Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

A Lyapunov Approach to Frequency Analysis

Tingshu Hu[1] Andrew R. Teel[2] Zongli Lin[1]

***Abstract*— This paper proposes a Lyapunov approach to frequency analysis for general systems. The notion of frequency response is extended to general systems through a connection in linear systems. Lyapunov approaches to the characterization of frequency response are established for linear systems, homogeneous systems and nonlinear systems, respectively. In particular, we show that for linear systems, quadratic Lyapunov functions are sufficient for the characterization; for homogeneous systems, homogeneous Lyapunov functions are sufficient; and for general nonlinear systems, locally Lipschitz Lyapunov functions will be used. We also develop a Lyapunov approach for the characterization of the peak of the output. This approach is demonstrated to be effective on linear systems. An LMI based method for performing frequency analysis on linear differential inclusions is developed. Through a numerical example, an interesting phenomenon is observed about the relation between the frequency response and the L_2 gain of linear differential inclusions.**

Keywords: Frequency analysis, transient analysis, Lyapunov functions, differential inclusions, LMI.

Basic Definitions

- $|x|$: Euclidean norm of x.
- $\|u\|$: L_∞^m-norm of an essentially bounded function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to class $\mathcal{K}_\infty$ if, in addition, it is unbounded.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s,t) = 0$, and for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$.

I. Introduction

A. Background

Frequency domain analysis and time domain analysis are equally important for linear systems. They provide different insights into systems characteristics and they complement each other in the development of linear systems theory [8]. For nonlinear systems, frequency analysis is also an important problem and has been attempted since 1950s (see, e.g., [4], [5], [6], [9], [14], [15] and [2] for a review of the early developments). Some of the early papers used the describing function method to obtain an approximate characterization of the frequency response [5], [6], [14], [15] and others used individual systems to demonstrate specific nonlinear phenomena such as jump phenomena, subharmonic oscillations and frequency entrainment. This paper attempts to establish a systematic frequency analysis approach for the study of the input-output properties of nonlinear systems. In particular, we would like to determine both the asymptotic behavior and the transient behavior of the output under the excitation of an input signal which has one or more frequency components.

Consider nonlinear systems of the form

$$\dot{x} = f(x,u), \quad y = h(x) \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are both locally Lipschitz continuous. Several notions of output stability for such systems were introduced in [11] and a Lyapunov approach was developed in [12] for the characterization of these output stability properties. Given an initial state $x_\circ \in \mathbb{R}^n$ and an input u, let $x(\cdot, x_\circ, u)$ be the solution of the system and let $y(\cdot, x_\circ, u)$ be the corresponding output. Assume that for every $x_\circ$ and u, the solution $x(t, x_\circ, u)$ is defined for all $t \geq 0$. Then the system is said to be input to output stable if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$|y(t, x_\circ, u)| \leq \beta(|x_\circ|, t) + \gamma(\|u\|), \quad \forall t > 0. \tag{2}$$

Now suppose that the system is input to output stable. One interesting problem is to find a function γ that characterizes the asymptotic bound of the output as sharply as possible. For linear systems, the problem of minimizing the asymptotic bound can be approached through reachable sets with unit-peak inputs, which are estimated with ellipsoids under the LMI framework (see page 82, [3]). In the literature, attempts have also been made to minimize the gain function γ for special classes of nonlinear systems (e.g., see [7], [10] for systems with saturation nonlinearities). Most often, the input u is not an arbitrary signal and more detailed information about it may be exploited to obtain a weaker stability condition or a sharper bound on the output. For instance, the bound on the derivatives of the input was used in [1] to obtain a weaker condition of stability.

In many situations, the input can be modeled as the output of an autonomous system. For instance, in rotating machinery, a disturbance/input u usually takes the form of a simple sinusoidal signal, a signal with several sinusoidal components, a periodic signal, a signal with a certain frequency band, or some output of an oscillator. If we ignore such frequency information and treat the signal as arbitrary, we may end up with an overly conservative gain function γ. In other words, using the frequency information properly

[1] Department of Electrical and Computer Engineering, University of Virginia, P. O. Box 400743, Charlottesville, VA 22904-4743, U.S.A. Email: {th7f,zl5y}@virginia.edu. Work supported in part by NSF under grant CMS-0324329.

[2] Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106-9560. Email: teel@ece.ucsb.edu. Work supported in part by NSF under grant ECS-0324679 and by AFOSR under grant F49620-03-1-0203.

has a potential to result in a better estimate of the asymptotic gain from the input to the output. This brings us back to the topic of frequency analysis for nonlinear systems.

B. Motivation and problem formulation

1) Modeling the input: Consider the system (1). An input u whose components have the same frequency ϕ can be modeled as

$$u = \Gamma w = \begin{bmatrix} I_m & 0_m \end{bmatrix} w, \;\; \dot{w} = \begin{bmatrix} 0 & -\phi I_m \\ \phi I_m & 0 \end{bmatrix} w. \qquad (3)$$

The magnitudes and the phases of all the components of u are determined by the initial condition of w. Moreover, the norm of u in the frequency domain equals the Euclidean norm of the initial condition, $|w_\circ|$. If u has several frequency components, we can model it as

$$u = \Gamma w, \quad \dot{w} = S w, \qquad (4)$$

with $w \in \mathbb{R}^\ell$ and $S + S^\mathrm{T} = 0$. A more general situation is that u is the output of an oscillator:

$$u = \Gamma w, \quad \dot{w} = g(w), \qquad (5)$$

where $w \in \mathbb{R}^\ell$ and $\langle w, g \rangle = 0$. In all the above cases, we have $|w(t)| \equiv |w_\circ|$.

Now we have the combined autonomous system

$$\begin{aligned} \dot{x} &= f(x, \Gamma w), \quad y = h(x), &(6)\\ \dot{w} &= g(w). &(7) \end{aligned}$$

Since the original input u is completely determined by the initial condition $w_\circ$, we would like to characterize the relationship between the output and $w_\circ$. Under the traditional frequency analysis framework, this relationship is described by the term “frequency response”.

2) Frequency analysis for linear systems: time domain interpretation: Consider a linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad x(0) = x_\circ, \qquad (8)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^q$. Assume that A is Hurwitz. Let u be an input whose elements have the same frequency ϕ, i.e., $u_i(t) = \bar{u}_i \cos(\phi t + \theta_i)$. Then u can be described with (3). Note that $|\bar{u}| = |w_\circ|$. Let y^∞ be the asymptotic component of y, i.e., $y(t) = y^\infty(t) + \tilde{y}(t)$, where $\lim_{t\to\infty} \tilde{y}(t) = 0$. Suppose that $y_i^\infty(t) = \bar{y}_i \cos(\phi t + \vartheta_i)$. Then the norms of u and y^∞ in the frequency domain are $|\bar{u}|$ and $|\bar{y}|$ respectively. For single input single output systems, we have $\|u\| = |\bar{u}| = |w_\circ|$ and $\|y^\infty\| = \bar{y}$. Hence

$$\sup_{w_\circ \neq 0} \frac{\|y^\infty\|}{|w_\circ|} = \sup_{|\bar{u}| \neq 0, \theta_i \in [0, 2\pi]} \frac{\|y^\infty\|}{\|u\|} = \|C(j\phi I - A)^{-1} B\|.$$

For multiple input multiple output systems, since $\|y^\infty\| \le |\bar{y}|$, we have

$$\sup_{w_\circ \neq 0} \frac{\|y^\infty\|}{|w_\circ|} \le \|C(j\phi I - A)^{-1} B\|. \qquad (9)$$

The discrepancy between the two sides of the above inequality can be handled by doubling the dimensions of the state and the output to form an augmented system.

In view of these arguments, we use the quantity on the left side of (9), i.e.,

$$\gamma_* := \sup_{w_\circ \neq 0} \frac{\|y^\infty\|}{|w_\circ|}$$

to represent the frequency response (at a particular frequency ϕ) of a linear system and will generalize it to nonlinear systems. To realize the generalization, let us now interpret the quantity γ_* in the state space. Consider the combined system

$$\dot{x} = Ax + Ew, \;\; \dot{w} = Sw, \;\; y = Cx, \;\; (E = B\Gamma). \qquad (10)$$

Let Π be the solution to

$$A\Pi - \Pi S = E. \qquad (11)$$

Claim 1: $\gamma_* = \|C\Pi\|$ and γ_* is the least positive number γ such that there exist $K > 0$ and $\eta > 0$ satisfying

$$|y(t)| \le K \left| \begin{matrix} x_\circ \\ w_\circ \end{matrix} \right| e^{-\eta t} + \gamma |w_\circ| \;\; \forall\, x_\circ \in \mathbb{R}^n, w_\circ \in \mathbb{R}^\ell. \qquad (12)$$

3) Frequency analysis for general systems: problem formulation: For a linear system, the “frequency response” can be easily obtained for all $\phi \in [0, \infty)$ and the term also represents the function itself. For a nonlinear system, we have to restrict our attention to a fixed frequency ϕ but we still use “frequency response” to describe the input-output relationship for simplicity. We also extend the term to the situation where the input has the general description of (5).

In view of Claim 1, for a general system

$$\dot{x} = f(x, \Gamma w), \qquad \dot{w} = g(w), \qquad y = h(x),$$

with $\langle w, g \rangle = 0$, any locally Lipschitz function $\gamma \in \mathcal{K}$ such that there exist $K > 0$ and $\eta > 0$ satisfying

$$|y(t)| \le K \left| \begin{matrix} x_\circ \\ w_\circ \end{matrix} \right| e^{-\eta t} + \gamma(|w_\circ|) \;\; \forall\, x_\circ \in \mathbb{R}^n, w_\circ \in \mathbb{R}^\ell \qquad (13)$$

is called an upper bound for the frequency response. Our first objective of frequency analysis is to characterize the set of locally Lipschitz functions $\gamma \in \mathcal{K}$ such that there exist $K > 0$ and $\eta > 0$ satisfying (13). [4] The infimum of this set of functions will be called the frequency response of the system.

The notion of frequency response can also be extended to differential inclusions. Consider the system

$$\dot{x} \in A(x, w), \quad \dot{w} \in G(w), \quad y = h(x), \qquad (14)$$

[4] The frequency response may also be developed replacing the exponential decay function in (13) with a more general decay function, such as the $\mathcal{KL}$-function β in (2). The tools needed to address this case are in place, and reasonable Lyapunov formulations can be stated. (Cf. [13].) However, for the particular Lyapunov structure we use in this paper, it is somewhat cumbersome to clarify the regularity of the Lyapunov function characterizing the frequency response. For this reason, in order to keep the presentation simple, we will restrict our attention to the exponential decay case in this paper.

where A and G are set-valued maps. Assume that $\langle w, g\rangle = 0$ for all $g \in G(w)$. Given a locally Lipschitz function $\gamma \in \mathcal{K}$, if there exist $K > 0$ and $\eta > 0$ satisfying (13) for all y in the set of solutions under the initial condition $x_\circ$ and $w_\circ$, then γ is called an upper bound for the frequency response. Similarly, the infimum of this set of functions will be called the frequency response of the system.

We will develop a Lyapunov approach to characterize such a set of functions γ for general systems. Specific results will be presented for linear systems, homogeneous systems and nonlinear systems, respectively. We will also present an LMI approach to estimate the frequency response for linear differential inclusions. Another objective of this paper is to characterize the transient behavior of the output, in particular, the peak of the output under a set of initial conditions. This problem is important for systems that are subject to state or output constraints.

II. Frequency Response: Main Results

Standing assumption: All systems considered are forward complete, i.e., there are no finite escapes.

A. Linear systems

Consider the linear system

$$\begin{array}{rcl} \dot{x} & = & Ax + Ew \\ \dot{w} & = & Sw \\ y & = & Cx \ , \end{array} \tag{15}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^q$.

Assumption 1: $S + S^{\mathrm{T}} = 0$ and A is Hurwitz. □

Under Assumption 1, we have $|w(t)| = |w_\circ|$ for all $t \geq 0$. Define

$$A_L := \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} \ . \tag{16}$$

We consider square matrices P and numbers $\gamma > 0$, $\eta > 0$ satisfying

$$\begin{array}{c} P = P^{\mathrm{T}} > 0 \\ \begin{bmatrix} C^{\mathrm{T}}C & 0 \\ 0 & 0 \end{bmatrix} \leq P \\ A_L^{\mathrm{T}}P + PA_L \leq -2\eta \left(P - \gamma^2 \begin{bmatrix} 0 & 0 \\ 0 & I_\ell \end{bmatrix} \right) \ . \end{array} \tag{17}$$

Theorem 1: Suppose Assumption 1 holds and let $\overline{\gamma} > 0$ be given. For each $\gamma > \overline{\gamma}$ there exist a matrix P and $\eta > 0$ satisfying (17) if and only if for each $\gamma > \overline{\gamma}$ there exist $K > 0, \overline{\eta} > 0$ such that

$$|y(t)| \leq K \left| \begin{array}{c} x_\circ \\ w_\circ \end{array} \right| e^{-\overline{\eta}t} + \gamma |w_\circ| \quad \forall\, x_\circ \in \mathbb{R}^n, w_\circ \in \mathbb{R}^\ell. \tag{18}$$

From Claim 1, we know that the least γ such that there exist $K > 0, \overline{\eta}$ satisfying (18) is the frequency response γ_*. By Theorem 1, we have

$$\gamma_* = \inf\{\gamma : \ \exists\, P, \eta > 0 \ \text{ satisfying (17)}\}. \tag{19}$$

B. Homogeneous systems

Let $M : \mathbb{R}^n \to$ (subsets of $\mathbb{R}^n$) be a set-valued map. We say that M is homogeneous of degree p if $M(\lambda x) = \lambda^p M(x)$ for all $\lambda \geq 0$ and $x \in \mathbb{R}^n$. Consider the system

$$\begin{array}{rcl} \dot{x} & \in & A(x, w) \\ \dot{w} & \in & G(w) \\ y & = & h(x) \ , \end{array} \tag{20}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^q$. $A : \mathbb{R}^n \times \mathbb{R}^\ell \to$ (subsets of $\mathbb{R}^n$) and $G : \mathbb{R}^\ell \to$ (subsets of $\mathbb{R}^\ell$) are set-valued maps. Define

$$\xi := \begin{bmatrix} x \\ w \end{bmatrix} \tag{21}$$

$$F(\xi) := \left\{ \begin{bmatrix} a \\ g \end{bmatrix} : \ a \in A(x, w), g \in G(w) \right\} . \tag{22}$$

Assumption 2: $\langle w, g\rangle = 0$ for all $g \in G(w)$. The set-valued map F and the function h are homogeneous of degree one and globally Lipschitz with nonempty compact, convex values. □

Under Assumption 2, it is clear that $|w(t)| = |w_\circ|$ for all $t > 0$.

We consider continuously differentiable functions $W : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geq 0}$ and numbers $\gamma > 0$, $\eta > 0$, $p > 1$ satisfying

$$\begin{array}{rcl} W(0) & = & 0 \\ |h(x)|^p & \leq & W(\xi) \\ \displaystyle\max_{f \in F(\xi)} \langle \nabla W(\xi), f\rangle & \leq & -p\eta \left(W(\xi) - \gamma^p |w|^p \right) \ . \end{array} \tag{23}$$

Theorem 2: Suppose Assumption 2 holds and let $\overline{\gamma} > 0$ be given. There exists $\eta > 0$ such that, for each $\gamma > \overline{\gamma}$ and $p > 1$, (23) has a continuously differentiable solution that is homogeneous of degree p if and only if there exists $\overline{\eta} > 0$ and for each $\gamma > \overline{\gamma}$ there exists $K > 0$ such that

$$|y(t)| \leq K|\xi_\circ| e^{-\overline{\eta}t} + \gamma |w_\circ| \ \ \forall\, x_\circ \in \mathbb{R}^n, w_\circ \in \mathbb{R}^\ell \ . \tag{24}$$

Remark 1: In fact, the necessary condition can be replaced with a seemingly weaker one: There exist $\eta > 0$ and $p > 1$ such that, for each $\gamma > \overline{\gamma}$, (23) has a continuously differentiable solution that is homogeneous of degree p. The equivalence of these conditions implies that we can restrict our attention to a fixed $p > 1$, such as an even integer.

By Theorem 2, the frequency response is given by the infimum of γ such that there exist $p > 1$, $\eta > 0$ and a continuously differentiable solution for (23) that is homogeneous of degree p.

C. Nonlinear Systems

Consider the system

$$\begin{array}{rcl} \dot{x} & \in & A(x, w) \\ \dot{w} & \in & G(w) \\ y & = & h(x) \ , \end{array} \tag{25}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^q$, and A and G are set-valued maps. Let ξ and F be defined as in (21)-(22).

Assumption 3: $\langle w, g\rangle = 0$ for all $g \in G(w)$, and the set-valued map F is locally Lipschitz with nonempty compact, convex values. $\square$

Under Assumption 3, we also have $|w(t)| = |w_\circ|$ for all $t > 0$.

Given $\overline{\gamma} \in \mathcal{K}$ locally Lipschitz, we consider locally Lipschitz functions $W : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geq 0}$ and numbers $\eta > 0$, $\varepsilon > 0$, $p > 1$ satisfying

$$\begin{aligned} W(0) &= 0 \\ |h(x)|^p &\leq W(\xi) \\ \max_{f\in F(\xi)} \langle \nabla W(\xi), f\rangle &\leq -p\eta\left(W(\xi) - (1+\varepsilon)^p \overline{\gamma}(|w|)^p\right) \text{ a.e.} \end{aligned} \tag{26}$$

Theorem 3: Suppose Assumption 3 holds and let $\overline{\gamma} \in \mathcal{K}$ be given and locally Lipschitz. There exists $\eta > 0$ such that, for each $\varepsilon > 0$ and $p > 1$, (26) has a locally Lipschitz solution if and only if there exists $\overline{\eta} > 0$ and for each $\varepsilon > 0$ there exists $\alpha_\varepsilon \in \mathcal{K}_\infty$ such that

$$\begin{aligned} |y(t)| \leq \alpha_\varepsilon(|\xi_\circ|)e^{-\overline{\eta}t} + (1+\varepsilon)\overline{\gamma}(|w_\circ|) \\ \forall\, x_\circ \in \mathbb{R}^n, w_\circ \in \mathbb{R}^\ell\ . \end{aligned} \tag{27}$$

By Theorem 3, we see that any bound on the frequency response can be arbitrarily closely approximated by a function $\overline{\gamma}$ satisfying (26), along with a locally Lipschitz function W and numbers $\eta > 0$, $\varepsilon > 0$ and $p > 1$.

The input-output description in (27) is a global relation. For some systems, such a relation may only be valid for initial states in a local region. This situation is addressed by the following corollary.

Corollary 1: Given $\overline{\gamma} \in \mathcal{K}$ locally Lipschitz. Suppose there exist locally Lipschitz function $W : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geq 0}$ and a function $\eta : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geq 0}$, and numbers $p > 1, \rho > 0, \overline{\eta} > 0$ such that $W(0) = 0$ and for all

$$\xi \in \Lambda(\rho) := \{\xi \in \mathbb{R}^{n+\ell} :\ \ W(\xi) \leq \rho,\ \overline{\gamma}(|w|)^p \leq \rho\}$$

the following is satisfied

$$\begin{aligned} \eta(\xi) &\geq \overline{\eta} \\ |h(x)|^p &\leq W(\xi) \\ \max_{f\in F(\xi)} \langle \nabla W(\xi), f\rangle &\leq -p\eta(\xi)\left(W(\xi) - \overline{\gamma}(|w|)^p\right) \ \text{ a.e.} \end{aligned}$$

Then there exists $\alpha \in \mathcal{K}_\infty$ such that,

$$|y(t)| \leq \alpha(|\xi_\circ|)e^{-\overline{\eta}t} + \overline{\gamma}(|w_\circ|) \quad \forall\, \xi_0 \in \Lambda(\rho).$$

Example 1: Consider a first-order system

$$\dot{x} = \text{sat}(-x) + d, \quad y = h(x) = x,$$

where $\text{sat}(u) = \text{sign}(u)\min\{1, |u|\}$ and d is the disturbance. If d is arbitrary, then the steady state gain from d to x is unbounded. For instance, a constant $d > 1$ will drive x unbounded. However, if d is a sinusoidal signal $d(t) = k\sin(\beta t)$, we will show that the steady state x is bounded by $2k/\beta$. We express d as

$$d = \begin{bmatrix} 1 & 0 \end{bmatrix} w, \quad \dot{w} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}, \quad |w_0| = k.$$

Choose $p = 2$, $\eta(\xi) = \text{sat}(x)/x$, $\overline{\gamma}(|w|) = 2|w|/\beta$ and

$$W(\xi) = 2x^2 + \frac{4}{\beta}xw_2 + \frac{2}{\beta^2}w_1^2 + \frac{4}{\beta^2}w_2^2$$

Then it can be verified that the condition of Corollary 1 is satisfied for every $\rho > 0$. Moreover, for all $\xi_0 \in \mathbb{R}^3$, the steady state x is bounded by $\overline{\gamma}(|w_0|) = 2|w_0|/\beta = 2k/\beta$.

III. Peak Estimation Through Lyapunov Approach

Evaluation of the peak of an output is an important problem, especially for systems that must operate under some state or output constraint. An LMI method for estimating a bound for the peak was presented in [3] for stable linear systems. In this section, we will develop a general method for peak evaluation by further exploring the Lyapunov approach.

A. A general result

Consider system (25) under the standing assumption and Assumption 3. Let $r : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a function that measures the size of the state x. We would like to determine the peak of the output in terms of $r(x_\circ)$ and $|w_\circ|$.

Theorem 4: Suppose that there exist a locally Lipschitz function $W : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geq 0}$, a function $\eta : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geq 0}$, and class $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfying

$$\begin{aligned} \alpha_1(|h(x)|^p) \leq W(\xi) &\leq \alpha_2(r(x)) + \alpha_3(|w|^p) \\ \max_{f\in F(\xi)} \langle \nabla W(\xi), f\rangle &\leq -\eta(\xi)\left(W(\xi) - \alpha_4(|w|^p)\right) \ \text{ a.e.} \end{aligned}$$

then

$$\begin{aligned} |y(t)|^p \leq \alpha_1^{-1}\left(\max\{\alpha_2(r(x_\circ)) + \alpha_3(|w_\circ|^p), \alpha_4(|w_\circ|^p)\}\right) \\ \forall\, x_\circ \in \mathbb{R}^n, w_\circ \in \mathbb{R}^\ell. \end{aligned}$$

B. Application to linear systems

Let the initial state of x be inside a set $X_\circ$, say $X_\circ = \{x \in \mathbb{R}^n :\ x^{\mathrm{T}}Rx \leq 1\}$, where $R = R^{\mathrm{T}} > 0$. Assume for simplicity that $w_\circ^{\mathrm{T}} w_\circ \leq 1$. We take $r(x) = x^{\mathrm{T}}Rx$. Consider quadratic type Lyapunov functions $W(\xi) = \xi^{\mathrm{T}}P\xi$ and $p = 2$. According to Theorem 4, if we can find a $P = P^{\mathrm{T}} > 0$, and numbers $\alpha_2, \alpha_3, \alpha_4, \eta > 0$ such that, for all $\xi \in \mathbb{R}^{n+\ell}$,

$$\xi^{\mathrm{T}}\begin{bmatrix} C^{\mathrm{T}}C & 0 \\ 0 & 0 \end{bmatrix}\xi \leq \xi^{\mathrm{T}}P\xi \leq \xi^{\mathrm{T}}\begin{bmatrix} \alpha_2 R & 0 \\ 0 & \alpha_3 I_\ell \end{bmatrix}\xi, \tag{28}$$

$$A_L^{\mathrm{T}}P + PA_L \leq -\eta\left(P - \alpha_4\begin{bmatrix} 0 & 0 \\ 0 & I_\ell \end{bmatrix}\right), \tag{29}$$

then for all $x_\circ \in X_\circ, |w_\circ| \leq 1$,

$$y(t)^{\mathrm{T}}y(t) \leq \max\{\alpha_2 + \alpha_3, \alpha_4\} \quad \forall\, t \geq 0. \tag{30}$$

Our objective is to compute

$$\gamma_{p*} := \inf_{\eta,\alpha_2,\alpha_3,\alpha_4>0,P} \max\{\alpha_2+\alpha_3,\alpha_4\}. \quad (31)$$

$$\begin{aligned}\text{s.t. } & a)\ P = P^{\mathrm{T}} > 0\\ & b)\ \begin{bmatrix} C^{\mathrm{T}}C & 0\\ 0 & 0\end{bmatrix} \le P\\ & c)\ P \le \begin{bmatrix} \alpha_2 R & 0\\ 0 & \alpha_3 I_\ell\end{bmatrix}\\ & d)\ A_L^{\mathrm{T}}P + PA_L \le -\eta\left(P - \alpha_4\begin{bmatrix} 0 & 0\\ 0 & I_\ell\end{bmatrix}\right)\end{aligned}$$

It is shown that the optimal solution can be obtained by restricting $\alpha_2+\alpha_3=\alpha_4$.

A degenerated case is where $X_0=\{0\}$. In this case, the initial conditions belong to the set

$$\{(x_\circ,w_\circ)\in\mathbb{R}^{n+\ell}:\ \ x_\circ^{\mathrm{T}}Rx_\circ\le 1,\ \ w_\circ^{\mathrm{T}}w_\circ\le 1\},$$

where $R=\infty I$. Because of this, the constraint (31c) should be replaced with

$$\begin{bmatrix} 0 & I_\ell\end{bmatrix} P \begin{bmatrix} 0\\ I_\ell\end{bmatrix} \le \alpha_3 I_\ell.$$

An optimization problem similar to (31) can be formulated to estimate the output bound.

Example 2: Consider a second order system

$$\dot{x}=Ax+Ew,\ A=\begin{bmatrix}-0.6 & -0.8\\ 0.8 & -0.6\end{bmatrix},\ E=\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix},\ x_\circ=0,$$

with

$$\dot{w}=Sw,\quad S=\begin{bmatrix}0 & -\beta\\ \beta & 0\end{bmatrix},\quad w_\circ^{\mathrm{T}}w_\circ\le 1.$$

The output is $y=Cx=[1\ \ 0]x$. If we don't use the frequency information of the disturbance, then the bound on y^2 computed from the method in [3] is $\lambda^*=2.3594$. By using the frequency information for different β, we obtained a much smaller bound on y^2, denoted as $y^2_{M,est}$ (see the dash-dotted curve in Fig. 1). The actual maximal value of $y(t)^2$, denoted y_M^2, is computed through simulation and is the solid curve in Fig. 1. As a comparison, we also plotted the exact asymptotic amplitude of y^2, denoted y_∞^2, as the dashed curve in Fig. 1. We see that the asymptotic amplitude of y^2 could be much smaller than its actual maximal value which is reached during the transient process.

IV. LDIs and an Observation on Frequency Response vs L_2 Gain

A. Numerical analysis with quadratic Lyapunov functions

Consider the following linear differential inclusion:

$$\dot{\xi}\in\{A_L\xi:\ \ A_L\in\Omega\},\quad y=\begin{bmatrix} C & 0\end{bmatrix}\xi, \quad (32)$$

where $\xi=\begin{bmatrix}x\\ w\end{bmatrix}\in\mathbb{R}^{(n+\ell)}$. Assume that Ω is a convex set in $\mathbb{R}^{(n+\ell)\times(n+\ell)}$ whose element A_L has the structure

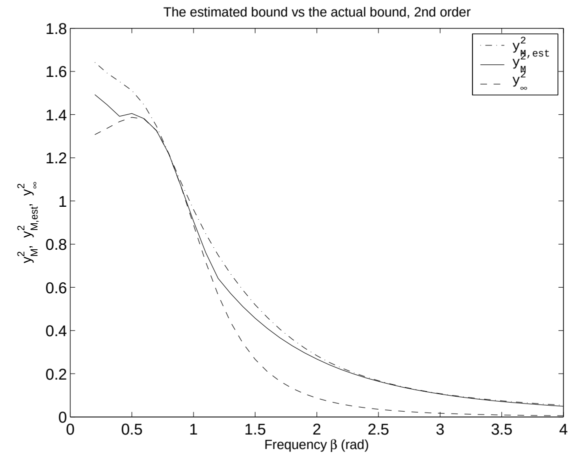

Fig. 1. The estimated bound and the actual bound: a 2nd order system

$A_L=\begin{bmatrix} A & E\\ 0 & S\end{bmatrix}$ and for each $A_L\in\Omega$, $S+S^{\mathrm{T}}=0$. Here we have two types of LDIs. For polytopic LDIs,

$$\Omega=\mathrm{co}\{A_{Li}:\ \ i=1,2,\cdots,N\},$$

and for structured LDIs,

$$\Omega=\{A_{L\circ}+U\Delta H:\ \ \|\Delta\|\le 1\},$$

where $A_{L\circ}$, U and H are given matrices. The polytopic LDI can be used to describe w with time-varying and uncertain frequency. Hence it allows nonperiodic signal w.

Theoretically, the exact frequency response can be obtained by optimizing over all the homogeneous Lyapunov functions and the numbers $\gamma>0,\eta>0,p>1$ satisfying (23). For computational simplicity, we would like to restrict our attention to quadratic Lyapunov functions. We consider Lyapunov functions of the type: $W(\xi)=\xi^{\mathrm{T}}\overline{P}\xi$ and $p=2$.

Then the condition (23) can be stated as

$$\begin{bmatrix} C^{\mathrm{T}}C & 0\\ 0 & 0\end{bmatrix}\le\gamma^2 P \quad (33)$$

$$A_L^{\mathrm{T}}P+PA_L\le -2\eta\left(P-\begin{bmatrix}0 & 0\\ 0 & I_\ell\end{bmatrix}\right),\ \forall A_L\in\Omega, \quad (34)$$

where $P=\overline{P}/\gamma^2$. We have replaced $\overline{P}$ with P for numerical simplicity. If there exist $P=P^{\mathrm{T}}>0$, $\gamma,\eta>0$ satisfying (33) and (34), then γ is an upper bound for the frequency response.

For polytopic LDIs, (34) is satisfied if and only if

$$A_{Li}^{\mathrm{T}}P+PA_{Li}\le -2\eta\left(P-\begin{bmatrix}0 & 0\\ 0 & I_\ell\end{bmatrix}\right),\ \forall i=1,\cdots,N. \quad (35)$$

Therefore, the bound on the frequency response can be sharpened by solving

$$\inf_{\gamma,\eta,P=P^{\mathrm{T}}>0}\gamma, \quad (36)$$
$$\text{s.t. } (33),(35)$$

For a fixed $\eta>0$, this is a standard "gevp" problem in LMI. For structured LDIs, similar optimization problem can be formulated.

B. An observation on frequency response vs L_2 gain

For linear systems, we know that the peak of the frequency response equals the L_2 gain. For LDIs, it may be expected that the peak of the frequency response is no greater than the L_2 gain. If this is the case, then the peak of the frequency response can be suppressed indirectly by minimizing the L_2 gain, which can be easily addressed by solving LMIs. However, the following example demonstrates that the L_2 gain of an LDI system could be less than the peak of the frequency response. This means that the frequency analysis has to be performed separately from the L_2 gain analysis to ensure that the output is below an admissible value.

Example 3: Consider the linear differential inclusion:

$$\dot{x} \in \mathrm{co}\{A_1 x + B_1 u, A_2 x + B_2 u\}, \tag{37}$$

where

$$A_1 = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.3 & -2.5 \\ 2.5 & -0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}.$$

The output is $y = Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x$. The L_2 gains of the two linear systems (C, A_1, B_1) and (C, A_2, B_2) are both 0.8333. An upper bound for the L_2 gain of the LDI is computed as $\gamma_\infty^u = 0.9906$ (with the algorithm in [3]).

Now assume that $u = \sin(\phi t + \theta)$. By using our method at $\phi = 1.26$, the upper bound for asymptotic output y^∞ is computed as 1.2169. An actual bound on y^∞, for a specific phase θ and a specific switching strategy within (37), is detected as 1.0917, which is greater than $\gamma_\infty^u = 0.9906$. Actually, under this particular switching strategy, we detected two "steady state" responses of the output, corresponding to different phase θ. These two asymptotic responses are plotted in Fig. 2. The solid response has a peak larger than 1 but the energy over the time interval is only 0.556 of the energy of the disturbance $u = \sin(1.26t + \theta)$ over the same interval. From Fig. 2, we see that the peaks of the solid curves are much sharper than those of a sinusoidal signal. This explains the low energy of the output even with a high peak. This cannot happen in linear systems [16].

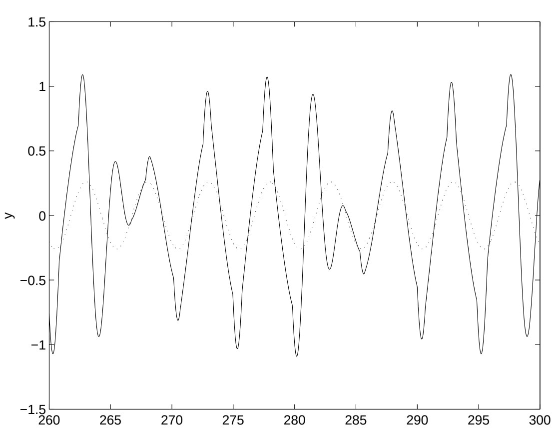

Fig. 2. Two asymptotic responses

V. CONCLUSIONS

Motivated by a state-space interpretation of frequency response for linear systems, we have given a trajectory-based definition of frequency response for general nonlinear systems and we have given an equivalent Lyapunov characterization. The Lyapunov characterization of frequency response uses quadratic functions, naturally, for linear systems, continuously differential homogeneous functions for homogeneous systems, e.g., linear differential inclusions, and locally Lipschitz functions for general nonlinear systems. We have also tailored the Lyapunov analysis to estimate the peak of the output as it converges to the frequency response. Finally, we have pointed out that, in contrast to the situation for linear systems, the frequency response may exceed the L_2 gain for nonlinear systems, in particular for a two dimensional LDI.

REFERENCES

[1] D. Angeli, E. D. Sontag and Y. Wang, "Input-to-state stability with respect to inputs and their derivatives," *Int. J. of Robust and Nonlinear Control*, **13**, pp. 1035-1056, 2003.

[2] D. P. Atherton, "Early developments in nonlinear control," *IEEE Control Systems*, Vol. 16, pp. 34-43, 1996.

[3] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, SIAM Studies in Appl. Mathematics, Philadelphia, 1994.

[4] E. M. Dewan and H. Lashinsky, "Asynchronous quenching of the van der Pol oscillator," *IEEE Trans. Automat. Contr.*, Vol. 14, pp. 212-214, 1969.

[5] A. Fukuma and M. Mattsubara, "Jump resonance in nonlinear feedback systems - Part I: Approximate analysis by the describing function method," *IEEE Trans. Automat. Contr.*, Vol. 23, pp.891-896, 1978

[6] P. E. W. Grensted, "The frequency response analysis of nonlinear systems," *Proc. IEE*, 102C, pp. 244-253, 1955.

[7] T. Hu, Z. Lin and B. M. Chen, "An analysis and design method for linear systems subject to actuator saturation and disturbance," *Automatica*, Vol. 38, No. 2, pp. 351-359, 2002.

[8] A. G. J. MacFarlane, "The development of frequency-response methods in automatic control," *IEEE Trans. Automat. Contr.*, Vol. 24, pp. 250-265, 1979.

[9] J. O'day and J. P. Hyde, "Subharmonics and jump resonance in a third-order nonlinear systems," *IEEE Trans. Automat. Contr.*, Vol. 17, pp. 721-722, 1972.

[10] A. Saberi, Z. Lin and A. R. Teel, "Control of linear systems with saturating actuators," *IEEE Trans. Automat. Contr.* Vol. 41, pp. 368-378, 1996.

[11] E. D. Sontag and Y. Wang, "Notions of input to output stability," *Systems and Control Letters*, Vol. 38, pp. 235-248, 1999.

[12] E. D. Sontag and Y. Wang, "Lyapunov characterizations of input to output stability," *SIAM J. Control and Optimization*, Vol. 39, 226-249, 2001.

[13] A. R. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESAIM: Control, Optimization and Calculus of Variations*, Vol. 5, 313-367, 2000.

[14] W. E. Wedlake and A. G. Vacroux, "A new method for finding the frequency response of nonlinear control systems," *IEEE Trans. Automat. Contr.*, Vol. 15, pp. 512-513, 1970.

[15] J. C. West and J. L. Douce, "The frequency response of a certain class of nonlinear feedback systems," *Br. J. Appl. Phys.*, Vol. 5, pp. 210-220, 1954.

[16] K. Zhou, J. C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, 1996.