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Global Stability Analysis of DC-DC Converters Using Sampled-Data Modeling

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***Abstract*— The paper presents stability analysis of a class of pulse-width modulated (PWM) systems which incorporates many different DC-DC converters. Two types of pulse-width modulation (digital and analog control) are considered. A procedure is developed for systematic search for Lyapunov functions. The state space is partitioned in such a way that stability is verified if a set of coupled Linear Matrix Inequalities (LMIs) is feasible. Global stability is considered as well as the computation of local regions of attraction.**

I. INTRODUCTION

This paper introduces a class of of pulse-width modulated (PWM) systems incorporating many different DC-DC converters [8]. Sampled-data modeling is then used to derive a systematic method for stability analysis. We consider two different PWM techniques which we refer to as digital and analog control respectively. The digital implementation has some advantages over the more common analog technique in that it is more reliable and less sensitive to aging and parameter variations. The disadvantage is that it introduces a time delay in the system since the duty ratio is calculated based on the sampled output, see [7] for further practical discussion on these issues.

Much of the reported analysis on DC-DC converters is based on the averaging approach [5], [10]. However, averaging is only an approximation of the low frequency converter dynamics and it requires sufficiently high switching frequency to be accurate. Furthermore, even if the averaged model is accurate, for many converter topologies it will be highly nonlinear and difficult to analyze. See for instance the example in Section V. In this paper the analysis is based on a sampled-data model where we take the switching frequency into account. Stability results are then derived using a quasi-linearization approach which can be applied to an arbitrary converter topology.

The literature on PWM systems is largely focused on analog DC-DC converters where the switching is determined using a comparator ramp function. Local stability results for such systems based on linearization of sampled-data models have appeared in e.g., [4], [11]. It has been shown that complicated dynamical phenomena such as chaos and various bifurcation phenomena can appear in some analog converters, see e.g. [3]. To avoid such complications we consider a class of analog converters where at most one switch take place during each period. Our previous results on digital multi phase converters in [1] are in this paper extended to this class of analog converters. Previously, global stability criteria have been discussed in [12] but for a class of converters where the switching frequency is not fixed.

To derive stability criteria for a stationary, typically periodic, solution of the PWM system we introduce an equivalent sampled-data model, similar to what is done in [9], [11]. By using quadratic Lyapunov functions we state stability conditions that can be verified by checking a set of coupled linear matrix inequalities (LMIs). We use the structure of the feedback control to partition the state-space and obtain different LMIs for different regions of the partition. In some of the regions it is necessary to sweep the duty rate over its domain of definition (typically $[0, 1]$). This is analogous to the time axis sweep used to compute quadratic surface Lyapunov function in [6], and related results in [12], which has proven to be a powerful approach for analysis of piecewise linear systems.

Criteria are presented for both global and local stability and in the local case we provide an estimate of the region of attraction.

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II. A CLASS OF PWM SYSTEMS

In this section we introduce a class of PWM systems which incorporates several DC-DC converter topologies of practical interest. The systems of the class are of the form

$$\dot{x}(t) = (A_0 + s(t)A_1)\, x(t) + (B_0 + s(t)B_1) \qquad (1)$$

where $x(t) \in R^n$, $A_i \in R^{n\times n}$, $B_i \in R^n$, $i = 0, 1$ and the switching function $s(t)$ takes values 0 and 1 only. Note that the state vector x contains the states of both converter and controller.

In DC-DC converters there are different ways of implementing the switching and hence different switching functions $s(t)$. In this paper we consider two different ways of controlling the switching - analog and digital control - and develop corresponding sampled-data models.

The first switching technique considered is digital control and the corresponding switching function is denoted $s_d(t)$. In digital control the state is sampled at times $kT,\ k \in \mathbb{N}$ to determine a duty ratio $d_k \in [0, 1]$ for each time interval $[kT, (k+1)T]$. The k^{th} duty ratio is determined by feedback

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according to

$$d_k = \begin{cases} 1, & F(x(kT) - x_r) > 1 \\ F(x(kT) - x_r), & 0 \le F(x(kT) - x_r) \le 1 \\ 0, & F(x(kT) - x_r) < 0 \end{cases} \quad (2)$$

where x_r denotes the reference trajectory of the state and F is a vector.

For every time interval $[kT, (k+1)T]$ the duty ratio determines the portion of time the switch is on ($s_d(t) = 1$) and off ($s_d(t) = 0$). More precisely

$$s_d(t) = \begin{cases} 1, & t \in [kT, (k + d_k)T) \\ 0, & t \in [(k + d_k)T, (k+1)T) \end{cases} \quad (3)$$

The second switching technique considered in the paper is analog control and the corresponding switching function is denoted $s_a(t)$. In analog control, the output $u(t)$ of a feedback controller is monitored continuously and compared to a periodic ramp function. The system switches when the trajectory crosses the ramp. The most basic model of the switching function is the following

$$s(t) = \begin{cases} 1, & u(t) - \frac{1}{T}\text{mod}(t, T) > 0 \\ 0, & u(t) - \frac{1}{T}\text{mod}(t, T) \le 0 \end{cases} \quad (4)$$

where $u(t) = F(x(t) - x_r)$ is the feedback and $\text{mod}(\cdot, \cdot)$ denotes the modulo function. However, as was noted in the introduction, analog converters have been known to exhibit chaotic behavior [3]. In this paper we avoid this phenomena by forcing there to be at most one switch in each time period. This is a natural assumption if the frequency of the ramp function is high compared to the control signal $u(t)$. Under this constraint, the switching signal $s_a(t)$ is described in (11) - (13)

III. SAMPLED-DATA MODELING

In order to prove stability of periodic solutions to the system (1) with switching functions $s_d(t)$ and $s_a(t)$ respectively, we develop corresponding sampled-data models.

A. SAMPLED-DATA MODELING, DIGITAL CONTROL

Consider the system (1) with the switching function $s_d(t)$ defined in (2) - (3). Over a time period $[kT, (k+1)T]$ with corresponding duty ratio $d_k \in [0, 1]$ the state evolves according to

$$\dot{x} = \begin{cases} (A_0 + A_1)\, x + (B_0 + B_1), & t \in [kT, (k + d_k)T) \\ A_0 x + B_0, & t \in [(k + d_k)T, (k+1)T) \end{cases} \quad (5)$$

The solutions on the two intervals are stacked to yield a discrete-time dynamic system. Using the notation $x_k = x(kT)$ we write

$$x_{k+1} = \bar{\Phi}(d_k) x_k + \bar{\Gamma}(d_k) \quad (6)$$

where the matrices $\bar{\Phi}(\cdot)$ and $\bar{\Gamma}(\cdot)$ are found in the appendix.

We assume that there exists a T-periodic solution $x^0(t)$ to the system in (1)-(3). For the sampled-data model this means that the point $x^0(0)$ is stationary, i.e $x_k = x^0(0) \quad \forall\ k \in \mathbb{N}$. We write $x^0 = x^0(0)$ and denote the corresponding stationary duty ratio $d^0 = F(x^0 - x_r)$. The stationary point (x^0, d^0) can be found by solving the following set of nonlinear equations

$$\begin{aligned} x^0 &= \bar{\Phi}(d^0) x^0 + \bar{\Gamma}(d^0) \\ d^0 &= F(x^0 - x_r) \end{aligned} \quad (7)$$

To prove either local or global stability of the discrete-time system we consider the deviation $\Delta x_k = x_k - x^0$ from the fixed point. For ease of notation we write $y_k = \Delta x_k$ and note that the error dynamics satisfy

$$y_{k+1} = \Phi(d_k) y_k + \Gamma(d_k) \quad (8a)$$

$$d_k = d^0 + \psi(F y_k) \quad (8b)$$

where $\psi(\cdot)$ is the saturation

$$\psi(Fy) = \begin{cases} -d^0, & Fy < -d^0 \\ Fy, & -d^0 \le Fy \le 1 - d^0 \\ 1 - d^0, & Fy > 1 - d^0 \end{cases}$$

and

$$\Phi(d) = \bar{\Phi}(d) \quad (9)$$

$$\Gamma(d) = (\bar{\Phi}(d) - \bar{\Phi}(d^0)) x^0 + \bar{\Gamma}(d) - \bar{\Gamma}(d^0) \quad (10)$$

Note that $(y, d) = (0, d^0)$ is a stationary point of the error model (8). If $(0, d^0)$ is an exponentially stable equilibrium of (8) then the limit cycle $x^0(t)$ is exponentially stable. Whether or not $(y, d) = (0, d^0)$ is a unique stationary point of (8) is a question of great importance to any further stability analysis. A necessary condition for uniqueness is given in [1].

B. SAMPLED-DATA MODELING, ANALOG CONTROL

Consider the system (1) with the switching function $s_a(t)$ described above. The duty ratio d_k is no longer calculated at time kT. Instead it is determined during the corresponding time interval when the controller output crosses the ramp function. Nevertheless, having forced the system to switch at most once per time interval, we can write d_k as a function of $x(kT)$. This means the analog switching function $s_a(t)$ can be written in the same form as the digital one and we can develop a sampled-data model analogous to the one above. We write

$$s_a(t) = \begin{cases} 1, & t \in [kT, (k + d_k)T) \\ 0, & t \in [(k + d_k)T, (k+1)T) \end{cases} \quad (11)$$

The expression above is identical to the expression (3) describing $s_d(t)$. However, the duty ratio d_k is now calculated in a different way

$$d_k = \begin{cases} 0, & F(x(kT) - x_r) \le 0 \\ d^*, & F(x(kT) - x_r) > 0 \end{cases} \quad (12)$$

where d^* is the solution to

$$\min d$$
$$s.t. \begin{cases} F(\bar{\Phi}_1(d)x_k + \bar{\Gamma}_1(d) - x_r) = d \\ 0 \leq d \leq 1 \end{cases} \quad (13)$$

and the matrices $\bar{\Phi}_1(\cdot)$ and $\bar{\Gamma}_1(\cdot)$ are found in the appendix. If (13) is infeasible, we set d^* to 1.

The expressions above should be understood as follows: If the controller output $u = F(x-x_r)$ is less than or equal to zero at time kT the duty ratio is set to $d_k = 0$ and the switch remains open during the entire interval $[kT, (k+1)T]$. If the controller output is greater than zero, problem (13) is solved to see when the controller output intersects the ramp. This yields the duty ratio d_k. If the minimization problem does not have a solution we conclude that the output does not cross the ramp during the interval in question and the switch remains closed during the entire interval.

Writing the switching function as in (11) enables us to derive a discrete-time dynamic system identical to (6)

$$x_{k+1} = \bar{\Phi}(d_k)x_k + \bar{\Gamma}(d_k) \quad (14)$$

only now the duty ratio is not a linear function of the state x_k but is given by (12)- (13).

The error model is also derived in analogy with the discrete control case. We assume there is a fixed point (x^0, d^0) solving the system of nonlinear equations

$$\begin{aligned} & x^0 = \bar{\Phi}(d^0)x^0 + \bar{\Gamma}(d^0) \\ & F(\bar{\Phi}_1(d^0)x + \bar{\Gamma}_1(d^0) - x_r) = d^0 \\ & 0 \leq d^0 \leq 1 \end{aligned} \quad (15)$$

and consider the deviation $\Delta x_k = x_k - x^0$. Again using the notation $y_k = \Delta x_k$ the error dynamics satisfy

$$y_{k+1} = \Phi(d_k)y_k + \Gamma(d_k) \quad (16a)$$

$$d_k = \begin{cases} 0, & F(x^0 - x_r) + Fy_k \leq 0 \\ d^*, & F(x^0 - x_r) + Fy_k > 0 \end{cases} \quad (16b)$$

where $\Phi(\cdot)$ and $\Gamma(\cdot)$ are given in (9)- (10) and d^* is the solution to (13) or $d^* = 1$ if (13) is infeasible.

IV. STABILITY ANALYSIS

The discrete time error models in (8) and (16) describe how a perturbation $x - x^0$ of a stationary periodic solution of (1) evolves at the switching instants kT. If we prove stability of the error models, this also implies stability of the continuous time solutions since the dynamics between the switching instances is affine.

To obtain sufficient conditions for the error models to be exponentially stable we use discrete time Lyapunov theory. The following result is fundamental to our analysis

Lemma 1: Consider a sequence $\{y_k\} \in R^n$ and a function $V_k := V(y_k) : R^n \to R$. If there are real numbers α_1, α_2, $\beta > 0$ such that $\alpha_1\|y_k\|^2 \leq V_k \leq \alpha_2\|y_k\|^2$ and

$$V_{k+1} - V_k \leq -\beta \parallel y_k \parallel^2$$

for all k then the sequence is exponentially stable (about the origin)

Now consider a sequence $\{y_k\}$ satisfying the error dynamics $y_{k+1} = \Phi(d_k)y_k + \Gamma(d_k)$. Using a quadratic Lyapunov function $V_k = y_k^T P y_k$ the inequality $V_{k+1} - V_k \leq -\beta \parallel y_k \parallel^2$ of lemma 1 can be written in the quasi-quadratic form.

$$\bar{y}_k^T \Pi(d_k, \beta, P)\bar{y}_k \leq 0 \quad (17)$$

where $\bar{y}_k = [y_k, 1]^T$ and

$$\Pi(d_k, \beta, P) = \begin{bmatrix} \Phi(d_k)^T P\Phi(d_k) - P + \beta I & \Phi(d_k)^T P\Gamma(d_k) \\ \Gamma(d_k)^T P\Phi(d_k) & \Gamma(d_k)^T P\Gamma(d_k) \end{bmatrix}$$

In the sections below, we use a partition of the state space to derive stability conditions from inequality (17).

A. STABILITY ANALYSIS, DIGITAL CONTROL

To test the inequality (17) we utilize a partition of the state space. The fact that the duty ratio d is confined to the interval $[0, 1]$ gives a natural partition of the state space into three (unbounded) polyhedral sets with pairwise disjoint interior. We write

$$\mathbf{R}^n = \bigcup_{i \in \{0,1,2\}} \mathbb{S}_i \quad (18)$$

where

$$\mathbb{S}_i = \{y \mid \bar{E}_i\bar{y} = \begin{bmatrix} E_i & e_i \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} \leq 0\}$$

and the vector inequality $z \leq 0$ is component-wise. In particular we have

$$\bar{E}_0 = \begin{bmatrix} F & d^0 \end{bmatrix}, \quad \bar{E}_1 = \begin{bmatrix} -F & 1 - d^0 \end{bmatrix}$$
$$\bar{E}_2 = \begin{bmatrix} F & d^0 - 1 \\ -F & -d^0 \end{bmatrix}$$

Thus, $\mathbb{S}_0$ and $\mathbb{S}_1$ are the regions where the feedback $d_k = d^0 + \psi(Fy_k)$ saturates and $\mathbb{S}_2$ is the non-saturated region.

When testing (17) we also make use of a parameterization of $\mathbb{S}_2$. Let the matrix Z be the orthogonal complement of the feedback matrix F, i.e.

$$Z = F^\perp \quad (19)$$

where $F^\perp$ satisfies $FF^\perp = 0$, $(F^\perp)^T F^\perp = I$. For any $d \in R$, the set of points $y \in R^n$ such that $d = d^0 + Fy$ is a hyperplane in R^n. If we let $\gamma(d) = \frac{d-d^0}{FF^T}F^T$ then the set $\mathbb{S}_2$ can be expressed as

$$\mathbb{S}_2 = \{y \mid y = Zx + \gamma(d), x \in R^{n-1}, d \in [0, 1]\} \quad (20)$$

This means that as d varies in the interval $[0, 1]$ we sweep the entire non-saturated region.

We now consider the inequality (17) on each of the regions $\mathbb{S}_i$. In the regions $\mathbb{S}_0$ and $\mathbb{S}_1$ we apply the S-procedure [2] to obtain a lossless relaxation of the inequalities $\bar{E}_i\bar{y} \leq 0$. In the region $\mathbb{S}_2$ we make use of the parameterization described in (20). By Lemma 1 we have

Theorem 1: Consider the error model (8). If there are real numbers $\beta > 0$, $\tau_i \geq 0$ and a matrix $P = P^T > 0$ such that

$$\Pi(i, \beta, P) - \tau_i \begin{bmatrix} 0 & E_i^T \\ E_i & 2e_i \end{bmatrix} \leq 0$$

for $i = 0, 1$ and

$$\begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix}^T \Pi(d, \beta, P) \begin{bmatrix} Z & \gamma(d) \\ 0 & 1 \end{bmatrix} \leq 0$$

for $d \in [0, 1]$, then (8) is exponentially stable about the origin.

Remark 1: In order to check the last condition of Theorem 1 we partition the interval $[0, 1]$ into points d_i. For a fixed d_i the condition is an LMI and verifying the theorem thus amounts to checking a set of coupled LMIs. To deduce feasibility of the inequality above from the result on a partition is a non-trivial matter, but we omit the details here. See [1] for a thorough treatment of the subject.

B. STABILITY ANALYSIS, ANALOG CONTROL

Just as in the digital control case, we test stability by checking the inequality (17). We again use a partition of the state space, but because of the new way of calculating the duty ratio the result is slightly different.

Looking at equation (16) and how the duty ratio d_k is determined it is clear that for any point y_k, one and only one of the following statements holds.

- $F(x^0 - x_r) + Fy_k \leq 0$. In this case $d_k = 0$.
- $F(x^0 - x_r) + Fy_k > 0$ and the equation $F(\bar{\Phi}_1(d)(x^0 + y_k) + \bar{\Gamma}_1(d) - x_r) = d$ has a solution for $d \in [0, 1]$. In this case $d_k = d$.
- $F(x^0 - x_r) + Fy_k > 0$ and the equation $F(\bar{\Phi}_1(d)(x^0 + y_k) + \bar{\Gamma}_1(d) - x_r) = d$ has no solution for $d \in [0, 1]$. In this case $d_k = 1$.

This defines a similar partition of the state space as in (18). The first and the third bullet points correspond to the sets

$$\mathbb{S}_0 = \{y \mid \bar{E}_0 \bar{y} \leq 0\}, \quad \mathbb{S}_1 = \{y \mid \bar{E}_1 \bar{y} < 0\}$$

where the matrices $\bar{E}_i = [E_i, e_i],\ i = 0, 1$, are

$$\bar{E}_0 = \begin{bmatrix} F & F(x^0 - x_r) \end{bmatrix}$$
$$\bar{E}_1 = \begin{bmatrix} -F & -F(x^0 - x_r) \\ -F\bar{\Phi}_1(1) & 1 - F(\bar{\Phi}_1(1)x^0 + \bar{\Gamma}_1(1) - x_r) \end{bmatrix}$$

The set S_1 takes this form since $F(\bar{\Phi}_1(0)(x^0 + y) + \bar{\Gamma}_1(0) - x_r) = F(x^0 + y - x_r) > 0$ and the equation $F(\bar{\Phi}_1(d)(x^0 + y) + \bar{\Gamma}_1(d) - x_r) = d$ has no solution for $d \in [0, 1]$, which by continuity implies $F(\bar{\Phi}_1(1)(x^0 + y) + \bar{\Gamma}_1(1) - x_r) > 1$.

The non-saturated region corresponding to the second bullet point above is denoted by $\mathbb{S}_2$ and it is described using a parameterization. Let

$$Z(d) = (F\bar{\Phi}_1(d))^\perp$$

and

$$\gamma(d) = \frac{d - F(\bar{\Phi}_1(d)x^0 + \bar{\Gamma}_1(d) - x_r)}{F\bar{\Phi}_1(d)\bar{\Phi}_1(d)^T F^T} \bar{\Phi}_1(d)^T F^T$$

then $\mathbb{S}_2$ can be expressed as

$$\mathbb{S}_2 = \{y \mid \bar{E}_{11}\bar{y} < 0 \text{ and } y = Z(d)x + \gamma(d), \quad x \in R^{n-1}, d \in [0, 1]\} \qquad (21)$$

where $\bar{E}_{11}$ denotes the first row of $\bar{E}_1$.

Again, we consider the inequality (17) on each region $\mathbb{S}_i$. The S-procedure is used on all regions to obtain relaxations of the corresponding inequalities and on $\mathbb{S}_2$ we also make use of the parameterization described in (21). We have

Theorem 2: Consider the error model (16). Let $\bar{E}_{ij} = [E_{ij}\ e_{ij}]$ denote the j^{th} row of $\bar{E}_i$ and let $\Pi = \Pi(d, \beta, P)$. If there are real numbers $\beta > 0$, $\tau_{ij} \geq 0$ and a matrix $P = P^T > 0$ such that

$$\Pi(i, \beta, P) - \sum_j \tau_{ij} \begin{bmatrix} 0 & E_{ij}^T \\ E_{ij} & 2e_{ij} \end{bmatrix} \leq 0$$

for $i = 0, 1$ and

$$\begin{bmatrix} Z(d) & \gamma(d) \\ 0 & 1 \end{bmatrix}^T \left(\Pi - \tau_{21} \begin{bmatrix} 0 & E_{11}^T \\ E_{11} & 2e_{11} \end{bmatrix} \right) \begin{bmatrix} Z(d) & \gamma(d) \\ 0 & 1 \end{bmatrix} \leq 0$$

for $d \in [0, 1]$, then (16) is globally exponentially stable.

V. LOCAL RESULTS

If the error models (8) or (16) are not globally stable, we confine our attention to some ellipsoid $E_Q = \{y \mid y^T Q y \leq 1\}$. The idea is to prove local stability by finding a contractive set in E_Q.

A. LOCAL RESULTS, DIGITAL CONTROL

Let y_k be a sequence satisfying (8) and let $V(y) = y^T P y$ be a quadratic Lyapunov candidate. First we require $V(y)$ to be strictly decreasing on E_Q, i.e.

$$\bar{y}^T \Pi(d, \beta, P) \bar{y} \leq 0 \quad \forall\ y \in E_Q \qquad (22)$$

The inequality above is not enough to prove stability on E_Q. This is because there is no guarantee that y_{k+1} will stay in E_Q. However, if (22) is satisfied, then a level surface of $V(y)$ contained in E_Q is the boundary of a contractive set. Therefore we impose a second condition on $V(y)$, namely that

$$\{y \mid V(y) \leq 1\} \subseteq E_Q \Leftrightarrow$$
$$y^T Q y \leq 1 \quad \forall\ y \text{ s.t. } \quad y^T P y \leq 1 \qquad (23)$$

If (22) and (23) are both satisfied, then the error model (8) is exponentially stable on the set $E_P = \{y \mid y^T P y \leq 1\}$.

With Z and $\gamma(d)$ as in equation (20), any point $y \in R^n$ can be represented as $y = Zx + \gamma(\bar{d}), x \in R^{n-1}, \bar{d} \in R$. (Note that $\bar{d} \neq d$, but $d = d^0 + \psi(\bar{d} - d^0)$ is in fact a saturated function of $\bar{d}$). We let D be the solution to the problem

$$\begin{array}{ll} \max & \bar{d} \\ \text{s.t} & (Zx + \gamma(\bar{d}))^T Q (Zx + \gamma(\bar{d})) \leq 1 \end{array}$$

and this means that we can cover E_Q by sweeping $\bar{d}$ over the interval $[-D, D]$.

Using the parameterization above the condition (22) is written

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Z & \gamma(\bar d) \\ 0 & 1 \end{bmatrix}^T \Pi(d,\beta,P) \begin{bmatrix} Z & \gamma(\bar d) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0 \quad \text{(24a)}$$

$\forall\ (x, \bar d)$ s.t

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Z & \gamma(\bar d) \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z & \gamma(\bar d) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0 \quad \text{(24b)}$$

For a fixed $\bar d \in [-D, D]$ we apply the S-procedure to the condition (24a)- (24b). In doing so we obtain a bilinear matrix inequality and can state the following

Theorem 3: Consider the error model (8) on the ellipsoid $E_Q = \{y \mid y^T Q y \leq 1\}$. Let $\Pi = \Pi(d, \beta, P)$ and $d = d^0 + \psi(\bar d - d^0)$. If there are real numbers $\beta, \lambda > 0$, a function $\tau(\bar d) \geq 0$ and a matrix $P = P^T > 0$ such that

$$\lambda \begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} \leq 0$$

$$\begin{bmatrix} Z & \gamma(\bar d) \\ 0 & 1 \end{bmatrix}^T \left(\Pi - \tau(\bar d) \begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} Z & \gamma(\bar d) \\ 0 & 1 \end{bmatrix} \leq 0$$

for $\bar d \in [-D, D]$, then the error model (8) is exponentially stable on $E_P = \{y \mid y^T P y \ \leq \ 1 \ \}$.

Remark 2: To verify Theorem 3 we grid the entire interval $[-D, D]$ and this can be computationally burdensome when D is large. A less work intensive approach would be to grid only the non-saturated region and relax the inequalities on the saturated half-spaces. However, in this case the S-procedure may be lossy, causing a conservative estimate of the stability region. In the numerical example of this paper the "full grid" approach resulted in a larger region of attraction.

B. LOCAL RESULTS, ANALOG CONTROL

In analogy with the discrete control case we want to verify conditions (22) and (23), thereby proving exponential stability on E_P. We consider the partition of the state-space described in Section IV-B and use different parameterizations to cover different regions. Let

$$\begin{aligned} \kappa &= F(x^0 - x_r) \\ G(d) &= F\bar\Phi_1(d) \\ g(d) &= d - F(\bar\Phi_1(d)x^0 + \bar\Gamma_1(d) - x_r) \end{aligned}$$

and let $\bar d = [d_1, d_2]^T$ be the parameter vector. We use these expressions to define

$$Z_0 = F^\perp, \quad Z_1 = \begin{bmatrix} F \\ G(1) \end{bmatrix}^\perp, \quad Z_2(\bar d) = \begin{bmatrix} F \\ G(d_2) \end{bmatrix}^\perp$$

$$\gamma_0(\bar d) = \frac{d_1 - \kappa}{FF^T} F^T$$

$$\gamma_1(\bar d) = \begin{bmatrix} F \\ G(1) \end{bmatrix}^T \left(\begin{bmatrix} F \\ G(1) \end{bmatrix} \begin{bmatrix} F \\ G(1) \end{bmatrix}^T \right)^{-1} \begin{bmatrix} d_1 - \kappa \\ g(1) + d_2 \end{bmatrix}$$

$$\gamma_2(\bar d) = \begin{bmatrix} F \\ G(d_2) \end{bmatrix}^T \left(\begin{bmatrix} F \\ G(d_2) \end{bmatrix} \begin{bmatrix} F \\ G(d_2) \end{bmatrix}^T \right)^{-1} \begin{bmatrix} d_1 - \kappa \\ g(d_2) \end{bmatrix}$$

The sets $\mathbb{S}_i$ can now be written

$$\begin{aligned} \mathbb{S}_0 &= \{y \mid y = Z_0 x + \gamma_0(\bar d),\ x \in R^{n-1},\ d_1 \leq 0\} \\ \mathbb{S}_1 &= \{y \mid y = Z_1 x + \gamma_1(\bar d),\ x \in R^{n-2},\ d_1 \geq 0, d_2 \geq 0\} \\ \mathbb{S}_2 &= \{y \mid y = Z_2(\bar d) x + \gamma_2(\bar d),\ x \in R^{n-2},\ d_1 \geq 0, \\ &\qquad d_2 \in [0, 1]\} \end{aligned}$$

We can now cover the regions $E_Q \cap \mathbb{S}_i$ by sweeping the corresponding parameters over suitable areas. Let $\Omega_i \subset R^2$ be the sets satisfying

$$\begin{aligned} &\forall \bar d \in \Omega_i\ \exists\ x \text{ s.t } y = Z_i x + \gamma_i(\bar d) \in E_Q \cap \mathbb{S}_i \\ &\forall \bar d \notin \Omega_i\ \nexists\ x \text{ s.t } y = Z_i x + \gamma_i(\bar d) \in E_Q \cap \mathbb{S}_i \end{aligned}$$

These sets can easily be estimated numerically and this allows us to verify the following conditions.

Theorem 4: Consider the error model (16) on the ellipsoid $E_Q = \{y \mid y^T Q y \leq 1\}$. If there are real numbers $\beta, \lambda > 0$, functions $\tau_i(\bar d) \geq 0$ and a matrix $P = P^T > 0$ such that

$$\lambda \begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} \leq 0$$

$$\begin{bmatrix} Z_i & \gamma_i(\bar d) \\ 0 & 1 \end{bmatrix}^T \left(\Pi(i, \beta, P) - \tau_i(\bar d) \begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} Z_i & \gamma_i(\bar d) \\ 0 & 1 \end{bmatrix} \leq 0$$

for $\bar d \in \Omega_i,\ i = 0, 1$ and

$$\begin{bmatrix} Z_2(\bar d) & \gamma_2(\bar d) \\ 0 & 1 \end{bmatrix}^T \Bigg(\Pi(d_2, \beta, P) - \tau_2(\bar d) \begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix} \Bigg) \begin{bmatrix} Z_2(\bar d) & \gamma_2(\bar d) \\ 0 & 1 \end{bmatrix} \leq 0$$

for $\bar d \in \Omega_2$, then the error model (16) is exponentially stable on the ellipsoid $E_P = \{y \mid y^T P y \ \leq \ 1 \ \}$.

Remark 3: Verifying Theorem 4 may be computationally heavy. In Remark 2 we discussed a less work intensive approach which can be used also here.

Example 1: The converter in Fig. 1 is designed to transform a source voltage of 3000V into a load voltage of 1500V. The parameters of the system are

$$\begin{aligned} &L_1 = 23\text{mH}, \quad C_1 = 2\text{mF}, \quad L_2 = 11\text{mH}, \quad C_2 = 16\text{mF}, \\ &E = 3\text{kV}, \quad I_{load} = 200\text{A}, \quad T = 2.5\text{ms} \end{aligned}$$

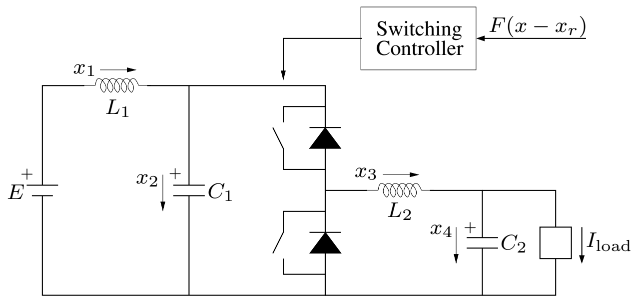

Fig. 1. Bidirectional DC-DC step-down converter with input filter.

The feedback controller contains a low pass filter to attenuate ripple in the output voltage. This adds an extra state to the converter dynamics ($x_5 = \frac{\omega}{s+\omega} x_4$ in the Laplace domain). The cutoff frequency of the filter is $\omega = 100 \cdot 2\pi$ rad/s. The feedback matrix is chosen as

$$F = \begin{bmatrix} 0 & 0 & 0 & -0.00369 & 0.00348 \end{bmatrix}$$

and the reference signal x_r is chosen such that $F x_r = -0.8114$.

To prove local stability of this system, we first numerically solve for fixed points (x^0, d^0) satisfying (7) and (15) respectively. Exponential stability is then proved on sets E_P by verifying the conditions in Theorems 3 and 4. The conditions can be formulated as checking the feasibility of a set of linear matrix inequalities. These are solved using the MATLAB LMI Control Toolbox. We choose the parameter $Q = \lambda I$ and solve the problem for gradually decreasing values of λ. The fixed points (x^0, d^0), parameters λ for which the problem is feasible and the smallest and largest eigenvalue of P, denoted ρ_{min} and ρ_{max} are given in the table below.

	Digital Control
x^0	$\begin{bmatrix} 100.395 & 3026.917 & 114.284 & 1504.073 & 1504.493 \end{bmatrix}$
d^0	0.5006
λ	$1 \cdot 10^{-7}$
ρ_{min}	$1 \cdot 10^{-7}$
ρ_{max}	$1.6 \cdot 10^{-5}$
	Analog Control
x^0	$\begin{bmatrix} 100.032 & 3026.949 & 114.287 & 1498.667 & 1499.094 \end{bmatrix}$
d^0	0.4988
λ	$1 \cdot 10^{-8}$
ρ_{min}	$1 \cdot 10^{-8}$
ρ_{max}	$1.8 \cdot 10^{-6}$

VI. CONCLUDING REMARKS

The stability analysis presented in this paper is systematic and easy to implement. It was applied to a non-trivial example and provided a region of attraction of considerable size. We suspect that for most DC-DC converters, quadratic stability cannot be proven globally. A topic for further research is therefore to improve the local results obtaining larger regions of stability. We believe this can be done by using extensions of the quadratic Lyapunov function that further exploits the structure of the system. One possibility is to use the piecewise quadratic Lyapunov function.

APPENDIX: DERIVATION OF DISCRETE-TIME DYNAMIC SYSTEM

Using the notation $x_k = x(kT)$ and $x_k^1 = x((k+d_k)T)$ we write

$$x_k^1 = \bar{\Phi}_1(d_k) x_k + \bar{\Gamma}_1(d_k)$$
$$x_{k+1} = \bar{\Phi}_2(d_k) x_k^1 + \bar{\Gamma}_2(d_k)$$

where

$$\bar{\Phi}_1(d_k) = e^{(A_0 + A_1) d_k T}$$
$$\bar{\Phi}_2(d_k) = e^{A_0 (1 - d_k) T}$$
$$\bar{\Gamma}_1(d_k) = (A_0 + A_1)^{-1} (e^{(A_0 + A_1) d_k T} - I)(B_0 + B_1)$$
$$\bar{\Gamma}_2(d_k) = A_0^{-1} (e^{A_0 (1 - d_k) T} - I) B_0$$

Note that the matrix operator $A^{-1}(e^{AT} - I)$ generally should be interpreted as the integral $\int_0^T e^{At} dt$. The solutions on the two intervals are stacked to yield the discrete-time dynamic system

$$x_{k+1} = \bar{\Phi}(d_k) x_k + \bar{\Gamma}(d_k)$$

where

$$\bar{\Phi}(d_k) = \bar{\Phi}_2(d_k) \bar{\Phi}_1(d_k)$$
$$\bar{\Gamma}(d_k) = \bar{\Phi}_2(d_k) \bar{\Gamma}_1(d_k) + \bar{\Gamma}_2(d_k)$$

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