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Minimal Positive Realizations of a Class of Third-Order Systems

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***Abstract*— Let $H(z)$ be a third-order discrete-time transfer function with complex poles. This paper considers the following question: under what conditions does there exist a positive realization whose dimension is equal to the McMillan degree of $H(z)$. A sufficient condition is established for such a realization, which is also necessary under some mild assumption on the behavior of the impulse response.**

I. INTRODUCTION

In this paper, we study the following problem: Let $H(z)$ be a third-order discrete-time strictly proper transfer function, under what conditions does there exist a positive realization with dimension 3, which is the McMillan degree of $H(z)$? As shown in [2], the problem is quite intriguing, since the positivity constraint on the system matrices, may "force" a given transfer function to have a minimal positive realization of order much greater than its degree, and this seems to be a typical feature of most systems, even for a third-order transfer function. In [1], the positive realization is considered for a third-order transfer function with distinct real positive poles. In this paper, we will solve the positive realization problem for a third-order transfer function with complex poles.

II. PRELIMINARIES AND LEMMAS

In this paper, we focus on the following transfer function

$$H(z) = \frac{m_0}{z-\lambda} + \frac{m_1}{z-x_1+y_1 j} + \frac{m_1}{z-x_1-y_1 j} \qquad (1)$$

with $\lambda, x_1, y_1, m_1, m_0$ are real, $y_1 m_1 m_0 \neq 0$ and $\lambda^2 > x_1^2 + y_1^2$.

Definition 1 Given a matrix P, then $\mathcal{P} \triangleq \text{cone}(P)$ is the set of all (finite) nonnegative linear combinations of the columns of P.

Definition 2 The sets $R = (-\infty, \infty), R_+ = [0, +\infty)$ are called the sets of real numbers, positive real numbers, respectively. Let $N = \{1, 2, \ldots\}$ and denote by R_+^n the set of $n-$tuples of the positive real numbers. The set $R_+^{n\times n}$ is called the set of positive matrices of size n by n.

Definition 3 A transfer function $H(z)$ is said to be positively realizable if it has an n^{th} order positive realization, i.e., there exist a triple $A \in R_+^{n\times n}$, $b \in R_+^n$, $c \in R_+^n$, such that $H(z) = c^T(zI-A)^{-1}b$, where n is an integer.

Lemma 1[4] Let $H(z)$ be a transfer function with nonnegative impulse response. Then $H(z)$ has a positive realization if and only if $c_1 H(c_2 z)$ has a positive realization for any positive constants c_1, c_2.

By Lemma 1, the transfer function given in (1) has a positive realization of order $M(\geq 3)$ if and only if

$$H(z) = \frac{1}{z-1} + \frac{m}{z-x+yj} + \frac{m}{z-x-yj} \qquad (2)$$

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has a positive realization of order M. Without loss of generality, take $y < 0$.

Definition 4 $\mathcal{R} = \text{cl.cone}(b, Ab, A^2b, \ldots)$, i.e. the closure of $\text{cone}(b, Ab, A^2b, \ldots)$; $\mathcal{S} = \{z : c^T A^k z \geq 0, k = 0, 1, \ldots\}$.

Lemma 2[4],[8] Let $H(z)$ be a transfer function with minimal realization $\{A, b, c\}$, i.e., $H(z) = c^T(zI-A)^{-1}b$. Then, $H(z)$ has a positive realization if and only if there exists a matrix P such that

$$\mathcal{R} \subset \mathcal{P}, \quad A\mathcal{P} \subset \mathcal{P}, \quad c \in \mathcal{P}^*$$

where $\mathcal{P} = \text{cone}(P)$ and $\mathcal{P}^* = \left\{\beta : \alpha^T\beta \geq 0, \forall \alpha \in \mathcal{P}\right\}$.

Lemma 3[8] Let (A, b, c) be an n-dimensional realization of $H(z)$. Then, $H(z)$ is positively realizable if and only if there exists a polyhedral cone $\mathcal{P}$ such that $\mathcal{R} \subset \mathcal{P} \subset \mathcal{S}$, $A\mathcal{P} \subset \mathcal{P}$. Moreover, a positive realization (A_+, b_+, c_+) is given by solving

$$AP = PA_+, \quad b = Pb_+, \quad c_+^T = c^T P$$

where P is a matrix such that $\mathcal{P} = \text{cone}(P)$.

Lemma 4[9] Consider the transfer function $H(z)$ defined as in (1), then (A, b, c) is a 3-dimensional realization of $H(z)$, where

$$A = \begin{pmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} m \\ m \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Lemma 5[9] Let $A_{11} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ and $A_{11}^k = \begin{pmatrix} \Delta_1(k) & -\Delta_2(k) \\ \Delta_2(k) & \Delta_1(k) \end{pmatrix}$ for $k \in N$, then $A = \begin{pmatrix} A_{11} & 0 \\ 0 & 1 \end{pmatrix}$ and $\Delta_i(k), \Delta_i(k-1)$ satisfy the following equations

$$\begin{matrix} \Delta_1(k) = x\Delta_1(k-1) + y\Delta_2(k-1) \\ \Delta_2(k) = x\Delta_2(k-1) - y\Delta_1(k-1) \end{matrix} \quad k \in N \qquad (3)$$

with $\Delta_1(0) = 1, \Delta_2(0) = 0$.

Lemma 6[9] The impulse response of the transfer function $H(z)$ in (2) is

$$h(k) = 1 + 2m\Delta_1(k-1), \quad k \in N$$

with $\Delta_1(0) = 1$. Moreover, $h(k)$ is nonnegative if and only if $2m\Delta_1(k-1) + 1 \geq 0,\ k \in N$.

Lemma 7[9] The vectors b, Ab and A^2b lie in the plane $\{(\xi_1, \xi_2, 1) :\ \xi_1, \xi_2 \in R\}$.

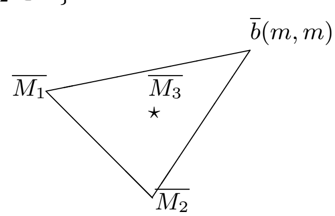

Fig. 1

Lemma 8[9] $b \in \mathcal{S}$ if and only if $1 + 2m\Delta_1(k-1) \geq 0, k \in N$.

Lemma 9 Let $\overline{X} = \text{cone}(\overline{b}, \overline{M_1}, \overline{M_2})$. Then $A_{11}\overline{M_2} \in \overline{X}$ if and only if

$$\left\{ \begin{array}{l} x^2+y^2+2x \leq 0, \\ x \in (-\frac{1}{2}, 0) \end{array} \right\} \bigcup \left\{ \begin{array}{l} x^2+y^2+2x < 0, \\ x = -\frac{1}{2} \end{array} \right\}.$$

III. Main Results

Theorem 1 Let $H(z)$ be a strictly proper transfer function and (A, b, c) be an $n-$ dimensional realization of $H(z)$. Then, $H(z)$ is positively realizable if and only if there exists a polyhedral cone $\mathcal{P}$ such that $b \in \mathcal{P}$, $c \in \mathcal{P}^*$, $A\mathcal{P} \subset \mathcal{P}$.

Theorem 2 The transfer function $H(z)$ defined in (2) has a third-order positive realization if

$$\begin{array}{l} a)\left\{\begin{array}{l} x^2+y^2+2x \leq 0, \\ x \in (-\frac{1}{2}, 0) \end{array}\right\} \bigcup \left\{\begin{array}{l} x^2+y^2+2x < 0, \\ x = -\frac{1}{2} \end{array}\right\}; \\ b) 2mx+1 \geq 0; \\ c) 2m+1 \geq 0. \end{array} \tag{4}$$

Theorem 3 If there exists at least one $k \in \{3, 4, \ldots\}$ such that $1 + 2m\Delta_1(k-1) = 0$, then the conditions in (4) are sufficient and necessary for the existence of a third-order positive realization of $H(z)$.

IV. Appendix

Remark Denote $b = \begin{pmatrix} \bar{b} \\ 1 \end{pmatrix}$, $Ab = \begin{pmatrix} \overline{M_1} \\ 1 \end{pmatrix}$ and $A^2 b = \begin{pmatrix} \overline{M_2} \\ 1 \end{pmatrix}$. Then,

$$\bar{b} = \begin{pmatrix} m \\ m \end{pmatrix}, \overline{M_1} = \begin{pmatrix} m(x+y) \\ m(x-y) \end{pmatrix}, \overline{M_2} = \begin{pmatrix} m(x^2-y^2+2xy) \\ m(x^2-y^2-2xy) \end{pmatrix}$$

Proof of Lemma 9: Rewrite

$$A_{11} = \sqrt{x^2+y^2} \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{pmatrix},$$

and define a linear translation as follows:

$$\begin{array}{l} \sigma : R^n \to R^n, \\ \sigma(p) = A_{11}p, \quad p \in R^n. \end{array}$$

Then the angle of $\sigma(p)$ is equal to the sum of that of p and the angle ϕ, and the magnitude of $\sigma(p)$ is equal to $\sqrt{x^2+y^2}$ multiple of that of p, where $\sin\phi = \frac{-y}{\sqrt{x^2+y^2}}$. Under the assumption of $y < 0$, we have $\phi \in (0, \pi)$.

In this case, the relationship between $\bar{b}, \overline{M_1}, \overline{M_2}$ is shown in Fig. 1. Denote $\overline{M_3} = A_{11}\overline{M_2}$. Let $|\cdot|$ denote the determinant operation, by classical geometry, the area of $\overline{X}$ is

$$\frac{1}{2}\begin{vmatrix} m & m & 1 \\ m(x+y) & m(x-y) & 1 \\ m(\Delta_1(2)-\Delta_2(2)) & m(\Delta_1(2)+\Delta_2(2)) & 1 \end{vmatrix},$$

where the endpoints of the triangle are in counterclockwise order.

Take

$$\det(T_1) = \frac{1}{2}\begin{vmatrix} m & m & 1 \\ m(x+y) & m(x-y) & 1 \\ m(\Delta_1(3)-\Delta_2(3)) & m(\Delta_1(3)+\Delta_2(3)) & 1 \end{vmatrix},$$

$$\det(T_2) = \frac{1}{2}\begin{vmatrix} m & m & 1 \\ m(\Delta_1(3)-\Delta_2(3)) & m(\Delta_1(3)+\Delta_2(3)) & 1 \\ m(\Delta_1(2)-\Delta_2(2)) & m(\Delta_1(2)+\Delta_2(2)) & 1 \end{vmatrix},$$

$$\det(T_3) = \frac{1}{2}\begin{vmatrix} m(\Delta_1(3)-\Delta_2(3)) & m(\Delta_1(3)+\Delta_2(3)) & 1 \\ m(x+y) & m(x-y) & 1 \\ m(\Delta_1(2)-\Delta_2(2)) & m(\Delta_1(2)+\Delta_2(2)) & 1 \end{vmatrix}.$$

If M_3 lies in $\overline{X}$, then $\det(T_i) \geq 0, i = 1, 2, 3$ and $\det(T_i) = 0$ for at most one unique $i \in \{1, 2, 3\}$; and if M_3 does not lie in $\overline{X}$, then $\det(T_i) < 0$ for some unique $i \in \{1, 2, 3\}$. Keeping this in mind, we obtain that

$$\begin{array}{c} \overline{M_3} \in \overline{X} \text{ if and only if } \det(T_i) \geq 0, i = 1, 2, 3, \\ \text{and } \det(T_i) = 0 \text{ for at most one } i \in \{1, 2, 3\}. \end{array}$$

Next, we compute $\det(T_i)$, which leads to

$$\begin{array}{l} \det(T_1) = -m^2 y(2x+1)(x^2+y^2-2x+1), \\ \det(T_2) = m^2 y(x^2+y^2+2x)(x^2+y^2-2x+1), \\ \det(T_3) = -m^2 y(x^2+y^2)(x^2+y^2-2x+1). \end{array}$$

Hence,

$$\overline{M_3} \in \overline{X} \text{ if and only if } \left\{\begin{array}{l} x^2+y^2+2x \leq 0 \\ 2x+1 \geq 0 \end{array}\right. \cup \left\{x = -\frac{1}{2}, y^2 \neq \frac{3}{4}\right\}$$

With the assumptions of $x^2 + y^2 < 1$ and $y \neq 0$, the condition above is equivalent to

$$\left\{\begin{array}{l} x^2+y^2+2x \leq 0 \\ x \in (-\frac{1}{2}, 0) \end{array}\right. \bigcup \left\{\begin{array}{l} x^2+y^2+2x < 0 \\ x = -\frac{1}{2} \end{array}\right.$$

Proof of Theorem 1: By Lemma 2, necessity follows immediately from the fact that $b \in \mathcal{R}$. To establish sufficiency, we assume that $b \in \mathcal{P}$, $c \in \mathcal{P}^*$, $A\mathcal{P} \subset \mathcal{P}$ and prove that $H(z)$ is positively realizable. Since $b \in \mathcal{P}$, $A\mathcal{P} \subset \mathcal{P}$, we have $A^k b \in \mathcal{P}$ for every $k \in N$. It follows that $\mathcal{R} \subset \mathcal{P}$. By Lemma 2, the sufficiency is proved.

Proof of Theorem 2: The following is a third-order realization of $H(z)$,

$$A_+ = \begin{pmatrix} 0 & 0 & x^2+y^2 \\ 1 & 0 & -x^2-y^2-2x \\ 0 & 1 & 2x+1 \end{pmatrix}, \; b_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$c_+ = \begin{pmatrix} 2m+1 \\ 2mx+1 \\ 2m(x^2-y^2)+1 \end{pmatrix}.$$

The proof of $2m(x^2-y^2)+1 \geq 0$ can refer to [9].

Proof of Theorem 3: To establish necessity, we assume that $H(z)$ has a third-order positive realization. By Theorem 1, there exists a polyhedral cone $\mathcal{P}$ with three edges such that

$$I) \;\; b \in \mathcal{P}; \; II) \;\; c \in \mathcal{P}^*; \; III) \;\; A\mathcal{P} \subset \mathcal{P}.$$

It follows that $b \in \mathcal{S}$. Application of Lemma 8 shows that conditions b) and c) are true. Lemma 7 claims that the vectors b, Ab, A^2b lie in a plane. Let $K = (b, Ab, A^2b)$, $\mathcal{K} = \text{cone}(K)$. Apparently, $\mathcal{K} \subset \mathcal{P}$; and b, Ab, A^2b are linear independent in view of $y \neq 0$. If there exists at least one $k \in \{3, 4, \ldots\}$ such that $1 + 2m\Delta_1(k-1) = 0$, then the first three vectors of the free evolution emanating from b lie on different edges of the observability cone $\mathcal{S}$. Hence, by Lemma 3, $\mathcal{K}$ is the polyhedral cone satisfying conditions I)-III) with minimal number of edges contained in $\mathcal{S}$. By Lemma 9, condition III) implies condition a). By Theorem 2, the statement is proved.

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