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Recursive Backstepping Control of Chaotic Duffing Oscillators

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Abstract - **In this paper, the dynamics of a forced Duffing oscillator are studied by means of modern nonlinear, bifurcation and chaos theories to show that the system is ultimately experiencing chaos. The main objective is to characterize and control this chaotic behavior. A nonlinear recursive Backstepping controller is proposed and the transient performance is investigated. Systematic following of a reference model is introduced. Robustness problems as well as ways to tune the controller parameters are examined. Simulation results are submitted for the uncontrolled and controlled cases, verifying the effectiveness of the proposed controller. Finally a discussion and conclusions are given with possible future extensions.**

1. Introduction

The route to chaos from regular periodicity through a sequence of period doubling bifurcations in nonlinear oscillators has been the subject of many analytical and numerical investigations, e.g. [1-9]. These studies and others have shown that this route to (or from) chaos can be adequately described by making use of approximate analytical methods to study instabilities of approximate periodic solutions along with computer simulations. By making use of variational Hill's type equations to examine various instabilities of corresponding approximate periodic solutions, these studies have shown that it is possible to determine and describe, with fair accuracy, the zones of period doubling bifurcations on resonance curves.

Methods of qualitative analysis, with the aid of simulations, were used in these studies to determine the locations of chaotic motion. Szemplinska [3,4] used this approach in connection with harmonic balance method to describe the period doubling bifurcations in a single equilibrium Duffing oscillator with asymmetric hardening nonlinearity [4], the symmetry breaking, and then period doubling bifurcations in single equilibrium symmetric Duffing oscillators with softening nonlinearity [3]. It is to be noted that for an oscillator with asymmetric nonlinearities, an approximate periodic solution is asymmetric which usually contains a bias and both even and odd harmonics. For such oscillators the stability analysis of the variational equation corresponding to an asymmetric periodic solution shows that, even for first order, the asymmetric periodic solution can undergo period doubling bifurcations.

The results of the analysis presented in [4], show that for an asymmetric Duffing oscillator the chaotic motion appears in a narrow zone, which is preceded by a wider period doubling zone close to the theoretical stability limit of the ½ subharmonic resonance, i.e. in the neighborhood of the frequency where the ½ resonance curve has a vertical tangent. On the other hand, symmetric nonlinear oscillators of the hardening type do not, as a general rule, admit at least in the first approximation asymmetric solution which is necessary for period doubling bifurcations, i.e. the stability analysis of the variational Hill's type equation corresponding to an approximate periodic solution in symmetric nonlinear oscillators shows that approximate solutions can undergo period doubling provided that it is asymmetric (e.g. see [3,4] for more details). The results presented in [4,6] for the classical Duffing oscillator and Duffing-Ueda oscillator, with hardening nonlinearities, which are obtained using a combination of harmonic balance and numerical simulations, show that for symmetric nonlinear oscillators of the hardening type in the transition to (or from) chaos is a sharp one and is associated with the loss of stability of the third superharmonic resonant. In this chase the chaotic motion appears at frequencies well below the principal resonance, i.e. in the region that is not adequately described by a first approximate harmonic solution. The results presented in [4-6] and in many other studies, indicate that for an asymmetric or a symmetric nonlinear oscillator of the hardening type with single equilibrium position oscillator, the chaotic motion is always associated with the loss of stability of secondary resonance (super – ultra super, sub, or ultra sub-harmonic). In such oscillators, the chaotic motion forms a transition zone that separates two periodic solutions having different periods (i.e. having different topological properties). On the other hand, in the symmetric Duffing oscillator with softening type elastic nonlinearity, numerical and analytical results presented in [3,7] show that in this oscillator the chaotic motion is preceded by a sequence of period doubling and appears near the peak of the principal resonance curve.

Using a combined harmonic balance and computer simulation, Szmplinska-Stupnika [3] showed that the stability analysis of the first approximate harmonic solution for this oscillator can predict the symmetry breaking and period doubling bifurcations provided that higher order instabilities of the corresponding variational Hill's type equation are examined. Unlike the classical symmetric, Duffing oscillator with hardening nonlinearity, the symmetric Duffing oscillator with softening nonlinearity can admit in the first approximation asymmetric solution that appears in pairs at usually relatively large response amplitude over a narrow frequency band [7].

Al-Qiasia et. al. [8] and Dooren [9], introduced a numerical procedure to study the transition from regular periodic motion to chaotic behavior of the Duffing oscillator with softening nonlinearity. The procedure used is based on the computation of accurate higher order approximate periodic solution of Galerkin's type in conjunction with the corresponding stability analysis of the first variational equation.

Traditional adaptive nonlinear schemes are classified as direct or indirect and as Lyapunov-based or estimation-based. They involve parameter identification [10] and adaptation algorithm. The direct-indirect classification reflects the fact that updated parameters are either those of the controller or the plant respectively. The distinction between Lyapunov-based and estimation-based schemes is more substantial and is indicated in part by proof of asymptotic stability and convergence. Recursive design procedures, referred to as Backstepping, can extend the applicability of Lyapunov-based deigns to non-linear systems [11]. When the true parameters of the systems are unknown, the controller parameters are either estimated directly (direct scheme) or computed by solving the same design equations with plant parameters estimates (indirect scheme). The resulting controller is called a certainty equivalence controller. Backstepping designs are flexible and do not force the designed system to appear linear. They also avoid cancellation of perhaps useful nonlinearities and often introduce additional nonlinear terms to improve transient performance [12,13,14]. The idea of Backstepping is to recursively design a nonlinear controller by considering some of the state variables as "virtual controls". When trying to deal with unknown parameters a conflict will exist between virtual controls and parameter update laws that can be sorted out using adaptation [15].

In this paper, a new controller based on Backstepping techniques [16] is presented. The paper is organized as follows; in Sec. 2, the mathematical model is presented. In Sec. 3, the bifurcation and chaos analysis is discussed. The methodology of Backstepping controller is discussed in Sec. 4. Finally, discussions and conclusions are presented in Sec. 5.

2. MATHEMATICAL MODEL OF DUFFING OSCILLATOR

A mathematical model of the well known forced Duffing oscillator is given by:

$$\ddot{x} + \delta\dot{x} + \left\{x + \varepsilon_1\left(x^2\ddot{x} + x\dot{x}^2\right) + \varepsilon_2 x^3\right\} = P\cos\Omega t + u \quad (1)$$

Using the following controller structure:

$$\begin{aligned} u &= u_{ff} + u_{fb} \\ &= -P\cos\Omega t + f(x_1, x_2) \end{aligned} \quad (2)$$

where:

u_{ff} is the feed-forward controller (assuming known P and Ω)

u_{fb} is the feedback controller (that needs to be found to stabilize the system)

Solving for $\ddot{x}$ yields:

$$\begin{aligned} \ddot{x} &= -\left\{\frac{\dot{x}(\delta + \varepsilon_1 x\dot{x}) + x\left(1 + \varepsilon_2 x^2\right)}{1 + \varepsilon_1 x^2}\right\} + \left\{\frac{1}{1 + \varepsilon_1 x^2}\right\} f \\ &= g(x) + h(x) f \end{aligned} \quad (3)$$

from which, we have the following:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g(x_1, x_2) + h(x_1, x_2) f \end{bmatrix} \text{ and}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (4)$$

3. BIFURCATION AND CHAOS ANALYSIS

The case without control signal, i.e. when u in (1) is equal to zero, is now investigated. Put $\varepsilon_1=10$, $\varepsilon_2=1$, $P=3.5$, and the control parameter $\Omega=1$, and for long time integration, a stable limit cycle is found as shown in Fig. 1-a., where the right side of Fig. 1. represents the state-plane, and the left side represents the time history.

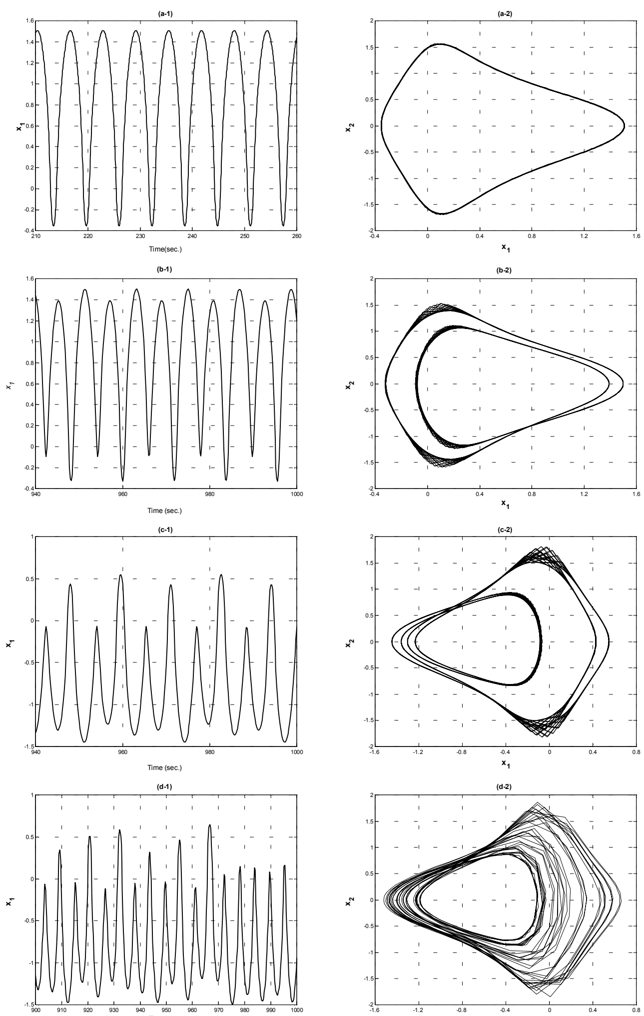

Fig. 1. Uncontrolled Simulations.

Increasing the forcing frequency, Ω, to *1.052*, doubles the period as shown in Fig. 1-b. As Ω increases, a period four appears as shown in Fig. 1-c. Using Ω=*1.12*, a chaotic solution took place as shown in Fig. 1-d. Fig. 2. shows the attractor-merging crisis.

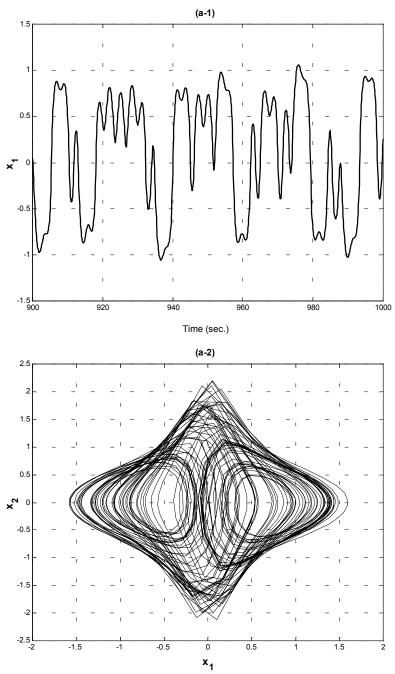

Fig. 2. Attractor-merging crisis.

Many researchers produced similar kind of results. But, our objective is to enable control in the unstable region, i.e. chaotic solution. That means: our intention is to design a nonlinear control based on nonlinear recursive method. This will be discussed in the next section.

4. BACKSTEPPING CONTROLLER DESIGN

A Backstepping controller is designed such that the Duffing oscillator system is stabilized. The second state of the system, x_2, is used as a virtual control to the system output, x_1. The following nonlinear dynamics will now be assumed for the desired virtual control:

$$x_{2_{des}} = -c_1 x_1 - c_2 x_1^3 \tag{5}$$

where c_1 and c_2 are design parameters that need to be carefully chosen such that the closed loop system is stable. The virtual error is given by:

$$e_v = x_2 - x_{2_{des}} = x_2 + c_1 x_1 + c_2 x_1^3 \tag{6}$$

Using (5) and (6), the system is transferred to the new state space:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{e} \end{bmatrix} = \begin{bmatrix} e - c_1 x_1 - c_2 x_1^3 \\ \dot{x}_2 + x_2\left(c_1 + 3c_2 x_1^2\right) \end{bmatrix} \tag{7}$$

Lyapunov second method will now be used to test for the system stability. Introducing the following augmented Lyapunov function:

$$V_a = \tfrac{1}{2} x_1^2 + \tfrac{1}{2} e^2 \tag{8}$$

results in:

$$\begin{aligned} \dot{V}_a &= x_1 \dot{x}_1 + e\dot{e} \\ &= -c_1 x_1^2 - c_2 x_1^4 + e\left\{x_1 + g(x) + h(x)\bar{u} + x_2\left(c_1 + 3c_2 x_1^2\right)\right\} \\ &= -c_1 x_1^2 - c_2 x_1^4 - c_3 e^2 \end{aligned} \tag{9}$$

where c_3 is a presumed constant and the closed loop control, f, is chosen in such a way that is guaranteed to be negative definite. Using (6) and (9), this is equivalent to:

$$\begin{aligned} & x_1 + g(x_1, x_2) + h(x_1, x_2) f + x_2\left(c_1 + 3c_2 x_1^2\right) \\ & = -c_3 (x_2 + c_1 x_1 + c_2 x_1^3) \end{aligned} \tag{10}$$

which can be put in the more compact form:

$$\begin{aligned} & g(x_1, x_2) + h(x_1, x_2) f \\ & = -x_1\left(1 + c_1 c_3\right) - x_2\left(c_1 + c_3\right) - c_2 x_1^2\left(c_3 x_1 + 3x_2\right) \\ & = w(x_1, x_2) \end{aligned} \tag{11}$$

from which, the following control low is deduced:

$$f = \frac{w(x_1, x_2) - g(x_1, x_2)}{h(x_1, x_2)} \tag{12}$$

Because of the special dynamic structure of the Duffing oscillator, the designed control, f, is causal and could be easily implemented by choice of the design parameters. The resulting closed loop system is consequently given by:

$$\begin{bmatrix} \dot{y} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 y^2 & 1 \\ -1 & -c_3 \end{bmatrix} \begin{bmatrix} y \\ e \end{bmatrix} \tag{13}$$

Eliminating e and its derivative from (13), the output dynamics is given by:

$$\ddot{y} + \dot{y}(c_1 + 3c_2 y^2 + c_3) + y(1 + c_1 c_3 + c_2 c_3 y^2) = 0 \tag{14}$$

which can be made linear by choosing $c_2 = 0$. Thus:

$$\begin{aligned} & \ddot{y} + (c_1 + c_3)\dot{y} + (1 + c_1 c_3) y = 0 \;\text{ and} \\ & y(0) = y_0 \;,\; \dot{y}(0) = 0 \end{aligned} \tag{15}$$

4.1 TRACKING

The designed control law guarantees the stability of the system for all system parameters and initial conditions. With reference to (9), there are no constraints on choosing the design parameters other that: $c_i > 0$, $i=1,2,3$. This offers more flexibility in the design that enables controlling the transient behavior of the Duffing oscillator to satisfy a certain performance criterion. We will now go one step further, and try to force the system output to follow that of a desired reference model. This reference model can be just a standard linear second order system whose performance is characterized by two parameters, damping ratio; ξ, and natural damping frequency; ω_n. This translates to:

$$\ddot{y} + (c_1 + c_3)\dot{y} + (1 + c_1 c_3)y = \ddot{y}_{des} + 2\xi\omega_n \dot{y}_{des} + \omega_n^2 y = 0 \tag{16}$$

such that:

$$Y_{des}(s) = y_0 \frac{(s + 2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{17}$$

Matching coefficients, we have:

$$c_{1,3} = \xi\omega_n \pm \sqrt{1 - \omega_n^2 (1 - \xi^2)} \tag{18}$$

Thus the overall constraints of the designed system are given by:

$$\begin{aligned} &c_1 \;,\; c_3 \text{ and } \xi > 0 \\ &\text{if } \xi < 1 \text{ then } 1 < \omega_n < \frac{1}{\sqrt{1-\xi^2}} \\ &\text{if } \xi > 1 \text{ then } 1 < \omega_n < \omega_{\max} \end{aligned} \tag{19}$$

This is represented graphically in Fig. 3., where it is seen that a unique set of design parameters is obtained for each desired performance.

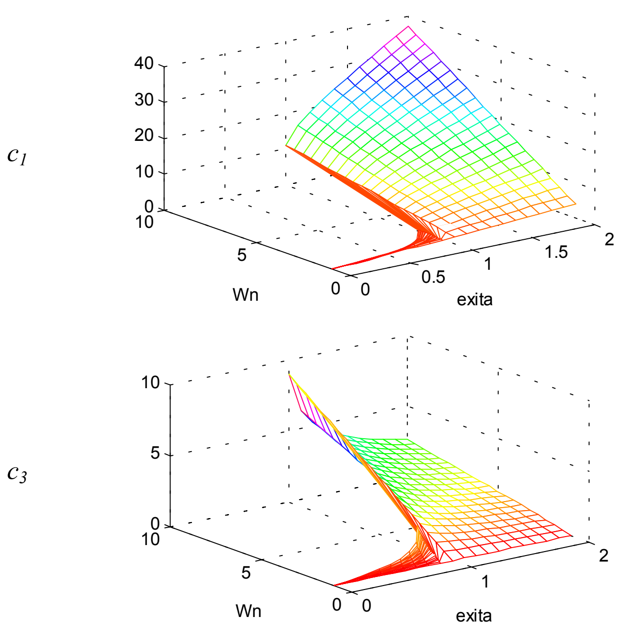

Fig. 3. Design parameters vs. performance parameters.

From (19), it is seen that the structure of both the desired reference model and the proposed controller puts some constraints in choosing the design parameters, e.g. the set corresponding to $\xi=0.5$ and $\omega_n=2$ is not feasible.

4.2 Simulation Results:

A Duffing oscillator with $\delta=0.5$, $\varepsilon_1=10$, $\varepsilon_2=1$, $P=3.5$, $\Omega=2$ *is* used. The reference model is assumed to have $\xi=0.707$ and $\omega_n=1.414$ which results in $c_1=c_3=1$. Using the recursive Backstepping technique with $y_0=1$ and $c_2=0$, we have the results shown in Fig. 4.

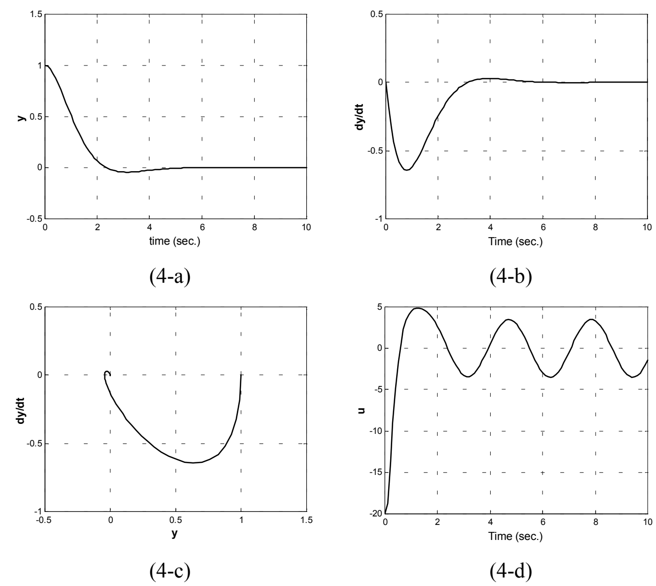

Fig. 4. Controlled simulations.

It is obvious, from Fig. 4., that not only does the system exhibit stable performance, but it also mimics the behavior of a standard well-known linear second order system. Fig. 4. (a), (b) verifies that the system settles down within four seconds, and exhibits a 4.3% overshoot which are typical results for the used reference model. Also, from Fig. 4. (d), it is shown that the control signal reduces to the counter of the original periodic signal so that the net forcing function for the Duffing oscillator is zeroed.

5. Discussion and Conclusion

Modern nonlinear control was applied to a forced Duffing oscillator that is experiencing chaotic behavior to have prescribed performance. Nonlinear recursive Backstepping controlled the undesirable chaotic behavior. The study shows the effectiveness of the technique in the unstable regions. The designed controller achieves two goals, first stabilizing the Duffing oscillator system, and second providing means of shaping the transient performance of the system. Usually a conflict occurs if these two goals are to be achieved simultaneously, but the designed controller resolves such conflict by introducing a flexible set of controller parameters that adds more freedom in the design. The dynamic structure of the virtual control can be carefully chosen to satisfy some design criteria, it need not be as given in (5). This adds more versatility to the designed controller. During the analysis and simulation, it was assumed that the Duffing oscillator parameters were deterministic, a crucial for the use of (2). If, however, this is not the case, robustness can be added to the system by incorporating states estimator, parameters estimator or both. The design of the update law mechanism for these estimators can be integrated into the design by introducing more virtual control parameters and changing the corresponding Lyapunov function, (8), accordingly [17].

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