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Maximum Likelihood Estimation on Mismatch for Stochastic Nearly Optimal Control

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Abstract: In a robot image processing system along with the visual feedback or in a TV telecommunication system along with signal processing, pattern recognition is most often confronted with a large state space representation requirement due to the complexity of unexpected shape, color and motion as well as environmental disturbance. It is inevitable that raw signals are affected by Gaussian noises. This problem in the presence of random noises can be modeled as a LQG model modulated by a finite state Markov chain. The optimal solution is achieved by dynamic programming and associated HJB equations. For large-scale systems, averaging approach is necessary to obtain consistent solutions to Riccati equations, which is the nearly optimal control scheme. The mismatch from time scale separation should be minimized. As a result, maximum likelihood estimation is proposed to optimize the total mismatch, which is a generally consistent and asymptotic Gaussian. In this article, the total mismatch and the convergence property within stochastic nearly optimal control problem are illustrated by a set of multi-dimensional numerical simulations and then maximum likelihood estimation scheme is derived and investigated on a basis of the multi-dimensional state space.

Index Terms: Maximum Likelihood Estimation, Stochastic Optimal Control, Near Optimality

I. INTRODUCTION

The hybrid LQG (Linear Quadratic Gaussian) models modulated by a finite state Markov chain have been widely used for pattern recognition, signal processing, economy investment, telecommunication, etc. It can be optimized by dynamic programming and associated HJB (Hamilton-Jacobi-Bellman) equations. For a large-scale system, simplification schemes are needed to reduce computation complexity and to get consistent solutions to Riccati equations. As an example, economy modeling is a large dimensional state space problem due to market trends and various other economic factors. To reduce computation complexity, nearly optimal control scheme can be applied. The weakly irreducible states of Markov chains in each class are aggregated into a single state by the scheme. Discrete-event variables are associated with long-term Macro Economy. Continuous state variables represent the short-term dynamic Micro Economy. Then the hybrid stock models modulated by a continuous-time Markov chain can be optimized by nearly optimal asset allocation strategy [1, 2, 3, 4].

By dynamic programming approach, a control problem for a class of stochastic diffusion systems can be studied and its transition smoothing effect is associated with the uncontrolled system. The corresponding HJB equation is solved by a fixed-point argument in a small time interval and it is extended to arbitrary time intervals by suitable priori estimates. The continuous time finite horizon optimal control problem can be investigated by Markov decision strategies. Its analytical solutions are of low computation complexity. A class of continuous Markov processes is described by a multidimensional nonlinear stochastic equation. Asymptotic convergence theorem is proved using Markov process and asymptotic normality theorem is formulated. A novel strategy by Bellman's principle of optimality and the short-time approximation is to obtain global solutions of stochastic optimal control problems. The Markov chain with a dependent transition probability matrix allows the systematic evaluation of transient and steady state responses. An excellent control performance can be eventually achieved [5, 6, 7].

Maximum likelihood approach is commonly used to approximate the stochastic model in pattern recognition. It is affected by uncertainty in the regression matrix and random gaussian noise. A worst case likelihood of the measurement has been used to solve the optimization problem and applied to the parameter identification. For a non-parametric motion model attached to the image sequence, the maximum likelihood criterion leads to the best fitting model. Model complexity reduction can be achieved to supply an informative representation of the motion. The probability of misclassification and proper classification is calculated using conditional probability of the error and the priori probabilities. The maximum likelihood estimation with desired asymptotic properties presents the parameter value that maximizes conditional probability density function [8, 9, 10]. Thus it is to be used to obtain optimal estimation in stochastic nearly optimal control problems in this research.

II. HYBRID PROBLEM FORMULATION

Given a linear stochastic system in a finite time horizon:

$$dx(t)=[A(\alpha(t))x(t)+B(\alpha(t))u(t)]dt + \sigma dw(t) \quad (1)$$

$$x(s)=x \ (s \le t \le T)$$

where x(t) is the state vector ($n_1 \times 1$); u(t) is the control ($n_2 \times 1$); A($n_1 \times n_1$) and B($n_1 \times n_2$) are matrices with finite values; w(t) is the Brown motion vector, which is a unity vector with a Gaussian random element coefficient σ. Considering one stationary finite state Markov chain $\alpha(t) \in M=\{1, \ldots, m\}$, of which the transition probability $P(\alpha(t)=j| \alpha(s)=i)$ depends only on (t-s).

The performance index is to be minimized:

$$J(s,x,\alpha,u(.)) = E\{x^T(T)Dx(T) + \int_s^T [x^T(t)M(\alpha(t))x(t) + u^T(t)N(\alpha(t))u(t)]dt\} \quad (2)$$

where E is the expectation given $\alpha(s)=\alpha$ and x(s)=x. Let $i=\alpha(t)$;. M(i) ($n_1 \times n_1$) is symmetric nonnegative definite matrix; N(i) ($n_1 \times n_1$) and D ($n_1 \times n_1$) are symmetric positive definite matrices.

To decompose a large dimensional system into a number of simple structure subsystems, a very small parameter ε>0 is introduced to display a two-time-scale behavior. The generator Q^ε consists of both a rapidly changing part and a slowly varying part.

$$Q^\varepsilon = Q_1/\varepsilon + Q_2 \quad (3)$$

where $Q^\varepsilon f(.)(i) = \Sigma q^\varepsilon_{ij}(f(j)-f(i))$, for suitable f(.) (i≠j). If the process α(.) in (1) and (2) is replaced by $\alpha^\varepsilon(.)$, we have the performance index J^ε (s, α, x, u(.)), where $\alpha^\varepsilon(.)$ and w(.) are independent.

A generator Q is weakly irreducible if vQ = 0 and $\Sigma_1^m v_i=1$ has a unique nonnegative solution $v=(v_1,\ldots, v_m)$. If v is strictly positive, then the generator Q is strongly irreducible. The solution v is referred to as the stationary distribution. The irreducibility is to classify the groups of states in which the fast transitions take place.

III. OPTIMAL CONTROLS

The optimal LQG control is accomplished by dynamic programming approach with the related HJB and Riccati equations for $0 \le s \le T$, $i=\alpha(s)\in M$ with $v(T, i, x) = x^TDx$. The solution to the HJB equations is:

$$v^\varepsilon(s, i, x) = \inf_{u(.)} J^\varepsilon(s, i, x, u(.)) = x^TK^\varepsilon(s,i)x+q^\varepsilon(s,i) \quad (5)$$

Let K^ε be a symmetric matrix (m×m) and q^ε is a scalar function. The Riccati equation for K^ε ($K^\varepsilon(T, i)=D$) is:

$$\dot{K}^\varepsilon(s,i) = - K^\varepsilon(s,i)A(i) - A^T(i) K^\varepsilon(s,i) - M(i) + K^\varepsilon(s,i)B(i)N^{-1}(i)B^T(i) K^\varepsilon(s,i) - Q^\varepsilon K^\varepsilon(s, .)(i) \quad (6)$$

The equation for q^ε ($q^\varepsilon(T, i)=0$) is:

$$\dot{q}^\varepsilon(s, i) = -tr(\sigma\sigma^T K^\varepsilon(s, i)) - Q^\varepsilon q^\varepsilon(s, .)(i) \quad (7)$$

The optimal control has the following formulation:

$$u^{\varepsilon,*}(s, i, x) = -N^{-1}(i)B^T(i) K^\varepsilon(s, i)x \quad (8)$$

The Riccati equations turn out to be extremely difficult to solve for most large-scale systems. The averaging approach with approximation schemes and the solutions to correspondent hybrid nearly optimal control problems has been derived analytically [1, 2], which will also be partially interpreted in the consecutive section VI. Multi-dimensional numerical solutions are now in necessity for various real world applications.

The closed loop state feedback is expressed as:

$$dx(t)=[A(i) -B(i)N^{-1}(i)B^T(i)K^\varepsilon(s, i)]x(t)dt+ \sigma dw(t) \quad (9)$$

It gives rise to a linear stochastic differential equation.

IV. SAMPLING THE STOCHASTIC DIFFERENTIAL EQUATION BY OPTIMAL FEEDBACK CONTROL

For computer-controlled systems, like data processing and pattern recognition, discrete-time control model can be derived using sampling. Equation (9) is simplified as:

$$dx(t)= A_C x(t)dt+ \sigma dw(t) \quad (10)$$

where $A_C=A(i)-B(i)N^{-1}(i)B^T(i)K^\varepsilon(s, i)$.

Assuming the optimal control signal is a constant over one sampling time and given that small sampling instants are $\{t_k$: k=0,1, …}. Integration of (10) over one sampling time is expressed as:

$$x(t_{k+1}) = e^{A_C(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A_C(t_{k+1}-s)} d\sigma w(s) \quad (11)$$

Define a random variable $v(t_k) = \int_{t_k}^{t_{k+1}} e^{A_C(t_{k+1}-s)} d\sigma w(s)$.

It has zero mean since w(t) has zero mean. $v(t_1)$ and $v(t_2)$ are uncorrelated when $t_1 \ne t_2$. Thus the discrete-time random sequence $\{x(t_k)$, k = 0,1,…} can be described by stochastic difference equation (12).

$$x(t_{k+1}) = e^{A_C(t_{k+1}-t_k)}x(t_k) + v(t_k) \quad (12)$$

where $v(t_k)$ is a unity uncorrelated vector with coefficient σ and zero mean, i.e. $v(t_k) = \sigma w(t_k)$. The covariance is expressed as follows.

$$E(v(t_k), v^T(t_k)) = \sigma^2 E\iint_{t_k}^{t_{k+1}} e^{A_C(t_{k+1}-s)} dw_c(s) dw_c^T(t) e^{A_C^T(t_{k+1}-t)}$$

V. DERIVATION OF MAXIMUM LIKELIHOOD FUNCTION

Let L(k, i) and λ(k, i) represent likelihood estimator and log-likelihood function. Given a fixed sampling time h ($h = t_{k+1}-t_k$, k = 1, 2, …), the discrete-time random sequence is simplified as:

$$x(k+1) = A_h(k, i)\, x(k) + v(k) \quad (13)$$

where $A_h(k,i) = e^{A_C(k,i)(t_{k+1}-t_k)} = e^{A_C(k,i)h}$

Assuming observation of x is contaminated by another stationary Gaussian noise sequence μ(k) with zero mean.

$$z(k) = x(k) + \mu(k) \quad (14)$$

where $\mu(k)= (\xi-1)v(k-1)$. Now we have:

$$z(k+1)=x(k+1)+\mu(k+1)=A_h(k, i)x(k)+v(k)+\mu(k+1)$$

$$=A_h(k, i)z(k)+ \xi v(k)-A_h(k, i)\mu(k) \quad (15)$$

where z(k) is the estimation of x(k) with mean value $\overline{z}$.

$$\overline{z}(k+1)=E[z(k+1)|z(k)]=A_h(k, i)z(k) \quad (16)$$

The conditional multivariate probability density function of z(k+1) can be determined, which has the conditional mean $\overline{z}$ (k+1) and covariance matrix Σ.

$$f_z[z_i(\mathrm{k}+1)|z_i(\mathrm{k})] = \frac{1}{(2\pi)^{n_1/2}\Delta_\Sigma^{1/2}} e^{-\frac{1}{2}(z_i-\overline{z}_i)^T\Sigma^{-1}(z_i-\overline{z}_i)} \quad (17)$$

where, z(k) ($n_1 \times 1$) is the state vector with independent elements; Σ ($n_1 \times n_1$) is the covariance matrix; Σ^{-1} is the inverse of matrix Σ and Δ_Σ is the determinant of Σ. Using zero covariance property of Gaussian Markov process, Σ is a diagonal matrix since all its covariance elements are zero and all of the diagonal terms are variances of states.

By zero covariance property in Gaussian process,

$$\Sigma_{ii} = E\{[z_i(k+1) - \overline{z}_i(k+1)]^2|z_i(k)\}$$
$$=E[\xi^2 v_i^2(k)+\mu_i(k)A_h(i,:)A_h^T(:, i)\mu_i(k)]$$
$$= \xi^2 [\sigma^2 + \|A_h(:, i)\|_2^2]$$
$$\Sigma_{ij}=E\{[z_i(k+1) - \overline{z}_i(k+1)][z_j(k+1)- \overline{z}_j(k+1)]\}=0$$
$$\Delta_\Sigma = \xi^{2n_1}\Pi_1^{n_1}[\sigma^2+ \|A_h(:, i)\|_2^2] \quad (18)$$

where Σ_{ii} is a diagonal term, Σ_{ij} is the i-j th element of Σ, $A_h(i,:)$ is the ith row of matrix A_h. Covariance matrix Σ is diagonal and positive semi-definite. Its determinant is determined by (18). The logarithm of the multivariate probability density function (19) can be easily derived.

$$\log f_z[z_i(\mathrm{k}+1)|z_i(\mathrm{k})] =$$
$$-\frac{1}{2}\log(2\pi)^{n_1}\Delta_\Sigma - \frac{1}{2}(z_i-\overline{z}_i)^T\Sigma^{-1}(z_i-\overline{z}_i) \quad (19)$$

Maximum likelihood estimation of multi-dimensional state is defined as the joint density. For all independent events, the joint probability density function is a product of the individual joint probability density functions. For n_1 independent state, the joint density is expressed as:

$$f_z[z(k+1)|\, z(k)] = \Pi_1^{n_1} f_{zi}[z_i(k+1)|\, z_i(k)]$$

As a result, the likelihood function is:

$$L(k, i) = \Pi_1^{n_1} f_z[z_i(k+1)|z_i(k)] \quad (20)$$

Correspondently, the log-likelihood function is:

$$\lambda(k, i) = -\Sigma_1^{n_1}\log f_z[z_i(k+1)|z_i(k)] \quad (21)$$

Since logarithm function is monotonically increasing, a multi-component vector that maximizes log-likelihood function will also maximize likelihood function.

$$L(k,i) = \frac{1}{(2\pi)^{n_1/2}\Delta_\Sigma^{1/2}}\Pi_{i=1}^{n_1} e^{-\frac{1}{2}(z_i-\overline{z}_i)^T\Sigma^{-1}(z_i-\overline{z}_i)} \quad (22)$$

$$\begin{aligned}\lambda(k,i) &= -\frac{1}{2}\sum_{i=1}^{n_1}[\log(2\pi)^{n_1}\Delta_\Sigma + (z_i-\overline{z}_i)^T\Sigma^{-1}(z_i-\overline{z}_i)]\\ &= -\frac{n_1}{2}\log(2\pi)^{n_1}\Delta_\Sigma - \sum_{i=1}^{n_1}(z_i-\overline{z}_i)^T\Sigma^{-1}(z_i-\overline{z}_i)\\ &= -\frac{n_1}{2}\log(2\pi)^{n_1}\Delta_\Sigma - \sum_{i=1}^{n_1}\frac{\|z_i-\overline{z}_i\|_2^2}{\xi^2[\sigma^2+\|A_h(:,i)\|_2^2]} \quad (23)\end{aligned}$$

Maximum likelihood estimation produces a variable value that maximizes the probability density function. It is equal to the value that in some sense best agrees with the sampling data. To maximize log-likelihood function, this differentiable problem is to be solved by standard differential calculus. Maximum likelihood estimation is described by a set of differential calculus equations.

$$\nabla\lambda=\Sigma\nabla\log f_z[z_i(k+1)|\, z_i(k)]=0 \quad (24)$$

where ∇ is the gradient operator. Intuitively, the optimal solution point has a same Mahalanobis distance to all mean element values of multi-dimensional state vector. The objective then is to find out if nearly optimal control for state space reduction is of convergence property. It is to be shown by numerical simulations.

VI. Recurrent And Transient States Approximation

The states of the Markov chain can be divided into a number of groups so that it fluctuates rapidly among different recurrent states within the same group but it jumps slowly as transient states among different groups.

Suppose Q_1 has a block-diagonal form and it controls the rapidly changing part.

$Q_1 = \mathrm{diag}(Q_1^1,\ldots, Q_1^L)$, $Q_1^k \in R^{m_k \times m_k}$ and $\Sigma m_k=m$.

$\Sigma q1_{ij}= 0$, for i=1, 2, … ,m_k

$$Q_1 = \begin{pmatrix} Q_{1r} & 0 \\ Q_{10} & Q_{1*}\end{pmatrix} \qquad Q_2 = \begin{pmatrix} Q_2^{11} & Q_2^{12} \\ Q_2^{21} & Q_2^{22}\end{pmatrix} \quad (25)$$

Q_2 is generator and it controls the slowly varying part.

$Q_2 = (q2_{ij})_{m\times m}$ and $\Sigma q2_{ij}= 0$, for i=1, … , m

All states in M_k are coupled strongly as a single state through matrix Q_2 and transitions from M_k to M_j, $k\neq j$ are weekly possible. By aggregating all states in M_k as one state k, an aggregated process $\{\alpha^c(.)$: $\alpha^c(t)=k\}$ is obtained when $\alpha^\varepsilon(t)\in M_k$. $\alpha^\varepsilon(.)$ converges weakly to $\alpha(.)$ by:

$$Q= \mathrm{diag}(v^1, \ldots, v^L)Q_2\mathrm{diag}(I_{m1}, \ldots, I_{mL}) \quad (26)$$

where v^k is the stationary distribution of Q_1^k, k=1, …, L and $I_n = (1, \ldots ,1)^T\in R^n$.

Consider transient state Markov chain cases.

$Q_{1r}=\mathrm{diag}(Q_1^1,\ldots,Q_1^L)$, $Q_{10}=(Q_{1*}^1,\ldots,Q_{1*}^L)$, for k=1,… , L.

Q_1^k is a generator, $Q_{1*}\in R^{mkxmk}$, $Q_{1*}^k \in R^{mkxmk}$, $\Sigma m_k=m$.

The state space of the underlying Markov chain is:

$M=M_1\cup\ldots\cup M_L\cup M^*$

$=\{S_{11},\ldots,S_{1m1},\ldots,S_{l1},\ldots,S_{lml}, S_{*1},\ldots, S_{*m*}\}$

where $M_*=\{S_{*1},\ldots S_{*m*}\}$ consists of the transient states.

Suppose for k=1, 2, …, L; Q_1^k is weakly irreducible. Q_{1*} has eigenvalues with negative real parts.

$Q_2^{11}\in R^{(m-m*)x(m-m*)}$,$Q_2^{12}\in R^{(m-m*)xm*}$

$Q_2^{21}\in R^{m*x(m-m*)}$, $Q_2^{22}\in R^{m*xm*}$

Define $Q_* = \mathrm{diag}(v^1,\ldots, v^L)(Q_2^{11}I+ Q_2^{12}(a_{m1},\ldots,a_{mL}))$, where $I=\mathrm{diag}(I_{m1}, \ldots ,I_{mL})$ and

$I_{mj}=(1,\ldots,1)^T\in R^{mjx1}$, for j=1, … , L,

$a_{mj}= (a_{mj(1)},\ldots,a_{mj(l)})= -Q_{1*}^{-1}Q_{1*}^j I_{mj}$.

These recurrent state and transient state approximation schemes will be applied to the nearly optimal control simulation using the dynamic programming approach.

Similarly, in a two-time-scale system introduced by a small parameter ε, the reduced state vector $x^{\varepsilon *}$, solution matrix $K^{\varepsilon *}$ of limit Riccati equations and value function of nearly optimal control $v^{\varepsilon *}$ are to be solved. The convergence property is to be presented by numerical simulations on a multi-dimensional basis.

VII. Numerical Simulations

Numerical simulations are conducted in this section. For optimal control problem, x^{ε} is the state, K^{ε} is the solution matrix of Riccati equation, v^{ε} is the value function, J^{ε} is an expected quadratic performance index. Parameter ε is set to 0.05. For a nearly optimal control problem, $x^{\varepsilon *}$, $K^{\varepsilon *}$, and $v^{\varepsilon *}$ are the correspondent terms.

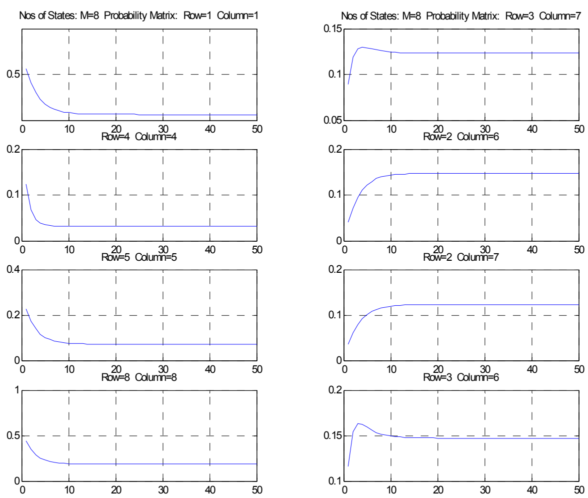

Fig. 1 Elements of Markov Chain Generator Matrix

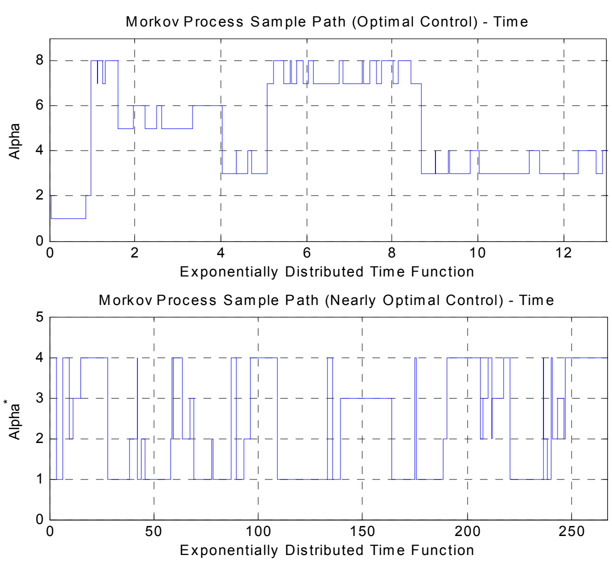

Fig. 2 Sample Paths of Markov Process in both Optimal Control and Nearly Optimal Control Problems

Various sample paths are shown in Fig. 1 to Fig. 5. An eight-th dimensional system is simulated where Markov chain $\alpha \in$ M={1, 2, …, 8} and also A(i) and B(i) are defined accordingly. Fig. 1 shows sample paths from different elements of probability matrix. The comparison has been made between Markov processes of the optimal control trajectory and nearly optimal trajectory in Fig. 2. There is a two-time-scale behavior in this hierarchical approach. As a result, time scales of two plots in Fig. 2 are different. The difference is owing to the small time scale parameter ε = 0.05. In following simulations, the matrices of M, N and D are all selected as symmetric positive definite.

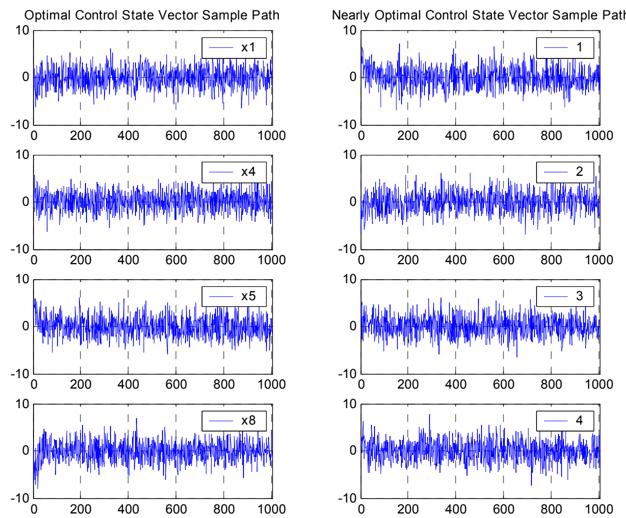

Fig. 3 Sample Paths of State Vector in Optimal Control and Nearly Optimal Control Problems

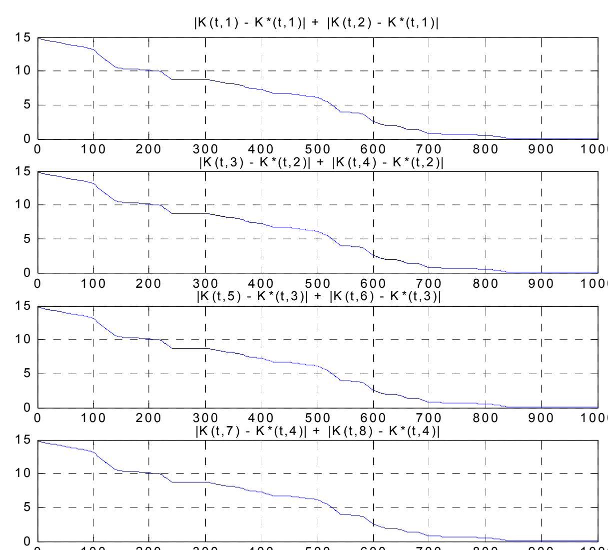

Fig. 4 Sample Path Mismatch of Solution Matrices of Riccati Equation and Limit Riccati Equation

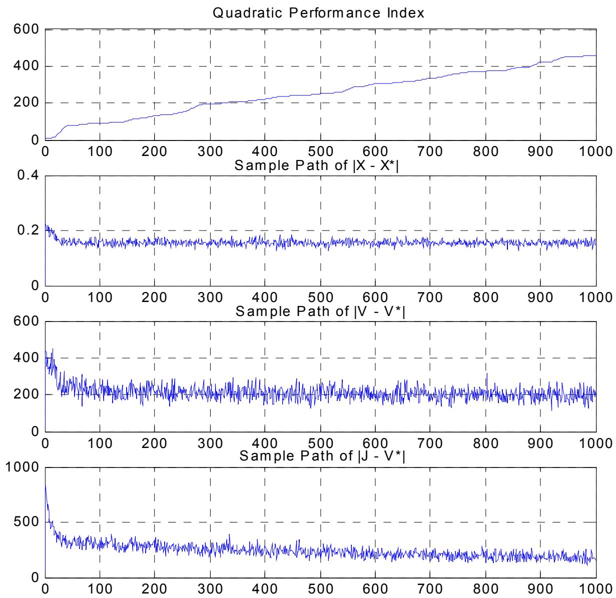

Fig. 5 Sample Paths of Performance Index, State Differences, Value Function and Cost Function Due to Nearly Optimal Control

The sample paths for the correspondent state vectors are illustrated in Fig. 3. Fig. 4 shows the trajectory differences on some elements from solution matrices. Sample paths of cost function and various comparison terms regarding convergence property of nearly optimal control approach are plotted in Fig. 5, respectively. The results are in fact convergent. All the associated error bounds are calculated in Table 1.

Table 1 Error Bounds

ε	$\lvert K^{\varepsilon}-K^{*}\rvert$	$\lvert x^{\varepsilon}-x^{*}\rvert$	$\lvert v^{\varepsilon}-v^{*}\rvert$	$\lvert J^{\varepsilon}-v^{\varepsilon}\rvert$
0.05	0.0277	0.0049	6.8049	7.9004

VIII. Conclusions

This article focuses on the mismatch estimation in stochastic nearly optimal control of a linear quadratic Gaussian model, where maximum likelihood estimation is employed. For a large scaled system, simplification schemes are in need to be applied for the computation reduction where a time scale partition is involved at the same time. Mismatch is consequently generated using averaging schemes in stochastic nearly optimal control. To eliminate approximation mismatch to an acceptable level, maximum likelihood estimation is proposed and satisfied results are achieved. Numerical simulations are also conducted and investigated in multi-dimensional stochastic optimal control systems in order to testify the schemes for feasibility and convergence.

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