide t as

$$t = p(\xi U_0 + qU_0\tau) + \bar{\xi} U_0 \tag{48}$$

where p is a nonnegative integer, $\bar{\xi} < \xi$ is a nonnegative scalar, and when the present subsystem is **DS**, we can always find a positive integer $\bar{q} < qU_0$ such that

$$p(\xi U_0+qU_0\tau)+\xi U_0+\bar{q}\tau \le t < p(\xi U_0+qU_0\tau)+\xi U_0+(\bar{q}+1)\tau\,. \tag{49}$$

In the case of (48), we have

$$\begin{aligned} x(t) &= e^{A_c\bar{\xi}U_0}(A_d^{qU_0} e^{A_c\xi U_0})^p x(0) \\ &= e^{A_c\bar{\xi}U_0}(e^{A_c\xi} A_d^q)^{pU_0} x(0) \end{aligned} \tag{50}$$

using $A_cA_d = A_dA_c$. Although A_c is not Hurwitz stable, we can always find two nonnegative scalars β_c and ζ_c such that

$$\|e^{A_c t}\| \le \beta_c e^{\zeta_c t} \tag{51}$$

holds for any $t \ge 0$. Thus, we obtain

$$\begin{aligned} \|x(t)\| &\le \beta_c e^{\zeta_c\bar{\xi}U_0} \gamma^{pU_0} \|x(0)\| \\ &= \beta_c e^{\zeta_c\bar{\xi}U_0} \left(\gamma^{\frac{1}{\xi+q\tau}}\right)^{pU_0(\xi+q\tau)} \|x(0)\| \\ &= \left(\beta_c e^{(\zeta_c+\hat{\lambda}_1)\bar{\xi}U_0}\right) e^{-\hat{\lambda}_1 t} \|x(0)\| \\ &\le \left(\beta_c e^{(\zeta_c+\hat{\lambda}_1)\xi U_0}\right) e^{-\hat{\lambda}_1 t} \|x(0)\| \end{aligned} \tag{52}$$

where $\hat{\lambda}_1 = \frac{-\ln\gamma}{\xi+q\tau} > 0$.

In the case of (49), since $x(t)$ is the same as that at $t_{\bar{q}} = p(\xi U_0 + qU_0\tau) + \xi U_0 + \bar{q}\tau$, we have

$$\begin{aligned} x(t) &= A_d^{\bar{q}} e^{A_c\xi U_0}(A_d^{qU_0} e^{A_c\xi U_0})^p x(0) \\ &= A_d^{\bar{q}} e^{A_c\xi U_0}(e^{A_c\xi} A_d^q)^{pU_0} x(0)\,. \end{aligned} \tag{53}$$

Although A_d is not Schur stable, we can always find two positive scalars β_d and $\zeta_d > 1$ such that

$$\|A_d^k\| \le \beta_d \zeta_d^k \tag{54}$$

holds for any $k \ge 1$. Thus, we obtain

$$\begin{aligned} \|x(t)\| &\le \beta_d \zeta_d^{\bar{q}} \beta_c e^{\zeta_c\xi U_0} \gamma^{pU_0} \|x(0)\| \\ &= \beta_d \left(\zeta_d^{\frac{1}{\tau}}\right)^{\bar{q}\tau} \beta_c e^{\zeta_c\xi U_0} \left(\gamma^{\frac{1}{\xi+q\tau}}\right)^{pU_0(\xi+q\tau)} \|x(0)\| \\ &= \left(\beta_c\beta_d e^{(\frac{\ln\zeta_d}{\tau}+\hat{\lambda}_1)\bar{q}\tau} e^{(\zeta_c+\hat{\lambda}_1)\xi U_0}\right) e^{-\hat{\lambda}_1 t} \|x(0)\| \\ &\le \left(\beta_c\beta_d e^{(\frac{\ln\zeta_d}{\tau}+\hat{\lambda}_1)qU_0\tau} e^{(\zeta_c+\hat{\lambda}_1)\xi U_0}\right) e^{-\hat{\lambda}_1 t} \|x(0)\|\,. \end{aligned} \tag{55}$$

Therefore, in both cases of (48) and (49), the switched system is exponentially stable. This completes the proof. ∎

Remark 5: In the case where there are more than three subsystems, for example, the switched system composed of (15) and (16), the condition (47) for existence of stabilizing switching law should be modified as: if all the subsystem matrices commute pairwise, and there are positive real number ξ_i and positive integers q_j such that

$$\left\| \left(e^{\sum_{i=1}^{N_c} A_{ci}\xi_i} \right) \left(\Pi_{j=1}^{N_d} A_{dj}^{q_j} \right) \right\| = \gamma < 1 \,, \qquad (56)$$

then a stabilizing switching law is: Activate the subsystems **CS-**i ($i = 1, \cdots, N_c$) in sequence and then **DS-**j ($j = 1, \cdots, N_d$) in sequence with their duration time being $\xi_i U_0$ and $q_j U_0 \tau$, respectively, where U_0 is an arbitrary positive integer. ∎

V. Conclusion

In this paper, we have studied stability property for a new type of switched systems which are composed of a continuous-time LTI subsystem and a discrete-time LTI subsystem. When the two subsystems are Hurwitz and Schur stable, respectively, we have shown that if the subsystem matrices commute each other, or if the subsystem matrices are symmetric, then a common Lyapunov function exists for the two subsystems and the switched system is exponentially stable under arbitrary switching. Without commutation or symmetricity assumption, we have shown that the switched system is exponentailly stable if the average dwell time between the subsystems is larger than a specified constant. When neither of the two subsystems is stable, we have established a sufficient condition in the form of a combination of the subsystem matrices, under which we have proposed a stabilizing switching law.

VI. References

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