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Robust H_∞ Control and Quadratic Stabilization of Uncertain Switched Linear Systems

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***Abstract*— In this paper, we focus on the robust H_∞ control analysis and synthesis for uncertain switched linear systems. It is shown that the quadratic stability of one associated non-switched uncertain combination linear system implies the feasibility of H_∞ control problem for an uncertain switched linear system. Then, the robust H_∞ control synthesis via switched state feedback is studied. A switched state feedback controller is designed to quadratically stabilize the plant and achieve a prescribed disturbance attenuation level γ for all admissible uncertainties simultaneously. All the results in this paper can be regarded as an extension of some existing results for both switched and non-switched systems.**

I. INTRODUCTION

As an important class of hybrid dynamic systems consisting of a family of linear time-invariant subsystems and a switching law specifying the switching among them, switched linear systems have been studied by a large number of papers (e.g., [1-19] and the references therein). Switched systems and switching control have recently gained a great deal of attention mainly because that many real-world systems in, for example, chemical processes, transportations, computer controlled systems and communication industries can be modelled as switched systems. And they also have lots of applications in many other fields [1-3]. It should be noted that some basic problems have been outlined in [4] and for recent progress in the field of switched systems, we refer to the survey papers of [4] and [5].

There are some existing results concerning quadratic stability of switched linear systems, see for instance [6-10]. For switched linear systems, although there are lots of papers concerning the stability, quadratic stabilization, controllability and so on, few results are concerned with the H_∞ control problem. However, H_∞ control is one of the most active subfields of research in control theory and H_∞ performance is also an extremely important performance. In [11], Hespanha presented a method to compute the slow-switching RMS gain of a switched linear system. The algorithm proposed uses the fact that a given constant γ provides an upper bound on the RMS gain whenever there is a separation between all the stabilizing and all the antistabilizing solutions to the algebraic Riccati equations of the systems being switched. Zhai $et\ al.$ [12] investigated the disturbance attenuation properties of time-controlled switched systems. They have showed that when all subsystems are Hurwitz stable and achieve a disturbance attenuation level γ_0, the switched system achieves a weighted disturbance attenuation level γ_0, which approaches normal disturbance attenuation if the average dwell time is sufficiently large. Several other papers such as [13], [14] and [15] are also dedicated to the study of H_∞ related problems for some kind of hybrid systems.

As to the study of disturbance attenuation of switched systems, Hespanha [11] and Zhai $et\ al.$ [12] have assumed that at least one subsystem must be Hurwitz stable. However, it is well known that the stability analysis of switched systems is complicated by the fact that there are switches of discrete states. Even if each subsystem is stable, the switched system will not necessarily be stable. Contrary, switching between individually unstable subsystems does not necessarily imply that the switched system is unstable [16], [17]. In this paper, we do not take the assumption made in [11] and [12] and focus on the following problem:

Is it possible for us to obtain a prescribed disturbance attenuation level γ via an appropriately designed switching rule when all subsystems are not Hurwitz stable ?

Here, we will show that the answer to this question is *YES*. The example in section IV is used to verify this fact. Moreover, the following two problems are also studied in this paper. One is the disturbance attenuation for the case that uncertainty enters state and input matrices of switched linear systems, the other is the H_∞ control synthesis problem via switched state feedback for uncertain switched systems.

Motivated by the above existing results on quadratic stabilization and disturbance attenuation properties of switched systems, we focuse on the robust H_∞ control problem of switched linear systems with uncertainties in both the state and input matrices via designing a switching rule and state feedback. It is assumed that the switching rule $r(x,t)$ used in this paper is state-dependent, which is also utilized in some papers (e.g., [6], [7], [18] and [19]). Our goal is to design switching rule $r(x,t)$ and an associated state feedback $u = K_{r(x,t)}x$ such that the resulting closed-loop system of uncertain switched systems is quadratically stabilizable with a prescribed level of H_∞ disturbance attenuation γ for all admissible uncertainties. First, the quadratic stability of uncertain switched linear systems is analyzed. Then, the disturbance attenuation property is investigated by taking advantage of a min-projection strategy. We show

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that for a given scalar $\gamma > 0$, the uncertain switched system is quadratically stabilizable with an H_∞ disturbance attenuation level γ via switching if one associated uncertain combination linear non-switched system is quadratically stable. Finally, these results are extended to cope with the switched state feedback H_∞ control problem. All the results in this paper can be regarded as an extension of some existing results for both switched and non-switched systems.

II. QUADRATIC STABILIZATION WITH DISTURBANCE ATTENUATION VIA MIN-PROJECTION STRATEGY

In this section, we study quadratic stabilization with disturbance attenuation for uncertain switched linear systems only via switching.

Consider the following uncertain switched linear systems:

$$\begin{cases} \dot{x}(t) = (A_r + \Delta A_r)x(t) + B_{1r}w(t) + (B_{2r} + \Delta B_{2r})u(t) \\ z(t) = C_r x(t) + D_r u(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $w(t) \in \mathbb{R}^h$ is the exogenous input which belongs to $L_2[0,\infty)$, $L_2[0,\infty)$ denotes the space of square integrable functions on $[0,\infty)$, $z(t) \in \mathbb{R}^q$ is the controlled output. The right continuous function $r := r(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \{1, 2, \cdots, l\}$ (denoted as $\underline{l}$) is the switching signal to be designed. Moreover, $r = i$ implies that the i-th subsystem is activated.

$$[\Delta A_i,\ \Delta B_{2i}] = E_i \Gamma [F_{1i},\ F_{2i}],\ \ \forall i \in \underline{l}. \tag{2}$$

A_i, B_{1i}, B_{2i}, C_i and D_i are constant matrices of appropriate dimensions that describe the nominal systems, E_i, F_{1i}, F_{2i} are given matrices which characterize the structure of uncertainty. Γ is the norm-bounded time-varying uncertainty,

$$\Gamma = \Gamma(t) \in \{\Gamma(t) : \Gamma(t)^T\Gamma(t) \le I, \Gamma(t) \in \mathbb{R}^{m\times k}, \}.$$

There are several reasons for assuming that the system uncertainty has the structure given in (2), which can be found in [20]. Without loss of generality, the following assumption is made for the sake of technical simplification. We refer to [21] for detailed discussion of this assumption.

Assumption 1: $D_i^T[C_i,\ D_i] = [0,\ I],\ i \in \underline{l}$.

Let us consider the following switched systems simplified from (1):

$$\begin{cases} \dot{x}(t) = A_r x(t) + B_{1r}w(t) \\ z(t) = C_r x(t) \end{cases} \tag{3}$$

Definition 1: The switched linear system (3) with $w = 0$ (i.e., the nominal system) is said to be **quadratically stabilizable via switching** if there exist a positive definite function $V(x) = x^T P x$, a positive scalar ε and a switching rule $r(x,t)$ such that

$$\frac{d}{dt}V(x) < -\varepsilon x^T x.$$

When there is uncertainty ΔA_r in the state matrix of (3), the system reads

$$\begin{cases} \dot{x}(t) = (A_r + \Delta A_r)x(t) + B_{1r}w(t) \\ z(t) = C_r x(t) \end{cases} \tag{4}$$

Definition 2: The uncertain switched system (4) with $w = 0$ is said to be **quadratically stabilizable via switching** if there exist a positive definite function $V(x) = x^T P x$, a positive scalar ε and a switching rule $r(x,t)$ such that

$$\frac{d}{dt}V(x) < -\varepsilon x^T x$$

holds for all admissible uncertainty Γ. Similarly, the uncertain switched linear system (1) is said to be **quadratically stabilizable via switched state feedback** if there exist a switching $r(x,t)$ and an associated gain matrix K_r such that with $u = K_r x$, the resulting closed-loop nominal system $(w = 0)$ is quadratically stabilizable.

The following four Lemmas are important to develop the main results.

Lemma 1: [22] Given any constant $\lambda > 0$ and any matrices M, Γ, N of compatible dimensions, then

$$2x^T M\Gamma N x \le \lambda x^T M M^T x + \frac{1}{\lambda} x^T N^T N x.$$

for $\forall x \in \mathbb{R}^n$, where Γ is an uncertain matrix satisfying $\Gamma^T\Gamma \le I$.

We now recall a result relating the quadratic stability of an autonomous non-switched uncertain linear system to a certain small gain condition. Consider the uncertain system

$$\dot{x}(t) = (A + E\Gamma F)x(t) \tag{5}$$

where A, E, F are given constant matrices of appropriate dimensions and Γ is the uncertainty with $\Gamma^T\Gamma \le I$.

Definition 3: [20] The system (5) is said to be quadratically stable if there exists a positive definite function $V(x) = x^T P x$ and a constant $\varepsilon > 0$ such that, for any admissible uncertainty Γ,

$$\frac{d}{dt}V(x) < -\varepsilon x^T x.$$

Lemma 2: [20] The system (5) is quadratically stable if and only if it satisfies the following conditions:

(a) A is a stable matrix,

(b) $\|F(sI - A)^{-1}E\|_\infty < 1$.

We introduce a linear time-invariant (LTI) system

$$\dot{x}(t) = (\widehat{A} + \widehat{E}_1\widehat{\Gamma}_1\widehat{F}_1)x(t), \tag{6}$$

where $\widehat{A} := \sum_{i=1}^{l} \alpha_i A_i$, $\sum_{i=1}^{l}\alpha_i = 1$, $\alpha_i \ge 0$, $\widehat{E}_1 = [\sqrt{\alpha_1}E_1, \cdots, \sqrt{\alpha_l}E_l]$, $\widehat{F}_1 = [\sqrt{\alpha_1}F_{11}^T, \cdots, \sqrt{\alpha_l}F_{1l}^T]^T$, E_i and F_{1i}, $i \in \underline{l}$ are matrices introduced in (2). $\widehat{\Gamma}_1 \in \mathbb{R}^{lm\times lk}$ and $\widehat{\Gamma}_1^T\widehat{\Gamma}_1 \le I$. For simplicity of presentation, system (6) is defined as **uncertain combination linear systems (UCLS)** in what follows. Clearly, for each uncertain switched linear system (4), there is a corresponding UCLS (6) except the selection of $\alpha_i, i \in \underline{l}$.

Lemma 3: System (4) is quadratically stabilizable via switching if there exist nonnegative scalars $\alpha_1, \cdots, \alpha_l$ with $\sum_{i=1}^{l}\alpha_i = 1$ such that UCLS (6) is quadratically stable. Furthermore, a stabilizing switching rule can be taken as

$$r(x,t) = \arg\min_{i\in\underline{l}}\{x^T(A_i^T P + PA_i + \lambda P E_i E_i^T P + \tfrac{1}{\lambda}F_{1i}^T F_{1i})x\}, \tag{7}$$

where P is the positive definite matrix associated with the quadratic stability of UCLS (6) and $\lambda > 0$ is a constant arbitrarily selected in advance.

Proof: By lemma 2, the quadratic stability of UCLS (6) implies that $\widehat{A}$ is Hurwitz stable and

$$\|\frac{1}{\sqrt{\lambda}}\widehat{F}_1(sI-\widehat{A})^{-1}\sqrt{\lambda}\widehat{E}_1\|_\infty = \|\widehat{F}_1(sI-\widehat{A})^{-1}\widehat{E}_1\|_\infty < 1.$$

It is well known that the above inequality is equivalent to the following matrix Riccati inequality

$$\widehat{A}^T P + P\widehat{A} + \lambda P\widehat{E}_1\widehat{E}_1^T P + \frac{1}{\lambda}\widehat{F}_1^T\widehat{F}_1 < 0 \tag{8}$$

and P is the positive definite matrix associated with the quadratic stability of UCLS (6). By (8), there always exists a positive scalar ε such that for $\forall x \in \mathbb{R}^n$,

$$\begin{aligned}&\sum_{i=1}^{l}\alpha_i \min_{i\in\underline{l}}\{x^T(A_i^T P + PA_i + \lambda PE_iE_i^T P + \frac{1}{\lambda}F_{1i}^T F_{1i})x\}\\ &\le \sum_{i=1}^{l}\alpha_i x^T(A_i^T P + PA_i + \lambda PE_iE_i^T P + \frac{1}{\lambda}F_{1i}^T F_{1i})x\\ &< -\varepsilon x^T x.\end{aligned} \tag{9}$$

Define the Lyapunov function for systems (4) as follows:

$$V(x(t)) = x^T(t)Px(t),$$

where P is the positive definite matrix satisfying (8). Noticing that $w = 0$, then by Lemma 1, (9) and the switching rule (7), for $\forall x \in \mathbb{R}^n, x \neq 0$,

$$\begin{aligned}\frac{d}{dt}V(x) &= x^T(A_r^T P + PA_r)x + 2x^T PE_r\Gamma F_{1r}x\\ &< -\varepsilon x^T x.\end{aligned}$$

By definition 2, the uncertain switched system (4) with $w = 0$ is quadratically stabilizable via switching. ■

Lemma 4: The following two types of switching rules are equivalent:

(a) $r(x,t) = \arg\min_{i\in\underline{l}}\{x^T H_i x\}$,

(b) $r(x,t) = \arg\min_{i\in\underline{l}}\{x^T(H_i + H)x\}$,

where $H_i, i \in \underline{l}$ are given symmetric matrices, H is a given positive definite matrix.

Proof: By the fact that

$$x^T H_i x \ge x^T H_j x, \quad i \neq j, \forall i,j \in \underline{l}$$

if and only if

$$x^T(H_i + H)x \ge x^T(H_j + H)x,$$

this lemma can be proved easily. ■

Definition 4: Given a constant $\gamma > 0$, the system (4) is said to be **quadratically stabilizable with H_∞ disturbance attenuation γ via switching** if there exists a switching rule $r(x,t)$ such that under this switching, it satisfies

(1) the system (4) with $w = 0$ is quadratically stabilizabile for all admissible uncertainties Γ,

(2) with zero-initial condition $x(0) = 0$, $\|z\|_2 < \gamma\|w\|_2$ for all admissible uncertainties Γ and all nonzero $w \in L_2[0,\infty)$.

Let us consider the following UCLS:

$$\dot{x}(t) = (\widehat{A} + \widehat{E}_2\widehat{\Gamma}_2\widehat{F}_2)x(t) \tag{10}$$

where $\widehat{E}_2 = [\widehat{E}_1,\ \gamma^{-1}\frac{\delta_{1B}}{\sqrt{\lambda}}I]$, $\delta_{1B} = \max_{1\le i\le l}\{\sigma_{\max}(B_{1i})\}$, $\sigma_{max}(\cdot)$ denotes the maximum singular value, $\gamma > 0$ is a scalar given in advance and $\lambda > 0$ is a constant scalar which is selected arbitrarily. $\widehat{F}_2 = [\widehat{F}_1^T,\ \sqrt{\lambda}\delta_C I]^T$, $\delta_C = \max_{1\le i\le l}\{\sigma_{\max}(C_i)\}$, $\widehat{\Gamma}_2 \in \mathbb{R}^{(lm+n)\times(lk+n)}$ and $\widehat{\Gamma}_2^T\widehat{\Gamma}_2 \le I$. $\widehat{A}, \widehat{E}_1, \widehat{F}_1$ are matrices defined in UCLS (6).

Theorem 1: Given a scalar $\gamma > 0$, system (4) is quadratically stabilizable with H_∞ disturbance attenuation γ via switching if there exist nonnegative scalars $\alpha_1, \cdots, \alpha_l$ with $\sum_{i=1}^{l}\alpha_i = 1$, such that the UCLS (10) is quadratically stable. Moreover, switching rule $r(x,t)$ is taken as (7).

Proof: By Lemma 2, the quadratic stability of UCLS (10) is equivalent to the following matrix Riccati inequality

$$\widehat{A}^T P + P\widehat{A} + \lambda P\widehat{E}_2\widehat{E}_2^T P + \frac{1}{\lambda}\widehat{F}_2^T\widehat{F}_2 < 0, \tag{11}$$

where P is the positive definite matrix associated with the quadratic stability of system (10). By (11)

$$\widehat{A}^T P + P\widehat{A} + \lambda P\widehat{E}_1\widehat{E}_1^T P + \frac{1}{\lambda}\widehat{F}_1^T\widehat{F}_1 < 0 \tag{12}$$

$$\begin{aligned}&\sum_{i=1}^{l}\alpha_i[A_i^T P + PA_i + \lambda PE_iE_i^T P + \frac{1}{\lambda}F_{1i}^T F_{1i}\\ &+\gamma^{-2}\delta_{1B}^2 PP + \delta_C^2 I] < 0.\end{aligned} \tag{13}$$

Define

$$V(x(t)) = x^T(t)Px(t),$$

$$\begin{aligned}r(x,t) \;=\; &\arg\min_{i\in\underline{l}}\{x^T(A_i^T P + PA_i + \lambda PE_iE_i^T P\\ &+\frac{1}{\lambda}F_{1i}^T F_{1i} + \gamma^{-2}\delta_{1B}^2 PP + \delta_C^2 I)x\},\end{aligned} \tag{14}$$

where P is the positive definite matrix satisfying (11). First of all, if we denote

$$\begin{aligned}&H_i := A_i^T P + PA_i + \lambda PE_iE_i^T P + \frac{1}{\lambda}F_{1i}^T F_{1i},\\ &H := \gamma^{-2}\delta_{1B}^2 PP + \delta_C^2 I\end{aligned}$$

then by Lemma 4, the switching rule (14) is equivalent to the switching rule (7). Consequently, by (12) and the proof of Lemma 3, it follows that systems (4) with $w = 0$ is quadratically stabilizable via switching rule (7) (or (14)), i.e., the first condition in Definition 4 is satisfied.

Secondly, let

$$J = \int_0^\infty (z^T z - \gamma^2 w^T w)dt.$$

Note that $x(0)=0$, then by Lemma 1 and combining (13) with switching rule (14), it follows that for $\forall w \in L_2[0,\infty)$,

$$\begin{aligned} J &= \int_0^\infty [z^T z - \gamma^2 w^T w + \frac{d}{dt}(x^T P x)]dt - x(\infty)^T P x(\infty) \\ &= \int_0^\infty [x^T(A_r^T P + PA_r + C_r^T C_r)x + 2x^T P E_r \Gamma F_{1r} x \\ &\quad + x^T \gamma^{-2} P B_{1r} B_{1r}^T P x \\ &\quad -(\gamma^{-1} B_{1r}^T P x - \gamma w)^T (\gamma^{-1} B_{1r}^T P x - \gamma w)]dt \\ &\le \int_0^\infty [x^T(A_r^T P + PA_r + \lambda P E_r E_r^T P + \frac{1}{\lambda} F_{1r}^T F_{1r} \\ &\quad + \delta_C^2 I + \gamma^{-2}\delta_{1B}^2 PP)x]dt \\ &< 0, \end{aligned} \tag{15}$$

that is, $\|z\|_2 < \gamma \|w\|_2$ holds for $\forall w \in L_2[0,\infty)$. This concludes the proof. ■

Remark 1: The direct application of min-projection strategy may result in sliding motions. We refer to [18] for detailed discussion of how this behavior can be avoided.

III. ROBUST H_∞ CONTROL VIA SWITCHED STATE FEEDBACK

In this section, we investigate H_∞ control problem for systems (1) via switched state feedback.

The **switched state feedback H_∞ control** problem addressed in this section is as follows: *for a given scalar $\gamma > 0$, design switching rule $r(x,t)$ and an associated state feedback $u = K_r x$ such that the resulting closed-loop system of (1) is quadratically stabilizable with H_∞ disturbance attenuation γ for all admissible uncertainties.*

The resulting closed-loop system of (1) can be written as

$$\begin{cases} \dot{x}(t) = (\widehat{A}_r + \Delta\widehat{A}_r)x(t) + B_{1r} w(t) \\ z(t) = \widehat{C}_r x(t) \end{cases} \tag{16}$$

where $\widehat{A}_r := A_r + B_{2r}K_r$, $\Delta\widehat{A}_r := E_r \Gamma \widehat{F}_r$, $\widehat{F}_r := F_{1r} + F_{2r}K_r$, $\widehat{C}_r := C_r + D_r K_r$.

The non-switched state feedback H_∞ control synthesis of the following UCLS relates to the switched state feedback H_∞ control problem of uncertain switched system (1).

$$\begin{cases} \dot{x}(t) = (\widehat{A} + \triangle\widehat{A})x(t) + \widehat{B}_1 \widehat{w}(t) + (\widehat{B}_2 + \triangle\widehat{B}_2)\widehat{u}(t) \\ \widehat{z}(t) = \widehat{C}x(t) + \widehat{D}\widehat{u}(t) \end{cases} \tag{17}$$

where $\triangle\widehat{A} := \widehat{E}_1\widehat{\Gamma}_1\widehat{F}_1$, $\triangle\widehat{B}_2 := \widehat{E}_1\widehat{\Gamma}_1\widehat{F}_D$, $\widehat{A}, \widehat{E}_1, \widehat{\Gamma}_1, \widehat{F}_1$ are matrices defined in UCLS (6), $\widehat{w}(t) \in L_2[0,\infty)$, $\widehat{u}(t) \in \mathbb{R}^{lp}$, $\widehat{z}(t) \in \mathbb{R}^s$ and positive integer s satisfy $s \ge n + lp$, $\widehat{B}_1 := \gamma^{-1}\delta_{1B} I_{n\times n}$, $\delta_{1B} = \max_{1\le i\le l}\{\sigma_{\max}(B_{1i})\}$, $\widehat{B}_2 := [\sqrt{\alpha_1}B_{21}, \cdots, \sqrt{\alpha_l}B_{2l}] \in \mathbb{R}^{n\times lp}$, $\widehat{F}_D := \mathrm{diag}\{F_{21},\cdots,F_{2l}\} \in \mathbb{R}^{lk\times lp}$ and $\widehat{C} \in \mathbb{R}^{s\times n}$, $\widehat{D} \in \mathbb{R}^{s\times lp}$,

$$\widehat{C} = \begin{bmatrix} \delta_C I_{n\times n} \\ 0 \end{bmatrix}, \widehat{D}^T [\ \widehat{C} \quad \widehat{D}\ ] = [\ 0 \quad I\ ], \tag{18}$$

$\delta_C = \max_{1\le i\le l}\{\sigma_{\max}(C_i)\}$.

Note that it is always possible for us to select matrix $\widehat{D}$ such that (18) is satisfied.

Theorem 2: Given a scalar $\gamma > 0$, the switched state feedback robust H_∞ control of systems (1) is feasible if there exist nonnegative scalars $\alpha_1,\cdots,\alpha_l$ with $\sum_{i=1}^l \alpha_i = 1$, such that the standard robust H_∞ design problem for UCLS (17) is solvable via non-switched state feedback. Furthermore, the switching rule is taken as

$$r(x,t) = \arg\min_{i\in\underline{l}}\{x^T H_i x\}, \tag{19}$$

where

$$\begin{aligned} H_i &= A_i^T P + PA_i + \lambda P E_i E_i^T P + \frac{1}{\lambda}F_{1i}^T F_{1i} \\ &\quad -(PB_{2i} + \frac{1}{\lambda}F_{1i}^T F_{2i})R_i^{-2}(PB_{2i} + \frac{1}{\lambda}F_{1i}^T F_{2i})^T, \\ R_i^2 &= I + \frac{1}{\lambda}F_{2i}^T F_{2i}, \quad i \in \underline{l}, \end{aligned} \tag{20}$$

P is the positive definite matrix associated with the quadratic stability of closed-loop UCLS (17), $\lambda > 0$ is a constant associated with the standard robust H_∞ design problem for UCLS (17) and the stabilizing controller for each subsystem is designed by

$$K_i = -R_i^{-2}(B_{2i}^T P + \frac{1}{\lambda}F_{2i}^T F_{1i}), \ i \in \underline{l}. \tag{21}$$

Proof: By Theorem 3.1 of [23], that the feasibility of standard robust H_∞ design problem via state feedback for UCLS (17) is equivalent to the fact that there exists a constant $\lambda > 0$ and a positive definite matrix P satisfying the following matrix inequality

$$\begin{aligned} &\widehat{A}^T P + P\widehat{A} + P(\widehat{B}_1\widehat{B}_1^T + \lambda\widehat{E}_1\widehat{E}_1^T)P + \frac{1}{\lambda}\widehat{F}_1^T\widehat{F}_1 + \widehat{C}^T\widehat{C} \\ &-(P\widehat{B}_2 + \frac{1}{\lambda}\widehat{F}_1^T\widehat{F}_D)\widehat{R}^{-2}(P\widehat{B}_2 + \frac{1}{\lambda}\widehat{F}_1^T\widehat{F}_D)^T < 0, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \widehat{R}^2 &= I_{lp\times lp} + \lambda^{-1}\widehat{F}_D^T\widehat{F}_D \\ &= \mathrm{diag}\{I + \lambda^{-1}F_{21}^T F_{21},\cdots, I + \lambda^{-1}F_{2l}^T F_{2l}\}. \end{aligned}$$

(22) can be written as

$$\begin{aligned} &\widehat{A}^T P + P\widehat{A} + \lambda P\widehat{E}_1\widehat{E}_1^T P + \frac{1}{\lambda}\widehat{F}_1^T\widehat{F}_1 \\ &+\gamma^{-2}\delta_{1B}^2 PP + \delta_C^2 I \\ &-(P\widehat{B}_2 + \frac{1}{\lambda}\widehat{F}_1^T\widehat{F}_D)\widehat{R}^{-2}(P\widehat{B}_2 + \frac{1}{\lambda}\widehat{F}_1^T\widehat{F}_D)^T < 0. \end{aligned} \tag{23}$$

Hence,

$$\begin{aligned} &\widehat{A}^T P + P\widehat{A} + \lambda P\widehat{E}_1\widehat{E}_1^T P + \tfrac{1}{\lambda}\widehat{F}_1^T\widehat{F}_1 \\ &-(P\widehat{B}_2 + \tfrac{1}{\lambda}\widehat{F}_1^T\widehat{F}_D)\widehat{R}^{-2}(P\widehat{B}_2 + \tfrac{1}{\lambda}\widehat{F}_1^T\widehat{F}_D)^T < 0. \end{aligned} \tag{24}$$

Noticing that

$$\begin{aligned} \widehat{R}^{-2} &= \mathrm{diag}\{(I + \frac{1}{\lambda}F_{21}^T F_{21})^{-1},\cdots,(I + \frac{1}{\lambda}F_{2l}^T F_{2l})^{-1}\}. \\ &= \mathrm{diag}\{R_1^{-2},\cdots,R_l^{-2}\} \end{aligned}$$

and by simple computation, (24) implies that

$$\sum_{i=1}^l \alpha_i H_i < 0, \tag{25}$$

where $H_i (i \in \underline{l})$ are matrices defined in (20). Take

$$V(x) = x^T P x,$$

where $P > 0$ is the matrix satisfying (22) and the switching rule $r(x,t)$ is taken as (19). By Lemma 4, (19) is equivalent to the following switching rule

$$r(x,t) = \arg\min_{i \in \underline{l}} \{x^T (H_i + H) x\}, \tag{26}$$

where $H = \gamma^{-2}\delta_{1B}^2 PP + \delta_C^2 I$. In subsequent arguments, we shall first verify the quadratic stabilization of nominal systems ($w = 0$) of (16). By Lemma 1, it follows that

$$\begin{aligned}
&\frac{d}{dt}V(x(t)) \\
\leq\ & x^T(A_r^T P + PA_r)x + 2x^T PB_{2r}K_r x \\
&+\lambda x^T PE_r E_r^T Px + \frac{1}{\lambda}x^T \widehat{F}_r^T \widehat{F}_r x \\
\leq\ & x^T(A_r^T P + PA_r + \lambda PE_r E_r^T P + \frac{1}{\lambda}F_{1r}^T F_{1r})x \\
&+x^T[K_r + R_r^{-2}(B_{2r}^T P + \frac{1}{\lambda}F_{2r}^T F_{1r})]^T R_r^2 \\
&\quad [K_r + R_r^{-2}(B_{2r}^T P + \frac{1}{\lambda}F_{2r}^T F_{1r})]x \\
&-x^T[(PB_{2r} + \frac{1}{\lambda}F_{1r}^T F_{2r})R_r^{-2} \\
&\quad (PB_{2r} + \frac{1}{\lambda}F_{1r}^T F_{2r})^T]x,
\end{aligned} \tag{27}$$

where $R_r^2 = I + \frac{1}{\lambda}F_{2r}^T F_{2r}$. Hence, by (21) and (27)

$$\begin{aligned}
&\frac{d}{dt}V(x) \\
\leq\ & x^T(A_r^T P + PA_r + \lambda PE_r E_r^T P + \frac{1}{\lambda}F_{1r}^T F_{1r})x \\
-\ & x^T[(PB_{2r} + \frac{1}{\lambda}F_{1r}^T F_{2r})R_r^{-2}(PB_{2r} + \frac{1}{\lambda}F_{1r}^T F_{2r})^T]x \\
=\ & x^T H_r x.
\end{aligned}$$

Combining (25) and (19) and following the similar arguments as that in the proof of Lemma 3, we see that the nominal system of (16) is quadratically stabilizable via switching rule (19) (or, equivalently, (26)) along with the state feedback gains given by (21).
Secondly, let

$$J = \int_0^\infty (z^T z - \gamma^2 w^T w)dt.$$

Then, by $x(0) = 0$, we have

$$\begin{aligned}
J\ =\ & \int_0^\infty [z^T z - \gamma^2 w^T w + \frac{d}{dt}(x^T Px)]dt \\
&-x(\infty)^T Px(\infty) \\
=\ & \int_0^\infty [x^T(\widehat{A}_r^T P + P\widehat{A}_r + P\Delta\widehat{A}_r + \Delta\widehat{A}_r^T P \\
&\quad +\widehat{C}_r^T \widehat{C}_r)x + x^T\gamma^{-2}PB_{1r}B_{1r}^T Px \\
&-(\gamma^{-1}B_{1r}^T Px - \gamma w)^T(\gamma^{-1}B_{1r}^T Px - \gamma w)]dt \\
\leq\ & \int_0^\infty \{x^T[A_r^T P + PA_r + K_r^T B_{2r}^T P + PB_{2r}K_r]x \\
&+x^T[\lambda PE_r E_r^T P + \frac{1}{\lambda}\widehat{F}_r^T \widehat{F}_r]x \\
&+x^T[C_r^T C_r + K_r^T K_r + \gamma^{-2}PB_{1r}B_{1r}^T P]x\}dt \\
\leq\ & \int_0^\infty \{x^T[A_r^T P + PA_r + P\lambda E_r E_r^T P + \frac{1}{\lambda}F_{1r}^T F_{1r} \\
&\quad +\gamma^{-2}\delta_{1B}^2 PP + \delta_C^2 I]x \\
&+x^T[K_r + R_r^{-2}(B_{2r}^T P + \frac{1}{\lambda}F_{2r}^T F_{1r})]^T R_r^2 \\
&\quad [K_r + R_r^{-2}(B_{2r}^T P + \frac{1}{\lambda}F_{2r}^T F_{1r})]x \\
&-x^T[(PB_{2r} + \frac{1}{\lambda}F_{1r}^T F_{2r})R_r^{-2} \\
&\quad (PB_{2r} + \frac{1}{\lambda}F_{1r}^T F_{2r})^T]x\}dt \\
=\ & \int_0^\infty \{x^T[H_r + H]x \\
&+x^T[K_r + R_r^{-2}(B_{2r}^T P + \frac{1}{\lambda}F_{2r}^T F_{1r})]^T R_r^2 \\
&[K_r + R_r^{-2}(B_{2r}^T P + \frac{1}{\lambda}F_{2r}^T F_{1r})]x\}dt.
\end{aligned} \tag{28}$$

Since

$$K_r = -R_r^{-2}(B_{2r}^T P + \frac{1}{\lambda}F_{2r}^T F_{1r}), R_r^2 = I + \frac{1}{\lambda}F_{2r}^T F_{2r}, \tag{29}$$

and (23) means that

$$\sum_{i=1}^{l} \alpha_i (H_i + H) < 0, \tag{30}$$

where H_i and H are matrices defined in (20) and (26), respectively, by switching rule (19) (equivalently (26)), (28) and (30), it follows that

$$J < 0,$$

which implies that for all uncertainty Γ and $w \in L_2[0,\infty)$

$$\|z\|_2 < \gamma \|w\|_2.$$

This concludes the proof. ■

IV. NUMERICAL EXAMPLE

The goal of the follwing example is to illustrate that although both the nominal subsystem matrices A_1 and A_2 are not Hurwitz stable, it is still possible for us to obtain the H_∞ performance of uncertain switched linear systems (4) only via designing a switching rule.

Example 1: Consider the following system

$$\begin{cases} \dot{x}(t) &= (A_r + \Delta A_r)x(t) + B_{1r}w(t) \\ z(t) &= C_r x(t) \end{cases} \tag{31}$$

$r(t) : \mathbb{R}^+ \to \underline{2} = \{1, 2\}$, $\Delta A_i = E_i \Gamma F_{1i}, \Gamma^T\Gamma \leq I$,

$$A_1 = \begin{bmatrix} 13 & 7 \\ -3 & -14 \end{bmatrix}, E_1 = \begin{bmatrix} 0.1414 \\ 0 \end{bmatrix}, B_{11} = \begin{bmatrix} 1.5 \\ 0.3 \end{bmatrix},$$

$$F_{11} = \begin{bmatrix} 0.2828 & 0 \end{bmatrix}, \quad F_{12} = \begin{bmatrix} 0 & 0.7071 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -25 & 1.2 \\ -5 & 2 \end{bmatrix}, E_2 = \begin{bmatrix} 0.4384 \\ 0.3960 \end{bmatrix}, B_{12} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Take $\alpha_1 = \alpha_2 = 0.5, \lambda = 1$ and the disturbance attenuation level γ is given by $\gamma = 1$, it can be verified that the following symmetric matrix

$$P = \begin{bmatrix} 0.7225 & -0.0095 \\ -0.0095 & 0.7418 \end{bmatrix}$$

is a positive definite matrix associated with the quadratic stability of UCLS (6). By Theorem 1, system (31) is quadratically stabilizable with the disturbance attenuation level $\gamma = 1$ only via switching, where the switching regions are described in Figure 1 and the switching rule can be

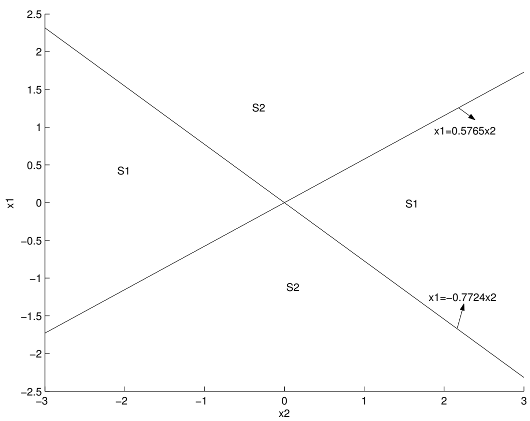

Fig. 1. Switching regions

implemented as follows:

$$r(x,t) = \begin{cases} 1, & if \;\; x \in S1, \\ 2, & if \;\; x \in S2. \end{cases}$$

V. CONCLUSION

This paper has studied robust H_∞ control problem via switching and state feedback for switched linear systems with uncertainties in both the state and input matrices. Although the obtained results are only sufficient, they provide us a new point of view on the quadratic stabilization along with the disturbance attenuation of uncertain switched linear systems. This is because a relationship between the H_∞ performance of uncertain switched systems and the quadratic stability of an associated uncertain combination linear systems (UCLS) is established. It should be noted that a common switching rule should be designed to guarantee the quadratic stability of nominal switched systems and the H_∞-norm bound constraint on disturbance attenuation for all admissible uncertainties simultaneously. We present a state dependent switching strategy in this paper to cope with this problem. How to design other suitable switching rules to better the performance of switched systems should be further studied in the future work. Our results may present a meaningful try to deal with the H_∞ control problem for switched systems.

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