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Fixed-Lag Smoothing as a Constrained Version of the Fixed-Interval Case

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***Abstract*— This paper studies the H^∞ fixed-lag smoothing problem in both continuous and discrete time. The central idea is to address it as a constrained version of the *fixed-interval* smoothing (L^∞ estimation) problem. This enables us to separate geometric (which are independent of the smoothing lag) and analytic constraints imposed by the problem data on the achievable performance. As a byproduct the technique provides an elegant means to find a minimal (finite) smoothing lag at which the optimal L^∞ performance level is achieved. State-space formulae are also derived in the continuous-time case.**

I. Introduction

This paper addresses the H^∞ fixed-lag smoothing problem in both continuous and discrete time. The fixed-lag smoothing formulation of a general estimation problem reflects the situation where some delay or latency between the measurement and the generation of estimation can be tolerated (e.g., in numerous signal processing applications) [1].

Most of the existing solutions to the H^∞ smoothing (or the dual preview tracking) problem resort to strictly sufficient conditions, system restrictions, iterative approximations and dimension increase. For example, the solution of the continuous-time H^∞ preview tracking, in [2], is derived in terms of the standard H^∞ algebraic Riccati equation (ARE), that is associated with the tracking problem without preview. That equation, however, might not admit a stabilizing solution under some performance levels γ, for which the preview problem *is* solvable. In other words, that solvability condition is only sufficient[1]. Similar situation takes place in the discrete-time case, where available low-dimensional solution procedures [4]–[6] might fail even when the problem is solvable.

The first complete solution to the H^∞ fixed-lag smoothing problem was obtained in [7] using the J-spectral factorization approach. In [8], [9], a game-theoretic approach was used to obtain a different form of the solvability conditions (also necessary and sufficient) and a (sub)optimal smoother in the continuous and discrete time, respectively.

In this paper we propose an alternative solution procedure. Following the idea of [10], [11], we shall treat the fixed-lag smoothing problem as a *constrained* version of a simpler estimation problem. In our case, that is the infinite-horizon *fixed-interval* smoothing problem, where the estimator has access to the entire future information about the measured signal, with no causality constraints. In other words, we view the fixed-lag problem as a constrained L^∞ estimation problem, and fixed-lag solutions are *extracted* from the complete parametrization of all fixed-interval (L^∞) solutions. We show that the extraction procedure reduces the original problem to an equivalent Nehari-type problem, which, in turn, is solved using the approach proposed in [12].

An attractive property of the proposed solution is the separation of geometric and analytic constraints imposed by the problem data on the achievable performance. The former constraints are independent of the smoothing lag h, so that they remain active even as $h \to \infty$. The latter constraints do depend on the smoothing lag h and vanish as $h \to \infty$. We show that generically there exists a finite smoothing lag for which the analytic constraints become inactive, so that no further increase of h could improve the achievable H^∞ performance. This phenomenon, known as the saturation phenomenon, was first quantified in the continuous-time case in [13] using a different (state-space) approach. Yet in the discrete-time case the question of whether the H^∞ performance saturates for a finite smoothing lag was still open.

In the continuous-time case we also provide the state-space solution. It is based on two ARE's of H^2 and H^∞ types, whose dimension is that of the (delay-free) problem. Both these equations are independent of the smoothing lag, so the proposed solution is better suited for the search of the minimal smoothing lag for a given performance level than those in [7], [8]. The smoothing lag appears only in an algebraic coupling condition, imposed on the resulting Riccati solutions. Also, the structure of the resulting estimator has some advantage over that in [7] as it only involves the matrix exponentials of stable (Hurwitz) matrices.

Notations

The nomenclature adopted here is driven by the need for a unified treatment of continuous- and discrete-time systems. We use λ for the Laplace transform variable in both the continuous (where $\lambda = s$) and the discrete (where $\lambda = z$) cases. The notation $\mathbb{B}$ stands for the stability boundary in the complex λ-plane: $\mathbb{B} = j(\mathbb{R} \cup \infty)$ (extended imaginary axis) in the continuous-time case and $\mathbb{B} = \mathbb{D}$ (unit circle) in the

[1]This problem was overcome in the recent paper [3] by the same authors.

discrete-time case. Furthermore, $\mathbb{U}$ denotes the "strictly" (i.e., excluding $\mathbb{B}$) unstable region in the λ-plane. Given a transfer function $G(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, its conjugate is denoted as $G^{\sim}(\lambda)$ and is defined as $G^{\sim}(s) = G'(-s)$ and $G^{\sim}(z) = G'(z^{-1})$, where $G'(\lambda) = \left[\begin{array}{c|c} A' & C' \\ \hline B' & D' \end{array}\right]$. By $D_h(\lambda)$ we denote the (inner) transfer function of the h time units delay operator: $D_h(s) = e^{-sh}$ and $D_h(z) = z^{-h}$.

II. Problem formulation and preliminaries

Let G_1 and G_2 be finite-dimensional proper transfer matrices given by their state-space realization

$$\begin{bmatrix} G_1(\lambda) \\ G_2(\lambda) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right]. \quad (1)$$

We assume that this realization satisfies the following standard assumptions:

$\boldsymbol{\mathcal{A}_1}$: (C_2, A) is detectable;
$\boldsymbol{\mathcal{A}_2}$: $\begin{bmatrix} A - \lambda I & B \\ C_2 & D_2 \end{bmatrix}$ has full row rank $\forall \lambda \in \mathbb{B}$.

Note that in the continuous-time case $\boldsymbol{\mathcal{A}_2}$ guarantees that $D_2 D_2' > 0$ (as it should hold at $\lambda = j\infty$). A general H^∞ fixed-lag smoothing problem is formulated as the problem of finding an estimator $K(\lambda) \in H^\infty$ such that

$$\|D_h(\lambda)G_1(\lambda) - K(\lambda)G_2(\lambda)\|_{H^\infty} < \gamma \quad (2)$$

for a given smoothing lag $h \geq 0$ and a constant $\gamma > 0$.

Note that no restriction on the stability of G_1 and G_2 is imposed. The treatment of the smoothing problem, however, is somewhat more clean when the realization in (1) is stable. We thus transform the original smoothing problem to an equivalent one with stable data. We use this opportunity also to ensure that the "G_2" part of the new problem is co-inner.

Lemma 1: Let $\boldsymbol{\mathcal{A}_{1,2}}$ hold. Then a left coprime factorization of (1) can be chosen in the form

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I & M_1 \\ 0 & M_2 \end{bmatrix}^{-1} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

with $M_1, M_2, N_1, N_2 \in RH^\infty$ and such that

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} N_2^{\sim} = \begin{bmatrix} V^{\sim} \\ I \end{bmatrix}$$

for some *strictly proper* $V \in H^\infty$. Moreover, $\tilde{K} \in H^\infty$ solves the fixed-lag smoothing problem

$$\|D_h(\lambda)N_1(\lambda) - \tilde{K}(\lambda)N_2(\lambda)\|_{H^\infty} < \gamma \quad (3)$$

iff $K = \tilde{K}M_2 - D_h M_1$ solves the original problem (2).

Proof: This result is a straightforward combination of [7, Lemma 5] (see also [14]) and factorization arguments, such as those in [15] (where §13.7 and §21.5.2 provide the continuous- and discrete-time results, respectively). ■

Henceforth we consider the equivalent H^∞ smoothing problem (3).

III. A frequency-domain solution

The purpose of this section is to solve the smoothing problem (3) using frequency-domain arguments. To this end we treat (3) as a constrained case of the infinite-horizon version of the fixed-interval smoothing problem.

The fixed-interval smoothing problem is the problem where the estimator $\tilde{K}$ has access to all future measurements of the output of N_2. Problem (3) can the be interpreted as the constrained problem in which $\tilde{K}$ has access to the output of N_2 only h time units ahead. It is thus clear that the solvability of the fixed-interval problem is necessary for the solvability of the fixed-lag problem, for any h. Moreover, solutions to the latter problem constitute a subset of the solutions to the former.

To be more precise, note that since $D_h(\lambda)$ is inner in both continuous and discrete time,

$$\|D_h N_1 - \tilde{K}N_2\|_{H^\infty} = \|N_1 - D_{-h}\tilde{K}N_2\|_{L^\infty}.$$

Thus, (3) is solvable iff there exists a $\tilde{K}_\alpha$ guaranteeing that $\|N_1 - \tilde{K}_\alpha N_2\|_{L^\infty} < \gamma$ and such that $D_h\tilde{K}_\alpha \in H^\infty$ (where the fact that $N_1, N_2 \in H^\infty$ is used). In other words,

$$\tilde{K}_\alpha \in D_{-h}H^\infty \subset L^\infty.$$

We shall therefore start with solving the fixed-interval smoothing problem and then impose an additional (causality) constraint on its solution to solve the original fixed-lag problem.

A. L^∞ solutions

The fixed-interval H^∞ smoothing problem was solved in [16] by state-space methods and in [17] by I/O methods (in the time domain). In this section we mainly follow the arguments in the latter reference with some minor modifications.

Consider the problem of finding $\tilde{K}_\alpha \in L^\infty$ such that

$$\|N_1 - \tilde{K}_\alpha N_2\|_{L^\infty} < \gamma. \quad (4)$$

This is essentially an algebraic problem over the range of $\lambda \in \mathbb{B}$, which is equivalently rewritten as

$$\big(N_1 - \tilde{K}_\alpha N_2\big)(N_1 - \tilde{K}_\alpha N_2)^{\sim} < \gamma^2 I, \quad \forall \lambda \in \mathbb{B}. \quad (5)$$

A standard completion of squares argument is used to characterize solutions of (5). Since $N_1 N_2^{\sim} = V^{\sim}$ and $N_2 N_2^{\sim} = I$ (Lemma 1), inequality (5) can be rewritten as

$$(V^{\sim} - \tilde{K}_\alpha)(V - \tilde{K}_\alpha^{\sim}) < \gamma^2 I - N_1(I - N_2^{\sim}N_2)N_1^{\sim}. \quad (6)$$

Since $(V^{\sim} - \tilde{K}_\alpha)(V - \tilde{K}_\alpha^{\sim}) \geq 0$, Eq. (4) is solvable only if

$$\gamma^2 I > N_1(I - N_2^{\sim}N_2)N_1^{\sim}, \quad \forall \lambda \in \mathbb{B}, \quad (7)$$

or, equivalently,

$$\gamma > \gamma_\infty \doteq \|N_1(I - N_2^{\sim}N_2)\|_{L^\infty}. \quad (8)$$

This condition is also sufficient as then $\tilde{K}_\alpha = V^{\sim}$ ($\in H_\perp^\infty$) obviously solves the problem.

Now, let $W_\gamma(\lambda)$ be any square transfer matrix such that its frequency response satisfies

$$W_\gamma W_\gamma^{\sim} = I - \tfrac{1}{\gamma^2}N_1(I - N_2^{\sim}N_2)N_1^{\sim}, \quad \forall \lambda \in \mathbb{B}. \quad (9)$$

By (7), such W_γ exists and has no zeros in $\mathbb{B}$ (equivalently, W_γ is invertible in L^∞). Having W_γ, (6) is equivalent to

$$W_\gamma^{-1}(V^\sim - \tilde{K}_\alpha)(V - \tilde{K}_\alpha^\sim)(W_\gamma^{-1})^\sim < \gamma^2 I, \quad \forall \lambda \in \mathbb{B}.$$

This, in turn, implies that (4) is equivalent to the following (one-block) L^∞ model-matching problem:

$$\big\| W_\gamma^{-1}(\lambda)\big(V(\lambda)^\sim - \tilde{K}_\alpha(\lambda)\big)\big\|_{L^\infty} < \gamma. \tag{10}$$

It is readily seen that all solutions to (10) are parametrized in the form $\tilde{K}_\alpha = V^\sim - W_\gamma Q$, where Q is any transfer function satisfying $\|Q\|_{L^\infty} < \gamma$. We, however, do not need this parametrization. Rather, in the analysis below we shall appeal directly to (10).

B. The one-block problem

As shown above, $\tilde{K}_\alpha$ solves (4) iff it solves (10). This obviously remains true when any causality constraint is imposed upon $\tilde{K}_\alpha$, which leads us to the conclusion that $\tilde{K} \in H^\infty$ is a solution of the original problem (3) iff it satisfies

$$\|W_\gamma^{-1}(D_h V^\sim - \tilde{K})\|_{L^\infty} < \gamma. \tag{11}$$

Now, note that since the parahermitian transfer function on the right-hand side of (9) has neither poles nor zeros in $\mathbb{B}$, the factor W_γ can always be chosen [15] such that $W_\gamma, W_\gamma^{-1} \in RH^\infty$ (spectral factorization).

Problem (11) would have been of a standard one-block Nehari-type had it not contained the delay D_h. Yet the presence of D_h can be circumvented using a trick from [12]. The key idea there is to decompose

$$D_h V^\sim = V_h^\sim + \Pi \tag{12}$$

for some strictly proper $V_h \in RH^\infty$ and $\Pi \in H^\infty$ such that $\Pi(\lambda)$ is an entire function of λ. The infinite-dimensional part of this decomposition, Π, could then be absorbed into $\tilde{K}$, so that (11) can be equivalently rewritten as the following delay-free (i.e., finite-dimensional) problem:

$$\|W_\gamma^{-1}(V_h^\sim - \tilde{K}_\pi)\|_{L^\infty} < \gamma, \tag{13}$$

where $\tilde{K}_\pi \doteq \tilde{K} - \Pi$ belongs to H^∞ iff so does $\tilde{K}$. Problem (13) can then be solved using standard methods. In particular, it is solvable iff

$$\|V_h(W_\gamma^{-1})^\sim\|_H < \gamma, \tag{14}$$

where the notation $\|\cdot\|_H$ stands for the Henkel norm.

The required transfer matrices V_h and Π will be constructed in Section IV in the continuous–time case using state-space machinery (the state-space solution of (13) will also be presented there). Here we describe only the basic rationale behind these choices, which will later help clarifying the effect of h on (14). Rewrite (12) as $V - D_h V_h = D_h \Pi^\sim$ and choose a rational V_h so that the impulse response of $D_h V_h$ cancels that of V in the interval $[h, \infty)$, which is clearly possible. Such a choice leads to an FIR (finite impulse response) $D_h \Pi^\sim$, supported over $[0, h]$ and, in turn, to an FIR Π.

In closing, it is observed that standard frequency-domain arguments, as in [15, §17.6], lead to the reduction to the one-block problem (11) without an appeal to the fixed-interval case. The proposed derivation has a conceptual advantages as it clearly separates geometric constraints, reflected by (7), which is independent of h, and analytic constraints, reflected by (14), where the effect of h comes to bear. To reveal this dependence, note that as the impulse response of V decays with the time, V_h can be made arbitrarily small (in any metric) by increasing h. This means, that for any γ satisfying (8) there exists a (sufficiently large) h_γ such that (14) holds $\forall h > h_\gamma$. In other words, the *analytic constraints vanish as* $h \to \infty$.

The last observation has an interesting spin-off. Unless there are no geometric constraints (i.e., $\gamma_\infty = 0$), there must exist a finite smoothing lag for which any $\gamma > \gamma_\infty$ is attainable. The question is whether this still happens as $\gamma \to \gamma_\infty$, i.e., whether there exists a *finite* smoothing lag for which the finite-interval performance is achievable (saturation of the performance). The problem here is that $W_{\gamma_\infty}(\lambda)$ is no longer invertible in L^∞, so that the situation in the limit requires careful treatment.

C. Including the optimal case

To address the situation when $\gamma \to \gamma_\infty$ we shall modify problem (2) as follows:

$$\|D_h(\lambda)G_1(\lambda) - K(\lambda)G_2(\lambda)\|_{H^\infty} \leq \gamma. \tag{$2_{\rm opt}$}$$

In that case, all arguments of Lemma 1 and §III-A remain unchanged modulo the replacement of "$<$" and "$>$" with "$\leq$" and "$\geq$", respectively, up to the factorization in (9). Indeed, if $\gamma = \gamma_\infty$, the right hand side of (9) is no longer invertible in $\lambda \in \mathbb{B}$. There are two different situations possible: *(a)* the right hand side of (9) still has full *normal* rank [15, §3.11] and *(b)* it is singular $\forall \lambda \in \mathbb{B}$, i.e., $\bar{\sigma}(\Phi(\lambda))$ is all-pass, where

$$\Phi \doteq N_1(I - N_2^\sim N_2)N_1^\sim.$$

We argue that the latter situation is rather non-generic. Indeed, it is known [17, §10.3.2] that Φ is the spectral density function of the L^2-optimal (non-causal H^2-optimal) solution. One would therefore expect that generically $\bar{\sigma}(\Phi)$ is not all-pass. For that reason we assume throughout this section that

$\boldsymbol{\mathcal{A}_3}$: $\gamma_\infty^2 I - \Phi(\lambda)$ is by-proper and has full normal rank.

This assumption also rules out the case when $\gamma_\infty = 0$.

When $\boldsymbol{\mathcal{A}_3}$ holds, the rational transfer matrix W_γ^{-1} exists for all $\gamma \geq \gamma_\infty$, though now it does not necessarily belong to L^∞ (it has at least one pole in $\mathbb{B}$ when $\gamma = \gamma_\infty$). We thus still can reduce (10) to a one-block problem of the form

$$\|W_\gamma^{-1}(D_h V^\sim - \tilde{K}_\pi)\|_{L^\infty} \leq \gamma. \tag{$11_{\rm opt}$}$$

The main difference from the suboptimal case in (11) is that now there might be a need to impose interpolation constraints on $\tilde{K}$ to guarantee that $W_\gamma^{-1}(D_h V^\sim - \tilde{K}) \in L^\infty$.

The following result is geared to that end:

Lemma 2: Let $W_\gamma \in RH^\infty$ be so that W_γ^{-1} has no poles in $\mathbb{U}$ and $V_h \in RH^\infty$. Then

1) there exist strictly proper $V_+, V_- \in RH^\infty$ so that

$$W_\gamma^{-1} V^\sim = V_+^\sim + W_\gamma^{-1} V_-;$$

2) $W_\gamma^{-1}(D_h V^\sim - \tilde{K}_\pi) \in L^\infty$ for a $\tilde{K}_\pi \in H^\infty$ iff

$$\tilde{K}_\pi = W_\gamma \tilde{K}_\nu + D_h V_- \tag{15}$$

for some $\tilde{K}_\nu \in RH^\infty$.

Proof: Since the spectra of W_γ^{-1} and $V_h^\sim$ are disjoint, the strictly proper $W_\gamma^{-1} V^\sim$ can always be decomposed as

$$W_\gamma^{-1} V^\sim = U_\mathrm{u} + U_\mathrm{s}$$

for some strictly proper $U_\mathrm{u} \in RH_\perp^\infty$ and U_s with no poles in $\mathbb{U}$. For the same reason, we can always decompose

$$W_\gamma U_\mathrm{u} = \hat{U}_\mathrm{u} + \hat{U}_\mathrm{s}$$

for some strictly proper $\hat{U}_\mathrm{u} \in RH_\perp^\infty$ and $\hat{U}_\mathrm{s} \in RH^\infty$. Substituting U_u from the first decomposition above to the left-hand side of the second decomposition, we get that $\hat{U}_\mathrm{u} = V^\sim$. Hence, the first statement of the Lemma is true for $V_+ = U_\mathrm{u}^\sim$ and $V_- = \hat{U}_\mathrm{s}$.

Now, if $\tilde{K}_\pi$ is chosen according to (15), then

$$W_\gamma^{-1}(D_h V^\sim - \tilde{K}_\pi) = D_h V_+^\sim - \tilde{K}_\nu \in L^\infty.$$

On the other hand, if some $\tilde{K}_\pi$ belonging to H^∞ guarantees that $W_\gamma^{-1}(D_h V^\sim - \tilde{K}_\pi) \in L^\infty$, then

$$\tilde{K}_\nu = D_h V_+ - W_\gamma^{-1}(D_h V^\sim - \tilde{K}_\pi) \in L^\infty.$$

At the same time, $\tilde{K}_\nu = W_\gamma^{-1}(\tilde{K}_\pi - D_h V_-)$ has no poles in $\mathbb{U}$. Therefore, $\tilde{K}_\nu \in H^\infty$, which completes the proof. ∎

Lemma 2 enables us to reduce (11) to the following Nehari problem:

$$\|D_h V_+^\sim - \tilde{K}_\nu\|_{L^\infty} \le \gamma,$$

which is solvable iff $\|D_{-h} V_+\|_H \le \gamma$. Since $V_+ \in RH^\infty$,

$$\lim_{h\to\infty} \|D_{-h} V_+\|_H = 0.$$

This proves the conjecture at the end of §III-B, that (under the generic $\boldsymbol{\mathcal{A}_3}$) smoothing performance saturates at a finite smoothing lag, even as $\gamma \to \gamma_\infty$.

Remark 3.1: The saturation property was first proved for the continuous-time case, in [13], using state-space techniques. In the discrete-time case, however, the existence of this property was an open problem. The solution method for the discrete H^∞ fixed-lag smoothing problem, in [9], is not readily suitable to address this issue.

IV. State-space solution in continuous time

In this section we present the state-space formulae for all steps described in the previous section. Because of space limitations, only continuous-time results are discussed. Discrete-time counterparts, the derivation of which is considerably more involved though follows exactly the same steps, will be presented in the full version of this paper.

A. Formulae of Lemma 1

The state-space realizations of the factors in Lemma 1 is derived by a standard state-space technique [15], as follows. Consider the standard continuous-time ARE

$$AY_2 + Y_2A' + BB' \\ - (Y_2C_2' + BD_2')R_2^{-1}(C_2Y_2 + D_2B') = 0, \tag{16}$$

where $R_2 \doteq D_2D_2' > 0$ by $\boldsymbol{\mathcal{A}_2}$. By $\boldsymbol{\mathcal{A}_{1,2}}$ there exists a stabilizing solution $Y_2 = Y_2' \ge 0$; i.e., such that the matrix $\tilde{A} = A + L_2C_2$ is Hurwitz, where $L_2 \doteq -(Y_2C_2' + BD_2')R_2^{-1}$. Defining $\tilde{B} \doteq B + L_2D_2$, $\begin{bmatrix} \tilde{C}_2 & \tilde{D}_2 \end{bmatrix} \doteq R_2^{-1/2}\begin{bmatrix} C_2 & D_2 \end{bmatrix}$, and

$$\begin{bmatrix} \tilde{C}_1 & \tilde{D}_1 \end{bmatrix} \doteq \begin{bmatrix} C_1 & D_1 \end{bmatrix} - D_1D_2'R_2^{-1}\begin{bmatrix} C_2 & D_2 \end{bmatrix}.$$

It is then the matter of a straightforward algebra to show that the transfer matrices satisfying the conditions of Lemma 1 have the following state-space realizations:

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C}_1 & \tilde{D}_1 \\ \tilde{C}_2 & \tilde{D}_2 \end{array}\right], \quad \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \left[\begin{array}{c|c} \tilde{A} & L_2 \\ \hline \tilde{C}_1 & -D_1D_2'R_2^{-1} \\ \tilde{C}_2 & R_2^{-1/2} \end{array}\right].$$

In particular, denoting $N \doteq \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$,

$$NN^\sim = \begin{bmatrix} N_1N_1^\sim & V^\sim \\ V & I \end{bmatrix} = \left[\begin{array}{cc|cc} \tilde{A} & -\tilde{B}\tilde{B}' & \tilde{B}\tilde{D}_1' & \tilde{B}\tilde{D}_2' \\ 0 & -\tilde{A}' & \tilde{C}_1' & \tilde{C}_2' \\ \hline \tilde{C}_1 & -\tilde{D}_1\tilde{B}' & \tilde{D}_1\tilde{D}_1' & 0 \\ \tilde{C}_2 & -\tilde{D}_2\tilde{B}' & 0 & I \end{array}\right] \\ = \left[\begin{array}{cc|cc} \tilde{A} & 0 & Y_2\tilde{C}_1' + \tilde{B}\tilde{D}_1' & 0 \\ 0 & -\tilde{A}' & \tilde{C}_1' & \tilde{C}_2' \\ \hline \tilde{C}_1 & -\tilde{C}_1Y_2 - \tilde{D}_1\tilde{B}' & \tilde{D}_1\tilde{D}_1' & 0 \\ \tilde{C}_2 & 0 & 0 & I \end{array}\right],$$

where the latter equality is obtained by the similarity transformation with $\left[\begin{smallmatrix} I & Y_2 \\ 0 & I \end{smallmatrix}\right]$, where Y_2 is the solution of (16) (note that it satisfies $\tilde{A}Y_2 + Y_2\tilde{A}' + \tilde{B}\tilde{B}' = 0$).

B. Spectral factorization in (9)

Arguably, the simplest way to derive the state-space realization of the system in the right-hand side of (9) is by applying the $\mathcal{S}$-transformation introduced in [18]. We have:

$$\mathcal{S}_\ell(NN^\sim) = \begin{bmatrix} N_1(I - N_2^\sim N_2)N_1^\sim & N_1N_2^\sim \\ -N_2N_1^\sim & I \end{bmatrix} \\ = \left[\begin{array}{cc|cc} \tilde{A} & 0 & Y_2\tilde{C}_1' + \tilde{B}\tilde{D}_1' & 0 \\ -\tilde{C}_2'\tilde{C}_2 & -\tilde{A}' & \tilde{C}_1' & \tilde{C}_2' \\ \hline \tilde{C}_1 & -\tilde{C}_1Y_2 - \tilde{D}_1\tilde{B}' & \tilde{D}_1\tilde{D}_1' & 0 \\ -\tilde{C}_2 & 0 & 0 & I \end{array}\right].$$

Denote $\tilde{C}_\gamma \doteq \frac{1}{\gamma}\tilde{C}_1$, $\tilde{D}_\gamma \doteq \frac{1}{\gamma}\tilde{D}_1$, and $\tilde{L}_\gamma \doteq -(Y_2\tilde{C}_\gamma' + \tilde{B}\tilde{D}_\gamma')$ (in the sequel, we write $\tilde{L}_1$ to denote $\tilde{L}_\gamma|_{\gamma=1}$) and then

$$\Psi_\gamma \doteq \begin{bmatrix} I - \frac{1}{\gamma^2}N_1(I - N_2^\sim N_2)N_1^\sim & \frac{1}{\gamma}N_1N_2^\sim \\ \frac{1}{\gamma}N_2N_1^\sim & I \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} \tilde{A} & 0 & \tilde{L}_\gamma & 0 \\ -\tilde{C}_2'\tilde{C}_2 & -\tilde{A}' & -\tilde{C}_\gamma' & \tilde{C}_2' \\ \hline \tilde{C}_\gamma & \tilde{L}_\gamma' & I-\tilde{D}_\gamma\tilde{D}_\gamma' & 0 \\ -\tilde{C}_2 & 0 & 0 & I \end{array}\right].$$

Let W_o be the observability Gramian of $(\tilde{A},\tilde{C}_2)$, i.e.,

$$W_o\tilde{A}+\tilde{A}'W_o+\tilde{C}_2'\tilde{C}_2=0.$$

Then

$$\Psi_\gamma = \left[\begin{array}{cc|cc} \tilde{A} & 0 & \tilde{L}_\gamma & 0 \\ 0 & -\tilde{A}' & -\tilde{C}_\gamma'-W_o\tilde{L}_\gamma & \tilde{C}_2' \\ \hline \tilde{C}_\gamma+\tilde{L}_\gamma'W_o & \tilde{L}_\gamma' & I-\tilde{D}_\gamma\tilde{D}_\gamma' & 0 \\ -\tilde{C}_2 & 0 & 0 & I \end{array}\right]. \tag{17}$$

We are now in the position to analyze the spectral factorization problem associated with (9). To this end, introduce now another Riccati equation:

$$\tilde{A}\tilde{Y}_\gamma+\tilde{Y}_\gamma\tilde{A}'+\big(\tilde{L}_\gamma-\tilde{Y}_\gamma(\tilde{C}_\gamma'+W_o\tilde{L}_\gamma)\big) \times \tilde{R}_\gamma^{-1}\big(\tilde{L}_\gamma'-(\tilde{C}_\gamma+\tilde{L}_\gamma'W_o)\tilde{Y}_\gamma\big)=0, \tag{18}$$

where $\tilde{R}_\gamma \doteq I-\tilde{D}_\gamma\tilde{D}_\gamma'$. A solution $\tilde{Y}_\gamma$ is said to be stabilizing if the matrix $\tilde{A}_\gamma \doteq \tilde{A}-\big(\tilde{L}_\gamma-\tilde{Y}_\gamma(\tilde{C}_\gamma'+W_o\tilde{L}_\gamma)\big)\tilde{R}_\gamma^{-1}(\tilde{C}_\gamma+\tilde{L}_\gamma'W_o)$ is Hurwitz. We have:

Proposition 1: Condition (7) holds iff $\tilde{R}_\gamma>0$ and there exists a stabilizing solution $\tilde{Y}_\gamma$ to the ARE (18). If this condition holds, then

$$W_\gamma = \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_\gamma\tilde{R}_\gamma^{-1} \\ \hline \tilde{C}_\gamma+\tilde{L}_\gamma'W_o & I \end{array}\right]\tilde{R}_\gamma^{1/2},$$

where $\tilde{B}_\gamma \doteq \tilde{L}_\gamma-\tilde{Y}_\gamma(\tilde{C}_\gamma'+W_o\tilde{L}_\gamma)$, is a bistable transfer matrix satisfying (9).

Proof: Let $\Psi_{\gamma 11}$ denote the $(1,1)$ subblock of Ψ_γ from (17). Following the standard bounded-real arguments [15] one can show that (7) holds iff $\tilde{R}_\gamma>0$ and the "A" matrix of $\Psi_{\gamma 11}^{-1}$, which is

$$\tilde{H}_\gamma \doteq \begin{bmatrix} \tilde{A} & 0 \\ 0 & -\tilde{A}' \end{bmatrix} + \begin{bmatrix} -\tilde{L}_\gamma \\ \tilde{C}_\gamma'+W_o\tilde{L}_\gamma \end{bmatrix}\tilde{R}_\gamma^{-1}\begin{bmatrix} \tilde{C}_\gamma+\tilde{L}_\gamma'W_o & \tilde{L}_\gamma' \end{bmatrix},$$

has no eigenvalues on the imaginary axis. On the other hand, it is readily seen that $\tilde{H}_\gamma'$ is the Hamiltonian matrix associated with (18). Since the pair $\big(\tilde{R}_\gamma^{-1/2}(\tilde{C}_\gamma+\tilde{L}_\gamma'W_o), \tilde{A}-\tilde{L}_\gamma\tilde{R}_\gamma^{-1}(\tilde{C}_\gamma+\tilde{L}_\gamma'W_o)\big)$ is detectable ($\tilde{A}$ is Hurwitz), Theorem 13.6 in [15] yields that the absence of the imaginary axis eigenvalues of H_γ is equivalent to the existence of a stabilizing solution of (18). This proves the first statement of the Proposition.

To derive the formula for the spectral factor W_γ, apply the similarity transformation $\left[\begin{smallmatrix} I & \tilde{Y}_\gamma \\ 0 & I \end{smallmatrix}\right]$ to the realization (17). Then,

$$\Psi_{\gamma 11} = \left[\begin{array}{cc|c} \tilde{A} & \tilde{B}_\gamma\tilde{R}_\gamma^{-1}\tilde{B}_\gamma' & \tilde{B}_\gamma \\ 0 & -\tilde{A}' & -\tilde{C}_\gamma'-W_o\tilde{L}_\gamma \\ \hline \tilde{C}_\gamma+\tilde{L}_\gamma'W_o & \tilde{B}_\gamma' & \tilde{R}_\gamma \end{array}\right],$$

which is $W_\gamma W_\gamma^\sim$. This $W_\gamma \in RH^\infty$ by construction and it can be easily verified that the "A" matrix of W_γ^{-1} is actually $\tilde{A}_\gamma$, so that $W_\gamma^{-1}\in RH^\infty$ as well. ■

C. Decomposition of $D_hV^\sim$

Given a transfer matrix $G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & 0\end{array}\right]$, define the transfer matrix

$$G_h(s) \doteq \left[\begin{array}{c|c} A & B \\ \hline Ce^{Ah} & 0\end{array}\right].$$

Then, the h-truncation of G [11], $\boldsymbol{\tau}_h\{G\}$, is defined as

$$\boldsymbol{\tau}_h\{G\}=G-e^{-sh}G_h=C(I-e^{-(sI-A)h})(sI-A)^{-1}B.$$

It is readily seen that $\boldsymbol{\tau}_h\{G\}$ is an entire function of s and its impulse response is zero in the whole interval $[h,\infty)$ as $e^{-sh}G_h$ cancels out the impulse response of G. Thus, $\boldsymbol{\tau}_h\{G\}$ is an FIR system, exactly how discussed in §III-B.

Therefore, we choose $D_h\Pi^\sim=\boldsymbol{\tau}_h\{V\}$ for the decomposition in (12). This results in

$$V_h(s)=\left[\begin{array}{c|c} \tilde{A} & -e^{\tilde{A}h}\tilde{L}_1 \\ \hline \tilde{C}_2 & 0\end{array}\right]$$

and

$$\Pi(s)=\left[\begin{array}{c|c} -\tilde{A} & e^{\tilde{A}'h}\tilde{C}_2' \\ \hline \tilde{L}_1' & 0\end{array}\right]-e^{-sh}\left[\begin{array}{c|c} -\tilde{A} & \tilde{C}_2' \\ \hline \tilde{L}_1' & 0\end{array}\right] \tag{19}$$

with $\xi_o=\Pi\xi_i$ implies that

$$\xi_0(t)=\tilde{L}_1'\int_{t-h}^{t} e^{\tilde{A}'(h+\tau-t)}\tilde{C}_2'\,\xi_i(\tau)d\tau.$$

It is worth emphasizing that the implementation of the relation above requires the computation of matrix exponentials of Hurwitz matrices only.

D. Solution to the one-block problem

We are now in the position to address (13). In its solution we will follow the arguments from the proof of Lemma 2 (though we address the suboptimal problem only). To this end, note that using the fact that $\tilde{A}_\gamma\tilde{Y}_\gamma+\tilde{Y}_\gamma\tilde{A}'=-\tilde{B}_\gamma\tilde{R}_\gamma^{-1}\tilde{L}_\gamma'$, the following decomposition can be performed:

$$W_\gamma^{-1}V_h^\sim=V_{h+}^\sim+W_\gamma^{-1}V_{h-},$$

where

$$V_{h+} \doteq \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_1 \\ \hline -\tilde{C}_2e^{\tilde{A}h} & 0\end{array}\right]\tilde{R}_\gamma^{-1/2},$$

$$V_{h-} \doteq \left[\begin{array}{c|c} \tilde{A} & \tilde{Y}_\gamma e^{\tilde{A}'h}\tilde{C}_2' \\ \hline \tilde{C}_1+\tilde{L}_1'W_o & 0\end{array}\right].$$

Problem (13) is reduced then to the standard Nehari problem

$$\|V_{h+}^\sim-\tilde{K}_\nu\|_{L^\infty}<\gamma, \tag{20}$$

where $\tilde{K}_\nu \doteq W_\gamma^{-1}(\tilde{K}_\pi-V_{h-})\in H^\infty$ iff $\tilde{K}_\pi\in H^\infty$. The solution to this problem is given by the following Proposition:

Proposition 2: Nehari problem (20) is solvable iff

$$\rho\big(\tilde{Y}_\gamma e^{\tilde{A}'h}W_oe^{\tilde{A}h}\big)<1. \tag{21}$$

Furthermore, if (21) holds, then

$$\tilde{K}_\nu=\tilde{R}_\gamma^{-1/2}\left[\begin{array}{c|c} \tilde{A}-Z_h\tilde{C}_2'\tilde{C}_2e^{\tilde{A}h} & Z_h\tilde{C}_2' \\ \hline \tilde{B}_1'e^{\tilde{A}'h}W_oe^{\tilde{A}h} & 0\end{array}\right],$$

where $Z_h \doteq \left(I - \tilde{Y}_\gamma e^{\tilde{A}'h} W_o e^{\tilde{A}h}\right)^{-1} \tilde{Y}_\gamma e^{\tilde{A}'h}$, solves (20).

Proof: Note that $e^{\tilde{A}'h} W_o e^{\tilde{A}h}$ and $\tilde{Y}_\gamma$ are the controllability and observability Gramians of $\frac{1}{\gamma} V_{h+}$, respectively. The proof then follows by a direct application of []. ■

To solve (3) it is only left to substitute the realizations of W_γ, V_{h-}, and $\tilde{K}_\nu$ to

$$\tilde{K} = W_\gamma \tilde{K}_\nu + V_{h-} + \Pi.$$

After some straightforward algebra the following result can be formulated:

Theorem 1: Let $\boldsymbol{\mathcal{A}}_{\mathbf{1,2}}$ hold. Then the fixed-lag smoothing problem (3) is solvable iff there exists a stabilizing solution to the ARE (18) and condition (21) holds. In this case, one solution of (3) is given by

$$\tilde{K}(s) = \left[\begin{array}{c|c} \tilde{A} - Z_h \tilde{C}_2' \tilde{C}_2 e^{\tilde{A}h} & Z_h \tilde{C}_2' \\ \hline -\tilde{C}_1 - \tilde{L}_1' W_h & 0 \end{array}\right] + \Pi(s) \tag{22}$$

where $\Pi(s)$ is as defined by (19),

$$W_h \doteq W_o - e^{\tilde{A}'h} W_o e^{\tilde{A}h} = \int_0^h e^{\tilde{A}'t} \tilde{C}_2' \tilde{C}_2 e^{\tilde{A}t} dt$$

and Z_h is as defined in Proposition 2.

The original problem, (2), is also solved:

Corollary 1: Let $\boldsymbol{\mathcal{A}}_{\mathbf{1,2}}$ hold. Then the fixed-lag smoothing problem (2) is solvable iff the conditions of Theorem 1 hold. In that case one solution is given by

$$K(s) = \tilde{K}(s) \left[\begin{array}{c|c} \tilde{A} & L_2 \\ \hline \tilde{C}_2 & R_2^{-1/2} \end{array}\right] - e^{-sh} \left[\begin{array}{c|c} \tilde{A} & L_2 \\ \hline \tilde{C}_1 & -D_1 D_2' R_2^{-1} \end{array}\right],$$

where $\tilde{K}$ is given by (22).

Some remarks are in order.

Remark 4.1: An important advantage of the solution of Theorem 1 over that in [7] is that the former does not involve matrix exponentials of matrices with eigenvalues in the RHP (including that in the last term of (22)). All exponentials are of the Hurwitz matrix $\tilde{A}$, so that the smoother (22) remains well-posed even as $h \to \infty$. However, after the transformation from $\tilde{K}$ back to K, as in Corollary 1, the structure of (22) is lost.

Remark 4.2: Another potential advantage of the proposed solution is that only the matrix Z_h might become ill-conditioned when γ approaches its minimal value (at least when $\det(\tilde{R}_\gamma) \neq 0$). Indeed, $\tilde{A}$ and W_o do not depend on γ and $\tilde{Y}_\gamma$ remains finite even for the optimal performance [19]. Note also that the singularity in Z_h can be handled easily by standard descriptor arguments [15, §16.9] and results in an order reduction of the first term on the right-hand side of (22). In contrast, in the solutions developed in [7], [8], the Riccati solution do become unbounded for the optimal performance.

Remark 4.3: The saturation phenomenon proved in §III-C can also be seen through the state-space solvability condition (21). To this end, note that neither $\tilde{Y}_\gamma$ nor W_o depends on the smoothing lag. Moreover, these matrices are bounded even for the optimal γ (more precisely, $\tilde{Y}_\gamma$ might become unbounded only if $\tilde{R}_\gamma$ is singular). It is then clear that the solvability condition (21) always holds if h is sufficiently large (since $\lim_{h\to\infty} e^{\tilde{A}h} = 0$). Thus, for a sufficiently large smoothing lag (21) is no longer restrictive and the solvability conditions reduce to the solvability of the ARE (18), which is independent of h. The minimal γ for which (18) is solvable is the minimal achievable performance in the fixed-interval smoothing (L^∞) case.

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