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A Mixed IQC Approach to Nonlinear Delay-Dependent System Analysis

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***Abstract*— Stability analysis techniques are presented for time-delay systems consisting of the feedback interconnection of a linear time-delay system, with a bounded and casual operator, featuring the nonlinearities, uncertainties, and/or time-varying components of the system. The delays considered are time-invariant but uncertain, residing within a bounded interval including zero. The theorem of Integral Quadratic Constraints (IQC Theorem) is employed in a novel fashion to formulate a stability criterion. In this method, the delay elements are replaced by parameter-dependent filters satisfying certain properties, while the nonlinearities are captured by IQCs. It is shown that satisfaction of the IQC Analysis Condition by the delay-differential system can be guaranteed by satisfaction of it by a finite-dimensional, parameter-dependent system. The KYP lemma is then applied to the latter to obtain a parameter-dependent LMI criterion.**

***Index Terms*— Time-delay systems; nonlinearity; stability; IQC.**

I. INTRODUCTION

An important aspect of many systems with delay elements within their dynamics is the presence of significant nonlinear terms. For instance, a class of interconnected systems arising in information networks, such as internet, is subject to delay and saturation nonlinearity. Because of potential queuing delays, the feedback is delayed, moreover the amount of delay is subject to uncertainty based on network traffic. In addition, saturation is an outcome of limited buffer sizes at the routers and the data transmission rates at sources [11]. Another interesting example is the heat release phenomenon in combustion instability which is described by a nonlinear delay differential equation [18].

While greater research effort has recently been brought to bear on the problem of nonlinear time-delay systems [1], [3], [17], [21], most of the analysis and synthesis methods display significant conservatism in practice. Many engineering systems of high practical interest can be modeled as the interconnection of a linear time-invariant system with a nonlinear (perhaps time-varying) component. Examples include the interaction of neurons in a continuous neural Hopfield network [4], the *delayed cellular network model* used in image processing and pattern recognition problems [16], and the thermoacoustic instability in combustors, where the system is linear time-invariant with the exception of the nonlinear heat release function [18].

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In this paper we are concerned with *delay-dependent* stability analysis of time-delay systems formed by the feedback interconnection of a linear delay-differential system and a causal and bounded operator, which may contain any and all nonlinear, uncertain, and/or time-varying components of the system, excluding the delay elements. The delays are assumed to be unknown but constant, residing within an interval $[0, \overline{\tau}]$. In this paper, we extend the results of [23] to nonlinear systems.

In general, for linear systems with feedback nonlinearities, a number of analysis techniques, by and large referred to as absolute stability theory [9], [15], [20], are available. Recently, the theory of Integral Quadratic Constraints [13] has been advocated for the unification and generalization of these results.

Previous research efforts have considered the use of IQCs for the analysis of linear and nonlinear time-delay systems [7], [8], [13], [2]. However, in these papers, IQCs were used to capture the delay elements as well as the nonlinearities (pure IQC analysis). It was demonstrated in [22] that the manner in which these type of IQCs cover the delay value set, usually results in a high degree of conservatism in stability analysis. In this paper, we advocate the use of IQCs only for the nonlinearities, with the delay elements tackled via parameter-dependent filters, tightly covering the value set of the delay elements (mixed IQC analysis). This new method brings the power of the IQC framework and the tight *value set* covering feature of our parameter-dependent filters together. In this fashion, we retain the high accuracy of previous results for linear systems [23], as we undertake the analysis of nonlinear system. As we shall see in the numerical example section, this method can be significantly less conservative than previous approaches to nonlinear delay-dependent stability analysis.

A. Problem Statement

In this paper, the standard feedback configuration, illustrated in Fig. 1, will be used when examining stability of the delay-differential system:

$$\begin{aligned} v &= Gw + f \\ w &= \Delta_d v + e \end{aligned} \tag{1}$$

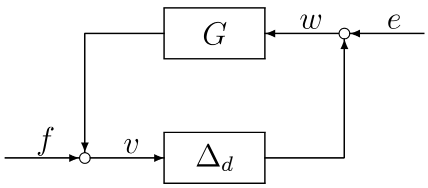

Fig. 1. Basic feedback configuration.

where $\Delta_d = \begin{bmatrix} \delta & 0 \\ 0 & d_\tau \end{bmatrix}$, $v = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$, $w = \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T$, $e = \begin{bmatrix} e_1^T & e_2^T \end{bmatrix}^T$ and $f = \begin{bmatrix} f_1^T & f_2^T \end{bmatrix}^T$. The linear operator G, has a transfer function $G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$, where $G_{11}(s) \in \mathbb{RH}_\infty^{l_1 \times m_1}$, $G_{12}(s) \in \mathbb{RH}_\infty^{l_1 \times m_2}$, $G_{21}(s) \in \mathbb{RH}_\infty^{m_2 \times m_1}$ and $G_{22}(s) \in \mathbb{RH}_\infty^{m_2 \times m_2}$. The perturbation operator δ is a bounded and causal operator on $\mathbb{L}_{2e}^{l_1}[0,\infty)$ to $\mathbb{L}_{2e}^{m_1}[0,\infty)$. $d_\tau := \operatorname{diag}\{d_{\tau_k} : v_k(t) \to v_k(t-\tau_k) - v_k(t)\}$, $k = 1, ..., N$ is the diagonal delay operator where each time-delay τ_k belongs to an interval $[0, \overline{\tau_k}]$. Using the state space notation, the feedback system (1) is described by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w_1(t) + B_2 w_2(t) \qquad (2) \\ v_1(t) &= C_1 x(t) + D_{11} w_1(t) + D_{12} w_2(t) + f_1(t) \\ v_2(t) &= C_2 x(t) + f_2(t) \\ w_1(t) &= \delta v_1(t) + e_1(t) \\ w_2(t) &= d_\tau v_2(t) + e_2(t) \end{aligned}$$

We will use $\sum_d^\delta(\tau)$ (or just $\sum_d^\delta$ whenever possible) to refer to the system (1) (or indifferently (2)). The system $\sum_d^\delta$ is the feedback interconnection of three components: (1) a linear, time-invariant operator $G : \mathbb{L}_{2e}^{m_1+m_2}[0,\infty) \to \mathbb{L}_{2e}^{l_1+m_2}[0,\infty)$ with transfer function $G(s) = \left[\begin{array}{c|c} [A] & \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} \end{array}\right]$; (2) the delay operator $d_\tau : v(t) \to d_\tau v(t)$ mapping $\mathbb{L}_{2e}^{m_2}[0,\infty)$ to $\mathbb{L}_{2e}^{m_2}[0,\infty)$; and (3) a bounded and causal operator $\delta : v(t) \to \delta v(t)$ mapping $\mathbb{L}_{2e}^{l_1}[0,\infty)$ to $\mathbb{L}_{2e}^{m_1}[0,\infty)$. Throughout this paper, it is assumed that $G(s)$ is stabilizable and detectable.

For clarity and convenience in notation, we prove the theory for the single delay case only ($N = 1$). All the theory presented here is readily extensible to the multiple-delay case and it should be clear how the extensions may be carried out (compare with [10]). Needless to say, δ itself can be a diagonal structured perturbation, where each diagonal element is a bounded, and causal operator.

B. Preliminaries

Notation 1: Let $\mathbb{R}^{n\times m}$ $(\mathbb{C}^{n\times m})$ be the set of all real (complex) $n \times m$ matrices, $\mathbb{C}_+$ $(\overline{\mathbb{C}}_+)$ be the open (closed) right-half of the complex plane. $\operatorname{He}(P)$ denotes $P + P^T$ and $P > 0$ indicates that P is a symmetric and positive definite matrix. $\mathbb{H}_\infty$ denotes the space of functions analytic in $\overline{\mathbb{C}}_+$. $\mathbb{RH}_\infty$ is the subspace of $\mathbb{H}_\infty$ consisting of all real rational functions with no poles in $\overline{\mathbb{C}}_+$. The set of $m \times n$ matrices with elements in $\mathbb{RH}_\infty$ will be denoted by $\mathbb{RH}_\infty^{m\times n}$. For $f \in \mathbb{R}^m$, $|.|$ denotes the Euclidean norm $|f| = \sqrt{f^T f}$. $\mathbb{L}_2^m[0,\infty)$ denotes the space of $\mathbb{R}^m$-valued functions $f : [0,\infty) \to \mathbb{R}^m$, for which the norm $\|f\|^2 = \int_0^\infty |f(t)|^2 dt$ exists and is bounded. $\mathbb{L}_2^m[0,\infty)$ is a subspace of $\mathbb{L}_{2e}^m[0,\infty)$ whose elements need to be square integrable on finite intervals only. The operator $\Delta : \mathbb{L}_{2e}^l[0,\infty) \to \mathbb{L}_{2e}^m[0,\infty)$ is said to be bounded if the gain $\|\Delta\|_{\mathcal{L}_2} := \sup\left\{\|\Delta f\| / \|f\| \mid f \in \mathbb{L}_2^l[0,\infty),\ f \neq 0\right\}$ exists and is finite. P_T denotes the past *truncation* operator, which is defined by: $P_T u(t) = \begin{cases} u(t) & \text{if} \quad t \in [0,T] \\ 0 & \text{if} \quad t \in (T,\infty] \end{cases}$. An operator F is *causal* if and only if $P_T F = P_T F P_T$. For a function $f \in \mathbb{L}_2^m[0,\infty)$, $\widehat{f}(j\omega)$ denotes the one-sided Fourier transform of f defined by: $\widehat{f}(j\omega) = \int_0^\infty e^{-j\omega t} f(t) dt$.
For any transfer matrix G, any measurable Hermitian-valued function Π, and any $\varepsilon > 0$, we denote:

(i) $\widetilde{\mathcal{Q}}(G(j\omega), \Pi(j\omega), \varepsilon) :=$

$$\lambda_{\max}\left(\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} + \varepsilon I\right),\ \omega \in \mathbb{R}$$

(ii) $\mathcal{Q}(G, \Pi, \varepsilon) :=$

$$\sup_{\omega\in\mathbb{R}} \lambda_{\max}\left(\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} + \varepsilon I\right)$$

Definition 1: Given a continuous function $g(q) : [0,\infty) \to \mathcal{D}$ where $\mathcal{D} = \{z \in \mathbb{C} | \ |z| = 1\}$, letting Γ_r be the path created by mapping the interval $q \in [0, r]$ via $g(q)$ to $\mathcal{D}$, we define a *continuous argument* (phase) function for the value $g(r)$ as $\operatorname{Arg}(g(r)) = \arg(g(r)) + 2\pi n(\Gamma_r, 0)$ where $\arg(z) \in (-2\pi, 0]$ is the unique argument of $z \in \mathbb{C}$, $z \neq 0$ and $n(\Gamma_r, 0)$ is the winding number of path Γ about a.

Definition 2: [13] The feedback system (1) is said to be well-posed if the map from $(v, w) \mapsto (e, f)$ has a causal inverse on $\mathbb{L}_{2e}^{l_1+m_2}[0,\infty) \times \mathbb{L}_{2e}^{m_1+m_2}[0,\infty)$.

Definition 3: [13] The feedback system (1) is said to be stable with finite-gain if it is well posed and there exists a constant $C > 0$, such that for any solution of (1) ,

$$\int_0^T \left(|v|^2 + |w|^2\right) dt \leq C \int_0^T \left(|f|^2 + |e|^2\right) dt, \ \forall T \geq 0. \qquad (3)$$

Definition 4: Let $\Pi : j\mathbb{R} \to \mathbb{C}^{l+m} \times \mathbb{C}^{l+m}$ be a bounded measurable function that takes Hermitian values. A bounded operator $\Delta : \mathbb{L}_{2e}^l[0,\infty) \to \mathbb{L}_{2e}^l[0,\infty)$ is said to satisfy the IQC defined by Π if:

$$\int_{-\infty}^{+\infty} \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta y}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta y}(j\omega) \end{bmatrix} d\omega \geq 0$$

for all $y \in \mathbb{L}_2^l[0,\infty)$. If Δ satisfies the IQC defined by Π, we denote $\Delta \in \mathrm{IQC}(\Pi)$. $\Delta \notin \mathrm{IQC}(\Pi)$ also means that there exists $y \in \mathbb{L}_2^l[0,\infty)$, for which the above integral is negative.

The following theorem is the well-known theorem of Integral Quadratic Constraints, which was developed in [13] and then extended and generalized in [19] and [5].

Theorem 1—IQC Theorem: Let $G(s) \in \mathbb{H}_\infty$, and Δ be a bounded causal operator. Consider the system:

$$\begin{aligned} v &= Gw + f \\ w &= \Delta v + e \end{aligned} \tag{4}$$

Assume that:
(i) Δ_λ is a parametrization of Δ, that satisfies:
 (a) $\Delta_\lambda : \mathbb{L}_{2e}^l[0,\infty) \to \mathbb{L}_{2e}^m[0,\infty)$ is bounded and causal for all $\lambda \in [0,1]$.
 (b) $\exists \gamma > 0$ such that
 $\|\Delta_{\lambda_1}(v) - \Delta_{\lambda_2}(v)\| \leq \gamma\,|\lambda_1 - \lambda_2|\,.\,\|v\|$
 for all $v \in \mathbb{L}_{2e}^l(0,\infty]$ and $\lambda_1, \lambda_2 \in [0,1]$
 (c) $\Delta_1 = \Delta$.
(ii) the feedback interconnection of G and Δ_λ is stable when $\lambda = 0$ and well-posed for $\lambda \in [0,1]$.
(iii) for every $\lambda \in [0,1]$, $\Delta_\lambda \in \mathrm{IQC}(\Pi)$.
(iv) there exists $\varepsilon > 0$ such that

$$\mathcal{Q}(G, \Pi, \varepsilon) \leq 0 \tag{5}$$

Then, the feedback interconnection of G and Δ_1 is stable with finite-gain.

II. System Transformation

Consider the Linear Fractional Transformation $\mathcal{F}(G, (e^{-\tau s} - 1)\, I_{m_2})$, defined as

$$\begin{aligned} &\mathcal{F}(G, \left(e^{-\tau s} - 1\right) I_{m_2}) \\ &:= G_{11} + G_{12}\left(I - \left(e^{-\tau s} - 1\right) G_{22}\right)^{-1}\left(e^{-\tau s} - 1\right) G_{21} \end{aligned} \tag{6}$$

Also, let the linear time-delay system (LTDS) $\sum_d(\tau)$ be defined as:

$$\begin{aligned} v_2 &= G_{22}w_2 + f_2 \\ w_2 &= (e^{-\tau s} - 1)w_2 + e_2, \qquad \tau \in [0,\overline{\tau}] \end{aligned}$$

We then, have the following lemma, regarding stability of the time-delay system $\sum_d^\delta$.

Lemma 1: Consider the system $\widetilde{\sum}_d^\delta(\tau)$:

$$\begin{aligned} \widetilde{v} &= \mathcal{F}(G, \left(e^{-\tau s} - 1\right) I_{m_2})\widetilde{w} + \widetilde{f} \\ \widetilde{w} &= \delta\widetilde{v} + \widetilde{e} \end{aligned} \tag{7}$$

Then, for every $\tau \in [0,\overline{\tau}]$, the system $\sum_d^\delta(\tau)$ is stable with finite-gain if the following two conditions hold:
 (i) $\sum_d(\tau)$ is stable with finite-gain.
 (ii) $\widetilde{\sum}_d^\delta(\tau)$ is stable with finite-gain.

Proof: Omitted for brevity. ■

III. Sufficient Stability Criterion

A. Application of the IQC Theorem

The following theorem establishes a sufficient stability criterion based on the IQC Theorem.

Theorem 2: Assume that:
(i) $\sum_d(\tau)$ is asymptotically stable for all $\tau \in [0,\overline{\tau}]$.
(ii) δ_α is a parameterization of δ that satisfies:
 (a) $\delta_\alpha : \mathbb{L}_{2e}^{l_1}[0,\infty) \to \mathbb{L}_{2e}^{m_1}[0,\infty)$ is bounded and causal for all $\alpha \in [0,1]$.
 (b) $\exists \gamma > 0$ such that
 $\|\delta_{\alpha_1}v - \delta_{\alpha_2}v\| \leq \gamma\,|\alpha_1 - \alpha_2|\,.\,\|v\|$
 for all $v \in \mathbb{L}_{2e}^{l_1}[0,\infty)$ and $\alpha_1, \alpha_2 \in [0,1]$
 (c) $\delta_\alpha|_{\alpha=1} = \delta$.
(iii) $\forall \tau \in [0,\overline{\tau}]$, the feedback interconnection of $\mathcal{F}(G, (e^{-\tau s} - 1)\, I_{m_2})$ and δ_α is well-posed for all $\alpha \in [0,1]$.
(iv) $\forall \alpha \in [0,1]$, $\delta_\alpha \in \mathrm{IQC}(\Pi)$.
(v) $\forall \tau \in [0,\overline{\tau}]$, there exists an $\varepsilon_\tau > 0$ such that

$$\mathcal{Q}\left(\mathcal{F}(G, (e^{-\tau s} - 1)I_{m_2}), \Pi, \varepsilon_\tau\right) \leq 0 \tag{8}$$

Then, the system $\sum_d^\delta(\tau)$ is stable with finite-gain for all $\tau \in [0,\overline{\tau}]$.

Proof: Condition (i) guarantees that $\sum_d(\tau)$ is asymptotically stable for all $\tau \in [0,\overline{\tau}]$ and therefore, $\mathcal{F}(G, (e^{-\tau s} - 1)\, I_{m_2}) \in \mathbb{H}_\infty$, $\forall \tau \in [0,\overline{\tau}]$. Conditions (ii) to (v), establish the four conditions of the IQC Theorem. Therefore, for every $\tau \in [0,\overline{\tau}]$, all the conditions of the IQC Theorem for the feedback interconnection of $\mathcal{F}(G, (e^{-\tau s} - 1)\, I_{m_2})$ and δ, are satisfied. That is, $\widetilde{\sum}_d^\delta(\tau)$ is stable with finite-gain for all $\tau \in [0,\overline{\tau}]$. Lemma 1, then completes the proof. ■

Remark 1: Verifying Conditions (i) and (v) of Theorem 2 is a nontrivial task. In this paper our main task is to provide non-conservative criteria that establish conditions (i) and (v) in an efficient way.

B. Filter Properties

Consider a rational polynomial parameter-dependent filter $h_o(\theta, s)$, with θ belonging to Θ, a bounded set of real numbers, and $h_o(\theta, s)$ having the following properties:

 P_O-1. $h_o(\theta, s)$ is Hurwitz for $\theta \in \Theta$
 P_O-2. The value set of $h_o(\theta, s)$, $\theta \in \Theta$, covers that of $e^{-\tau s}$, $\tau \in [0,\overline{\tau}]$, i.e.,

$$\begin{aligned} \Omega_d(\omega) &\subseteq \Omega_o(\omega), \forall \omega \geq 0, \text{ where,} \\ \Omega_o(\omega) &= \left\{c \in \mathbb{C}\,|c = h_o(\theta, j\omega),\ \theta \in \Theta\right\} \\ \Omega_d(\omega) &= \left\{c \in \mathbb{C}\,|c = e^{-j\tau\omega},\ \tau \in [0,\overline{\tau}]\right\} \end{aligned}$$

The two following Theorems establish the criteria to verify conditions (i) and (v) of Theorem 2.

C. Removing the Infinite Dimensionality

Theorem 3: Consider the linear comparison system $\sum_o(\theta)$:

$$v_2 = G_{22}w_2 + f_2 \quad (9)$$
$$w_2 = (h_o(\theta, s) - 1)\, v_2 + e_2$$

Then, if

$$\textstyle\sum_o(\theta) \text{ is asymptotically stable for all } \theta \in \Theta, \quad (10)$$

$\sum_d(\tau)$ is asymptotically stable for all $\tau \in [0, \overline{\tau}]$.

Proof: The result can be shown in a similar fashion to the one in [23]. ■

Theorem 4: If a Hermitian-valued bounded measurable matrix function Π exists, such that

$$\begin{array}{c}\forall \theta \in \Theta,\ \exists \varepsilon_\theta > 0,\ \text{such that} \\ \mathcal{Q}\left(\mathcal{F}(G, (h_o(\theta, s) - 1)\, I_{m_2}), \Pi, \varepsilon_\theta\right) \leq 0\end{array} \quad (11)$$

then

$$\begin{array}{c}\forall \tau \in [0, \overline{\tau}],\ \exists \varepsilon_\tau > 0,\ \text{such that} \\ \mathcal{Q}\left(\mathcal{F}(G, (e^{-\tau s} - 1)\, I_{m_2}), \Pi, \varepsilon_\tau\right) \leq 0\end{array} \quad (12)$$

Proof: Proof proceeds by contradiction. Assume that (12) is not true. Then,

$$\begin{array}{c}\exists \tau_0 \in [0, \overline{\tau}] \text{ and } \omega_0 \in \mathbb{R}, \text{ such that} \\ \widetilde{\mathcal{Q}}\left(\mathcal{F}(G(j\omega_0), \left(e^{-\tau_0 j\omega_0} - 1\right) I_{m_2}), \Pi(j\omega_0), \varepsilon\right) > 0, \\ \forall \varepsilon > 0\end{array}$$

From Property P_O-2, there exists $\theta_0 \in \Theta$, such that $e^{-\tau_0 j\omega_0} = h_o(\theta_0, j\omega_0)$, which immediately yields

$$\widetilde{\mathcal{Q}}\left(\mathcal{F}(G(j\omega_0), (h_o(\theta_0, j\omega_0) - 1)\, I_{m_2}), \Pi(j\omega_0), \varepsilon\right) > 0, \quad \forall \varepsilon > 0$$

contradicting (11). Proof is complete. ■

Remark 2: Theorems 3 and 4 provide the criteria that enable the verification of conditions (i) and (v) of Theorem 2 in an efficient way. As we will see in the sequel (10) and (11) can be verified by solving a finite number of linear matrix inequalities (LMIs).

D. Analysis with Candidate Filters

Let $p_l(s)$ denote the l^{th} order diagonal Pade approximation to e^{-s}. Now, consider the outer approximation (covering filter) to the delay transfer function $e^{-\tau s}$, $0 \leq \tau \leq \overline{\tau}$:

$$h_o(\theta, s) := p_l(\alpha_o \theta s), \qquad \theta \in \Theta \quad (13)$$

$$\text{where } \omega_o = \min\left(\omega > 0 \,|\, \mathrm{Arg}(p_l(j\omega)) = -2\pi\right) \quad (14)$$
$$\alpha_o = \frac{\omega_o}{2\pi},\ \Theta = [0, \overline{\tau}] \quad (15)$$

Lemma 2: Suppose that $h_o(\theta, s)$ is as specified in (13) with Pade order l chosen such that α_o and ω_o may be found from (14) and (15). Then $h_o(\theta, s)$ satisfies properties P_O-1 and P_O-2.

Proof: See [23]. ■

In robustness analysis with Integral Quadratic Constraints, it is desired to describe the perturbation operator δ as accurately as possible by IQC's. The class $\mathbf{\Pi}_\delta$ of all rational Hermitian valued matrix functions Π that define a valid IQC for a given δ is convex and is typically infinite-dimensional [13]. A suitable $\Pi \in \mathbf{\Pi}_\delta$ can therefore be found by numerical optimization over a finite-dimensional subset of $\mathbf{\Pi}_\delta$. In this case, Π can be expressed as:

$$\Pi(j\omega) = \sum_{r=1}^{r=r_0} x_r \Pi_r(j\omega) \quad (16)$$

where each x_r is a positive real scalar and $\delta \in \mathrm{IQC}(\Pi_r)$, $r = 1, ..., r_0$.

Now, assume that $h_o(\theta, s)$ is developed as in (13). We have the following corollary.

Corollary 1: Stability Employing Pade Filter. $\sum_d^\delta$ is stable with finite gain for all $\tau \in [0, \overline{\tau}]$ if:

(i) $\sum_o(\theta)$ is asymptotically stable for all $\theta \in [0, \overline{\tau}]$.

(ii) δ_α is a parameterization of δ that satisfies:

- (a) $\delta_\alpha : \mathbb{L}_{2e}^{l_1}[0, \infty) \to \mathbb{L}_{2e}^{m_1}[0, \infty)$ is bounded and causal for all $\alpha \in [0, 1]$
- (b) $\exists \gamma > 0$ such that $\|\delta_{\alpha_1} v - \delta_{\alpha_2} v\| \leq \gamma\, |\alpha_1 - \alpha_2| \,.\, \|v\|$ for all $v \in \mathbb{L}_{2e}^{l_1}[0, \infty)$ and $\alpha_1, \alpha_2 \in [0, 1]$
- (c) $\delta_\alpha|_{\alpha=0} = 0,\ \delta_\alpha|_{\alpha=1} = \delta$

(iii) $\forall \tau \in [0, \overline{\tau}]$, the feedback interconnection of $\mathcal{F}(G, (e^{-\tau s} - 1)\, I_{m_2})$ and δ_α is well-posed for all $\alpha \in [0, 1]$.

(iv) $\forall \alpha \in [0, 1]$, $\delta_\alpha \in \mathrm{IQC}(\Pi_r)$, $r = 1, .., r_0$.

(v) There exist positive real scalars x_r, $r = 1, .., r_0$, such that $\forall \theta \in [0, \overline{\tau}]$, there exists an $\varepsilon_\theta > 0$, such that

$$\mathcal{Q}\left(\mathcal{F}(G, (h_o(\theta, j\omega) - 1)\, I_{m_2}), \sum_{r=1}^{r=r_0} x_r \Pi_r(j\omega), \varepsilon_\theta\right) \leq 0 \quad (17)$$

Proof: The result follows from Theorems 2, 3 and 4. ■

Remark 3: The assumption $\delta_\alpha|_{\alpha=0} = 0$ is not necessary and can be replaced by the alternative assumption that the feedback interconnection of $\mathcal{F}(G, (e^{-\tau s} - 1)\, I_{m_2})$ and δ_0 is stable for all $\tau \in [0, \overline{\tau}]$. This assumption is made here, just to simplify presentation of the results and to avoid distracting details.

IV. Stability Analysis

A. Preliminaries

In this section, we provide LMI criteria that verify Conditions (i) and (v) of Corollary 1. Let $\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right]$ be a minimal realization of $[p_l(s) - 1]I_{m_2}$. Define:

$$A_{11} := A + B_2 D_p C_2 \qquad A_{12} := B_2 C_p$$
$$A_{21} := \alpha_o^{-1} B_p C_2 \qquad A_{22} = \alpha_o^{-1} A_p$$

Then, it can be verified that a minimal realization of $\mathcal{F}(G,(h_o(\theta,s)-1)I_{m_2})$ is given by

$$\mathcal{F}(G,(h_o(\theta,s)-1)I_{m_2})=\left[\begin{array}{c|c} A_L(\theta) & B_L \\ \hline C_L & D_L \end{array}\right] \tag{18}$$

where

$$A_L(\theta):=\begin{bmatrix} A_{11} & A_{12} \\ \theta^{-1}A_{21} & \theta^{-1}A_{22}\end{bmatrix} \quad B_L:=\begin{bmatrix} B_1 \\ 0\end{bmatrix}$$
$$C_L:=\begin{bmatrix} C_1+D_{12}D_pC_2 & D_{12}C_p\end{bmatrix} \quad D_L:=D_{11} \tag{19}$$

Remark 4: It should be clear that $A_L(\theta)$, is the Kernel matrix of the LTI system $\sum_o(\theta)$. Therefore, $\sum_o(\theta)$ is stable for all $\theta\in(0,\overline{\tau}]$, if and only if $A_L(\theta)$ is Hurwitz for all $\theta\in(0,\overline{\tau}]$.

Next, it is shown through the well-known KYP Lemma that the frequency-dependent Linear Matrix Inequality (17) is equivalent to a frequency-independent LMI condition.

B. Canonical Factorization and KYP Lemma

Without loss of generality we may assume that each Π_r has the canonical factorization

$$\Pi_r=\begin{bmatrix}(j\omega I-A_\pi)^{-1}B_\pi \\ I\end{bmatrix}^* M_{\pi_r}\begin{bmatrix}(j\omega I-A_\pi)^{-1}B_\pi \\ I\end{bmatrix}$$

where $B_\pi=\begin{bmatrix} B_{\pi v} & B_{\pi w}\end{bmatrix}$ and A_π is Hurwitz. Then (17) is equivalent to [6]:

$$\begin{bmatrix}(j\omega I-\overline{A})^{-1}\overline{B} \\ I\end{bmatrix}^* \left(\sum_{r=1}^{r=r_0} x_r\begin{bmatrix} Q_r & S_r \\ S_r^T & R_r\end{bmatrix}\right)\begin{bmatrix}(j\omega I-\overline{A})^{-1}\overline{B} \\ I\end{bmatrix}<0,\forall\omega\in\mathbb{R}\cup\{\infty\} \tag{20}$$

where

$$\overline{A}=\begin{bmatrix} A_\pi & B_{\pi v}C_L \\ 0 & A_L(\theta)\end{bmatrix},\ \overline{B}=\begin{bmatrix} B_{\pi v}D_L+B_{\pi w} \\ B_L\end{bmatrix} \tag{21}$$

$$\begin{bmatrix} Q_r & S_r \\ S_r^T & R_r\end{bmatrix}:=$$
$$\left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_L & D_L \\ \hline 0 & 0 & I\end{array}\right]^T M_{\pi_r}\left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_L & D_L \\ \hline 0 & 0 & I\end{array}\right]\triangleq M_r \tag{22}$$

Lemma 3—KYP Lemma: Given $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $M=M^T\in\mathbb{R}^{(n+m)\times(n+m)}$, with $\det(j\omega-A)\neq 0$ for $\omega\in\mathbb{R}$ and (A,B) controllable, the following two statements are equivalent:

(i) $\begin{bmatrix}(j\omega I_n-A)^{-1}B \\ I_m\end{bmatrix}^* M\begin{bmatrix}(j\omega I_n-A)^{-1}B \\ I_m\end{bmatrix}<0,\forall\omega\in\mathbb{R}\cup\{\infty\}$.

(ii) $\exists P=P^T\in\mathbb{R}^{n\times n}$ such that $M+\begin{bmatrix}A^TP+PA & PB \\ B^TP & 0\end{bmatrix}\leq 0$.

Therefore, through the canonical factorization of Π_r and the KYP Lemma, (17) can be reduced to a frequency-independent LMI condition.

C. LMI Analysis

Theorem 5: $\sum_o(\theta)$ is asymptotically stable for all $\theta\in[0,\overline{\tau}]$, if there exist symmetric matrices X_2 and X_3, a positive definite matrix X_1, a negative definite matrix X_4 and matrix Z, all of appropriate sizes, such that

$$\Lambda(0)<0,\qquad \Lambda(b)<0 \tag{23}$$
$$X(b)>0 \tag{24}$$

where

$$X(\theta)=\begin{bmatrix} X_1+\theta X_2 & \theta Z \\ \theta Z^T & \theta X_3+\theta^2X_4\end{bmatrix}$$
$$\Lambda(\theta)=\begin{bmatrix}\Lambda_{11}(\theta) & \Lambda_{12}(\theta) \\ * & \Lambda_{22}(\theta)\end{bmatrix}$$

$\Lambda_{11}(\theta)=\mathrm{He}\left(ZA_{21}+(X_1+\theta X_2)A_{11}\right)$,
$\Lambda_{12}(\theta)=(X_1+\theta X_2)A_{12}+ZA_{22}+\theta A_{11}^TZ+A_{21}^T(X_3+\theta X_4)$,
$\Lambda_{22}(\theta)=\mathrm{He}\left(\theta Z^TA_{12}+(X_3+\theta X_4)A_{22}\right)$.

Proof: Omitted for brevity. ■

Theorem 6: Let δ_α and Π_r, $r=1,..,r_0$ be such that conditions (ii), (iii) and (iv) of Corollary 1 are satisfied. Suppose that it has been established through Theorem 5 that for all $\theta\in[0,\overline{\tau}]$, $\sum_o(\theta)$ is asymptotically stable. Then, the system $\sum_d^\delta(\tau)$ is stable for all $\tau\in[0,\overline{\tau}]$, if there exist symmetric matrices X_1, X_2, X_3, X_4, P_{11} and matrices Z, P_{12}, P_{13} of appropriate size, and positive scalars x_r, $r=1,...,r_0$ such that the following conditions hold:

$$\Upsilon(\overline{\tau})+\sum_{r=1}^{r_0}x_rM_r<0 \tag{25}$$
$$\Upsilon(0)+\sum_{r=1}^{r_0}x_rM_r<0 \tag{26}$$

where each M_r corresponds to Π_r and is defined in (22) and

$$\Upsilon(\theta)=\begin{bmatrix}\Upsilon_{11}(\theta) & \Upsilon_{12}(\theta) & \Upsilon_{13}(\theta) & \Upsilon_{14}(\theta) \\ * & \Upsilon_{22}(\theta) & \Upsilon_{23}(\theta) & \Upsilon_{24}(\theta) \\ * & * & \Upsilon_{33}(\theta) & \Upsilon_{34}(\theta) \\ * & * & * & 0\end{bmatrix},\ \text{where}$$

$\Upsilon_{11}(\theta)=\mathrm{He}\left(P_{11}A_\pi\right)$
$\Upsilon_{12}(\theta)=A_\pi^TP_{12}+P_{11}B_{\pi v}(C_1+D_{12}D_pC_2)+P_{12}A_{11}+P_{13}A_{21}$
$\Upsilon_{13}(\theta)=\theta A_\pi^TP_{13}+P_{12}A_{12}+P_{13}A_{22}+P_{11}B_{\pi v}D_{12}C_p$
$\Upsilon_{14}(\theta)=P_{11}(B_{\pi v}D_{11}+B_{\pi w})+P_{12}B_1$
$\Upsilon_{22}(\theta)=\mathrm{He}(P_{12}^TB_{\pi v}(C_1+D_{12}D_pC_2)+(X_1+\theta X_2)A_{11}+ZA_{21})$
$\Upsilon_{23}(\theta)=\theta(C_1+D_{12}D_pC_2)^TB_{\pi v}^TP_{13}+\theta A_{11}^TZ+A_{21}^T(X_3+\theta X_4)+(X_1+\theta X_2)A_{12}+ZA_{22}+P_{12}^TB_{\pi v}D_{12}C_p$
$\Upsilon_{24}(\theta)=P_{12}^T(B_{\pi v}D_{11}+B_{\pi w})+(X_1+\theta X_2)B_1$
$\Upsilon_{33}(\theta)=\mathrm{He}(\theta Z^TA_{12}+(X_3+\theta X_4)A_{22}+$

$\theta P_{13}^T B_{\pi v} D_{12} C_p)$
$\Upsilon_{34}(\theta) = \theta P_{13}^T (B_{\pi v} D_{11} + B_{\pi w}) + \theta Z^T B_1$

Proof: First, notice that $\Upsilon(\theta)$ is affine in θ and is therefore convex in θ. Inequalities (25) and (26) then imply that $\Upsilon(\theta) + \sum_{r=1}^{r_0} x_r M_r < 0, \forall \theta \in [0, \overline{\tau}]$, where

$$\Upsilon(\theta) = \begin{bmatrix} P(\theta)\overline{A} + \overline{A}^T P(\theta) & P(\theta)\overline{B} \\ \overline{B}^T P(\theta) & 0 \end{bmatrix}$$

$$P(\theta) = \begin{bmatrix} P_{11} & P_{12} & \theta P_{13} \\ P_{12}^T & X_1 + \theta X_2 & \theta Z \\ \theta P_{13}^T & \theta Z^T & \theta X_3 + \theta^2 X_4 \end{bmatrix}$$

$\overline{A}$ and $\overline{B}$ are defined in (21) . It follows from (20) and the KYP Lemma that for each θ, there exists an $\varepsilon_\theta > 0$, such that

$$\mathcal{Q}\left(\mathcal{F}(G, (h_o(\theta, j\omega) - 1) I_{m_2}), \sum_{r=1}^{r_0} x_r \Pi_r(j\omega), \varepsilon_\theta\right) \leq 0$$

The result follows immediately from Corollary 1, since all conditions are satisfied. Proof is complete. ■

Remark 5: Assuming that all suppositions of Theorem 6 are true, and assuming that (26) is satisfied, the maximum $\overline{\tau}$, for which, Theorem 6 affirms finite-gain stability of $\sum_d^\delta$ over $[0, \overline{\tau}]$, can be found by implementing a bisection algorithm over (25) .

V. Numerical Example

Example 1: Consider the system (2) with $A = \begin{bmatrix} -0.25 & -0.875 \\ 4.0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.125 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0.15 & 0 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Consider the case where δ is a norm-bounded time-varying uncertainty, i.e. $\delta = \delta(t) I_q$ with $\|\delta\|_\infty \leq 1$. Then, δ satisfies IQC s defined by $\Pi(j\omega) = \begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$, where $X = X^T \geq 0$, and $Y = -Y^T$. The maximum delay margin provided by our LMI criterion (Theorem 6) is $\overline{\tau} = 1.643$ with Pade order 4, and 1.685 with Pade order 5. As comparisons, the maximum delay margin for the same system was found to be $\overline{\tau} = 0.200$ when the software IQC β [14] was used and $\overline{\tau} = 0.198$ when the result of [7] was used.

VI. Conclusion

A new method for delay-dependent stability analysis of nonlinear time-delay systems was presented. In contrast to a pure IQC analysis, in the mixed IQC approach only the nonlinearity is tackled via IQCs and the value set of the delay element is covered by the outer parameter-dependent filter. LMI conditions were developed for stability analysis. A numerical example demonstrates that this criterion may be much less conservative than a pure IQC analysis. The method can be applied to nonlinear systems with multiple delays in a very straightforward fashion.

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