Inequality-Based Estimates of Systems Reliability

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***Abstract*—Reliability testing of large-scale systems is often infeasible or very costly. Thus, when estimating system reliability, it is desirable to employ a method that uses subsystem tests, which are often less expensive and more feasible. This paper discusses a method for calculating full-system reliabilities based on subsystem tests. The method does not require that subsystems be independent. It accounts for dependencies through use of certain probability inequalities. The inequalities provide the basis for valid reliability calculations while not requiring full-system tests or information on greater-than-pairwise subsystem failure modes.**

I. INTRODUCTION

IT is often infeasible or very costly to assess the performance of complex systems through full-system tests. Further, full-system testing may sometimes involve the destruction of expensive system assets. There is a critical need for alternate approaches to estimating full system reliability or other performance characteristics, especially for systems that require non-destructive tests such as bridges, machines, aircraft, and satellites. In addition to the need to minimize the number of full system tests is the desire to exploit valuable information from subsystem tests (which tend to be less expensive than full system tests) as a means of quantifying full system reliability. Of the various system performance characteristics, system reliability—the focus of this paper—is one that is usually difficult to quantify without the benefit of full system testing. Many approaches exist for quantifying system reliability from subsystem tests (see, e.g., [1], [2], [3]). These methods, however, assume that the subsystems are *statistically independent* or that the system configuration is completely specified. Under the assumption of independence, reliability is calculated as the product of all the critical subsystem reliabilities. Independence is frequently not met in complex systems, which may contain many interdependent subsystems that interact in subtle and not-so subtle ways. For such systems, reliability calculations that rely on independence will be misleading.

A simple example illustrates the point. Suppose the system consists of component subsystems S_i, i = 1, 2, 3, … , 20 which fail if some parameter X_i exceeds a specified threshold, $X_i > 2.5$ say. Assume that the X_i's are zero mean, normal random variables with $\text{var}(X_i) = 1$ for all i and $\text{cov}(X_i, X_j) = 0.75$ when $i \neq j$. The probability of system failure is $p = 1 - P\left(\bigcap \{X_i \le 2.5\} \right) = 0.042$. If it were assumed, erroneously, that the X_i's were independent, then we would obtain $p = 1 - \prod P\{X_i \le 2.5\}$ = 0.117. Hence, in assuming independence, one would produce a probability of system failure that is greater than twice the true failure probability. If a significant number of the subsystem-to-subsystem covariances were negative (rather than positive as in this example), then a reliability calculation based on the assumption of independence would underestimate the probability of failure. Obviously, such errors can have potentially serious consequences in system design or analysis.

We discuss a method of quantifying system reliability that overcomes the major shortcoming of previous approaches. The method, called *Inequality-Based Reliability* (IBR), makes use of results from subsystem tests and (if available) full-system tests. IBR combines estimates of two quantities to estimate system reliability: an estimate of an upper bound on the system failure probability (derived from the subsystem tests) and a point estimate of system reliability (computed from system-level tests). The IBR estimate is a combination of these two estimates defined to minimize a certain mean-square error.

The upper bound estimate—which provides an initial upper bound estimate on the system failure probability—makes use of information about the probability of failure for the individual subsystems and the joint probability of failure for specified pairs of subsystems. (The inequality does not require greater than pairwise failure probabilities, nor does it require that all pairs of subsystem interactions.) The use of pairwise information is what distinguishes IBR.

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II. Probability Inequalities

We present probability inequalities and results useful in defining the IBR estimate (Section 3). Suppose that a system consists of $m > 1$ critical subsystems that have two states—operating or failure. The system fails if one or more of its subsystems fails. In other words, the subsystems can be viewed as being serially connected. Let F_i denote the event that subsystem *i* fails, p_i its probability of failure, and $F = \bigcup F_i$ the event of system failure. The probability p of system failure is $p = P\left(\bigcup F_i \right) = 1 - P\left(\bigcap F_i^c \right)$.

In general, $p \neq 1 - \prod_{i=1}^{m} (1 - p_i)$, since the F_i 's are not assumed to be independent. The exact expression for *p* contains sums and differences involving up to $2^m - 1$ joint probabilities $P_J = P\left(\bigcap_{i \in J} F_i \right)$, where the index set *J* varies over all the non-empty subsets of $1, 2, \ldots, m$. For complex systems, obtaining estimates of all the joint probabilities will usually be infeasible. The IBR reliability estimate relies on upper bounds on *p* and requires, for its computation, at most *pairwise* failure probabilities; hence, it only involves the p_i 's and $p_{ij} = P\left(F_i \bigcap F_j \right), \quad 1 \leq i, j \leq m$.

Upper bounds, for the probability of a union of sets, which depend on at most pairwise intersections, were derived in [4], [5], [6] using graph-theoretic results. We briefly summarize the relevant results.

Consider the set $T = \left\{ \{ i, j \} : i \neq j,\ 1 \leq i, j \ \ m \right\}$ consisting of all possible *edges* between the *vertices* $\{1, 2, \ldots, m\}$. A subset of *T* is a *graph*.

Proposition: ([4], [5]) Suppose τ is a graph, then

$$p \leq \sum_i p_i - \sum_{\{i,j\} \in \tau} p_{i,j} \tag{1}$$

if and only if τ satisfies:

i) for each $i = 1, \ldots, m$, there is a j ($j \neq i$) such that the edge $\{ i, j \}$ belongs to τ (i.e., each vertex is connected to at least one other vertex different from itself), and

ii) τ contains exactly $m - 1$ edges (i.e., the graph contains no *cycles*).

A graph satisfying conditions (i) and (ii) above is a *spanning tree* in $\{1, 2, \ldots, m\}$, which, by definition, is a graph that consists of exactly $m - 1$ branches such that at least one edge is incident on each vertex. The proposition states that inequality (1) holds *only* for spanning trees.

From (1) we have the following upper bounds for *p* ([4], [5]):

$$p \leq \min_{\tau} \left\{ \sum_i p_i - \sum_{(i,j) \in \tau} p_{i,j} \right\} \tag{2}$$

where the minimum are taken over all spanning trees in $\{1, 2, \ldots, m\}$.

In most applications, estimates of the probability $P\left(F_i \cap F_j \right)$ will only be available for some pairs of events. It is of interest, then, to consider the following inequalities, each of which requires only a subset of the pairwise probabilities for computing the least upper bound in (2):

$$p \leq \sum_i p_i - (2/m) \sum_{i<j} p_{ij} \quad ([7]) \tag{3}$$

$$p \leq \sum_i p_i - \sum_i p_{i,i+1} \quad ([5]) \tag{4}$$

$$p \leq \sum_i p_i - \max_j \sum_{i \neq j} p_{ij} \quad ([8]) \tag{5}$$

$$p \leq \sum_i p_i - \sum_{i=2}^{m} \max_{k<i} p_{ik} \quad ([9]). \tag{6}$$

All the inequalities (1) – (6) improve on the well-known Bonferroni inequality $p \leq \sum_i p_i$ due to the subtraction of non-negative terms.

It is easy to see that (4) holds. Indeed, the index set consisting of $\{ i, i+1 \}$, $i = 1, \ldots, m-1$, is a spanning tree, hence $\sum_i p_i - \max_{\tau} \left\{ \sum_{(i,j) \in \tau} p_{i,j} \right\} \leq \sum_i p_i - \sum_i p_{i,i+1}$, from which (4) follows. We show next that the other bounds—(3), (5), and (6)—are greater than or equal to the least upper bound in (2). In particular, we have the following result:

Proposition: Let $p^* = \min_{\tau} \left\{ \sum_i p_i - \sum_{(i,j) \in \tau} p_{i,j} \right\}$. The bounds (3), (5), and (6) satisfy:

(i) $p^* \leq \sum_i p_i - \max_j \sum_{i \neq j} p_{ij} \quad \leq \sum_i p_i - (2/m) \sum_{i<j} p_{ij}$

(ii) $p^* \leq \sum_i p_i - \sum_{i=2}^{m} \max_{k<i} p_{ik}$

Proof: To prove (i), note that the graph $\left\{ \{ i, j \} : i = 1, \ldots, m,\ i \neq j \right\}$ is a spanning tree for each *j*, hence

$$p^* \leq \sum_i p_i - \sum_{i \neq j} p_{ij}.$$

This inequality implies

$$p^* \le \sum_i p_i - \max_j \sum_{i \ne j} p_{ij} \le \sum_i p_i - \sum_{i \ne j} p_{ij} ,$$

which, when averaged over $j = 1, \ldots, m,$ yields

$$p^* \le \sum_i p_i - \max_j \sum_{i \ne j} p_{ij} \le \sum_i p_i - \frac{1}{m} \sum_j \sum_{i \ne j} p_{ij} .$$

The preceding inequality and the identity $\sum_j \sum_{i \ne j} p_{ij} = 2\sum_{i<j} p_{ij}$ complete the proof of (i).

Last, to prove (ii), note that the index set consisting of $\{ i, k_i \}$, $i = 2, \ldots, m$, is a spanning tree if for each *i,* $1 \le k_i < i$, $i = 2, \ldots, m$, in which instance $p^* \le \sum_i p_i - \sum_{i=2}^{m} p_{i,k_i}$. If, in addition k_i is chosen so that $p_{i,k_i} = \max \{ p_{i,j} : 1 \le j < i \}$, $i = 2, \ldots, m$, then $\sum_i p_i - \sum_{i=2}^{m} p_{i,k_i} = \sum_i p_i - \sum_{i=2}^{m} \max_{i<k} p_{i,k}$ from which the result follows.

Given an estimate $\hat{p}_i$ of p_i and $\hat{p}_{ij}$ of p_{ij}, $1 \le i, j \le m$, we can form an upper bound estimate $\hat{p}_{UB}$ by substituting the estimates in the right-side of (2) (or (3) through (6) and taking the smallest of the three bounds). Typically, $\hat{p}_i$ would be computed simply as the ratio $\{\text{\# failures}\}/\{\text{\# trials}\}$ for subsystem *i*. The estimation of the joint probabilities $\hat{p}_{ij}$ is typically more challenging and problem-dependent. Usually their derivation will involve physical modeling and system identification together with subsystem tests. In particular, failure detection and fault isolation methods ([10], [11], [12], [13]) may provide a means of estimating such joint probabilities. Further, fault isolation methods are valuable in providing a means for determining the specific cause of a failure.

Remark: Although the bound in (5) is sharper than (3), it is easier to derive uncertainties and confidence intervals for (3) than (5), since it does not involve finding a maximum.

III. The IBR Method

A. Computation of the Estimate

We now describe the IBR estimate. The approach recognizes the practical reality when one may have at least a few full system tests that contain valuable information to be combined with the subsystem tests. It is not necessary, however, to have full-system tests to implement the approach. If full-system test results are available, the IBR estimate here is a weighted combination of the upper bound estimate $\hat{p}_{UB}$ and the estimate $\hat{p}$ based on full-system testing. (The estimate $\hat{p}$ is usually computed simply as the ratio of number of failures to the number of tests.) In the absence of full system testing—i..e., $\hat{p}$ is unavailable—$\hat{p}_{UB}$ is defined to be the IBR estimate.

Fix a spanning tree, τ say, and let p_{UB} denote the bound on the right-side of (1). (For brevity, τ will sometimes be suppressed.) Assume that $\hat{p}_{UB}$ and $\hat{p}$ are derived from independent data. For each λ, $0 \le \lambda \le 1$, let $p_\lambda(\tau) = \lambda p + (1-\lambda) p_{UB}(\tau)$. For each λ, the quantity $\hat{p}_\lambda(\tau) = \lambda \hat{p} + (1-\lambda)\hat{p}_{UB}(\tau)$ is an (upper bound) estimator of p. The IBR estimate of p (based on $\hat{p}$, $\hat{p}_{UB}$ and, hence, τ) is obtained by suitably choosing λ. In particular, the IBR estimate is defined to be $\hat{p}_{\lambda^*}(\tau) = \lambda^* \hat{p} + (1-\lambda^*)\hat{p}_{UB}(\tau)$, where λ^* is chosen so that $\hat{p}_{\lambda^*}(\tau)$ minimizes the mean square error $E(\hat{p}_\lambda - p_\lambda)^2$, over all λ such that $0 \le \lambda \le 1$. In particular, as will be shown

$$\hat{p}_{\lambda^*}(\tau) = \lambda^* \hat{p} + (1-\lambda^*)\hat{p}_{UB}(\tau)$$

where $\lambda^* = \sigma_{UB}^2 / (\sigma^2 + \sigma_{UB}^2)$, $\sigma_{UB}^2 = E(\hat{p}_{UB} - p_{UB})^2$, and $\sigma^2 = E(\hat{p} - p)^2$.

B. Discussion

In this subsection, we provide justification of the loss function for defining the IBR estimate. According to the foregoing, the IBR estimate is the estimate of the form $\hat{p}_\lambda$ that minimizes the mean square error $E(\hat{p}_\lambda - p_\lambda)^2$. What typically is of interest in practice is the error $E(\hat{p}_\lambda - p)^2$, where *p* is the true system reliability. However, this latter quantity depends on the bias term $(p - p_{UB})^2$, which is unknown.

We can establish a bound on the error in using $E(\hat{p}_\lambda - p_\lambda)^2$ rather than $E(\hat{p}_\lambda - p)^2$ to define the IBR estimate.

Proposition: Let $\hat{p}_{\lambda^{**}}$ minimize $E(\hat{p}_\lambda - p)^2$ and let $\hat{p}_{\lambda^*} = \lambda^* \hat{p} + (1-\lambda^*)\hat{p}_{UB}$ be the IBR estimate of *p*. Then $\left| E(\hat{p}_{\lambda^{**}} - p)^2 - E(\hat{p}_{\lambda^*} - p_{\lambda^*})^2 \right| \le (p - p_{UB})^2$.

Proof: Observe that $p - \hat{p}_\lambda = p - p_\lambda + p_\lambda - \hat{p}_\lambda = (1-\lambda)(p - p_{UB}) + (p_\lambda - \hat{p}_\lambda)$, which implies

$$E(\hat{p}_\lambda - p)^2 = (1-\lambda)^2 (p - p_{UB})^2 + E(\hat{p}_\lambda - p_\lambda)^2 .$$

If λ^{**} minimizes $E(\hat{p}_\lambda - p)^2$, the last identity implies that

$$E\left(p-\hat{p}_{\lambda^{**}}\right)^2=\left(1-\lambda^{**}\right)^2\left(p-p_{UB}\right)^2+E\left(\hat{p}_{\lambda^{**}}-p_{\lambda^{**}}\right)^2\geq$$
$$\left(1-\lambda^{**}\right)^2\left(p-p_{UB}\right)^2+\min_{\lambda}\ E\left(\hat{p}_{\lambda}-p_{\lambda}\right)^2 .$$

Since $(1-\lambda)^2\leq 1$, we also have

$$\left(p-p_{UB}\right)^2+E\left(\hat{p}_{\lambda^*}-p_{\lambda^*}\right)^2\geq E\left(p-\hat{p}_{\lambda^{**}}\right)^2 .$$

Consequently,

$$\left|E\left(\hat{p}_{\lambda^{**}}-p\right)^2-E\left(\hat{p}_{\lambda^*}-p_{\lambda^*}\right)^2\right|\leq\ \left(1-\lambda^{**}\right)^2\left(p-p_{UB}\right)^2\leq\left(p-p_{UB}\right)^2 .$$

Note that the right most term in the above is simply the square of the error in p_{UB}.

C. Properties of the Estimate

We establish a connection between the IBR estimate $\hat{p}_{\lambda^*}(\tau)$ and the method of least squares by deriving the optimal value λ^* of λ. For the connection between IBR and least squares we prove the following result.

Proposition: Let $\hat{p}$ and $\hat{p}_{UB}$ be unbiased, independent estimates of p and p_{UB}, respectively. Then,

(i) $p\leq E\left(\min_{\tau}\hat{p}_{\lambda^*}(\tau)\right)\leq\min_{\tau}\ p_{\lambda^*}(\tau)$, where the τ's belong to a specified set of J of spanning trees.

(ii) If $\hat{p}_{\lambda^*}$ is the IBR estimate p, then $\lambda^*=\sigma_{UB}^2/\left(\sigma^2+\sigma_{UB}^2\right)$, where $\sigma_{UB}^2=E\left(\hat{p}_{UB}-p_{UB}\right)^2$ and $\sigma^2=E\left(\hat{p}-p\right)^2$.

Proof: The proof of (i) is omitted since it is straightforward. Now consider (ii). First,

$$E\left(\hat{p}_{\lambda}-p_{\lambda}\right)^2=\lambda^2E\left(\hat{p}-p\right)^2+2\lambda(1-\lambda)\ E\left\{\left(\hat{p}-p\right)\ \left(\hat{p}_{UB}-p_{UB}\right)\right\}$$
$$+(1-\lambda)^2E\left(\hat{p}_{UB}-p_{UB}\right)^2$$

$$=\lambda^2E\left(\hat{p}-p\right)^2+(1-\lambda)^2E\left(\hat{p}_{UB}-p_{UB}\right)^2, \qquad (8)$$

where the last identity follows from the independence of $\hat{p}$ and $\hat{p}_{UB}$. Using the method of Lagrange multipliers, it can be shown that the right-side of (8) attains its minimum at $\lambda^*=\sigma_{UB}^2/\left(\sigma^2+\sigma_{UB}^2\right)$.

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