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# A Linear Programming Approach to The Synthesis Of Fixed Structure Controllers

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***Abstract*— Many practical applications require the design of fixed order and structure feedback controllers. A broad class of fixed structure controller synthesis problems can be reduced to the determination of a real controller parameter vector (or simply, a controller), $K = (k_1,\ k_2,\ \dots, k_l)$, so that a given set of real or complex polynomials of the form $P(s, K) := P_o(s) + k_1P_1(s) + \cdots + k_lP_l(s)$ is Hurwitz. The stability of the closed loop system is addressed by requiring a real polynomial to be Hurwitz, while several performance criteria can be addressed by requriring a complex polynomial to be Hurwitz. In this paper, we consider only real polynomials, $P(s, K)$; the extension to complex polynomials is direct. A novel feature of this paper is the exploitation of the Interlacing Property (IP) of Hurwitz polynomials to synthesis, by systematically generating sets of linear inequalities in $K$. The union of the feasible sets of linear inequalities provides an approximation of the set of all controllers, $K$, which render these $P(s, K)$ Hurwitz. We show that this approximation can be made as accurate as desired. The main tools that are used in the construction of the sets of linear inequalities are the Hermite-Biehler theorem, Descartes' Rule of Signs and its generalization due to Poincare. We provide examples of the applicability of the proposed methodology to the synthesis and design of fixed order stabilizing controllers.**

## I. INTRODUCTION

The problem of fixed-order stabilization of a Linear-Time Invariant (LTI) dynamical system is one of the most important open problems in control theory [7]. It has attracted significant attention over the last four decades [33], [3]. This problem may be simply stated as follows: Given a finite-dimensional LTI dynamical system, is there a stabilizing proper, rational controller of a given order (a causal controller of a given state-space dimension)? The set of all the stabilizing controllers of fixed order is the *basic* set in which all design must be carried out. Despite many results concerning this problem, there is no systematic procedure for synthesizing a fixed-order controller.

A fixed-order stabilization problem arises when simplicity, hardware limitations or reliability in the implementation of a controller dictates low order of stabilization. For example, the pointing error controller of a Hubble Telescope is required to be of order 42 or less [9], [35] owing to computer hardware limitations. Applications requiring tuning of control parameters by an operator force a designer to minimize the number of controller parameters; some applications in this directions are in [12], [4].

Structural limitations on the controller can also result in a fixed-order stabilization problem. Such problems are encountered naturally in the decentralized control of a large-scale system. For example, [32] deals with the problem of decentralized control of a collection of Autonomous Underwater Vehicles (AUVs), where the dynamics of each AUV may be modeled as:

$$\dot{x}_i = A_i x_i + B_i u_i.$$

In the above equation, $x_i(t) \in \Re^n$ is the state of the $i^{th}$ AUV, $A_i \in \Re^{n\times n}, B_i \in \Re^{n\times m}$ are known system matrices. Every AUV has the same aggregated information, $q$, about the entire collection. This requirement is enforced for limiting the information to be communicated and the authors of [32] assume the existence of a sensor that can provide such a measurement without any communication with the AUVs. The aggregated information evolves according to a scalar differential equation:

$$\dot{q} = \sum_{j=1}^{r} H_j x_j,$$

where $H_j,\ \ j = 1, 2, \dots, r$ are known matrices of appropriate dimension. The problem of stabilization is that of finding appropriate controller gains, $K_i, i = 1, \dots, r$, so that the collection of AUVs is stabilized when each AUV employs a local control law of the form $u_i = K_i q$. This problem is clearly one of fixed-order (static output feedback) stabilization for a multi-input single-output LTI system.

Only recently, the set of all stabilizing PID controllers was found [11]. The issue of achievable performance is still open. In light of the pervasive use of fixed-order controllers in process control and the emergence of new applications such as formations of vehicles, it is imperative to understand whether fixed-order controllers that achieve a specified performance exist and if so, how one can find them and compute the set of all such stabilizing controllers that achieve a specified performance.

In this paper, we focus on the problem of determining the set of all real controller parameters, $K = (k_1,\ k_2, \dots k_l)$, which render a set of real polynomials Hurwitz, where each member of the set is of the form:

$$P(s, K) = P_o(s) + \sum_{l=1}^{N} k_l P_l(s). \qquad (1)$$

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The problem of finding a fixed-order controller can be posed as that of determining the feasibility of a set of polynomial inequalities in the controller parameters through the Routh-Hurwitz criterion. This problem has been shown to be decidable by Anderson et. al [2] using the Quantifier Elimination (QE) technique. However, this method is not computationally tractable. A good survey of the attempts to solve the fixed order control problem and the related Static Output Feedback (SOF) problem is given in [33], [7], [3] and the references therein. For recent work on the SOF problem using QE technique, see [16]. The associated problem of pole placement using SOF is presented in [23], [28], [13].

A necessary condition for a polynomial to be Hurwitz is that all the coefficients of the polynomial be of the same sign. In particular, if the coefficients of the closed loop polynomial are linear functions of the controller parameters, this necessary condition can be equivalently expressed as the feasibility of at most two linear programs. In [6], this approach was taken to arrive at a lower bound on the minimal order of stabilization.

The set of all fixed order/structure stabilizing controllers is *non-convex* and in general, *disconnected* in the space of controller parameters [1], [5]. This is a major source of difficulty in its computation. In [11], the Hermite-Biehler theorem is used in getting an approximation of the set of all stabilizing PID controllers for SISO plants. The basic idea is to make the Mikhailov plot [29], [5] of the characteristic polynomial, evaluated along the imaginary axis, go through an appropriate number of quadrants.

In [20], another interesting route to approximating the set of stabilizing controllers is presented. This approach combines ideas from Strict Positive Realness(SPRness), positive polynomials written as sum of squares (SOS) and LMIs. This approach also considers characterstic polynomials that are linear in the parameters of the controller. This approximation is an inner approximation of the set of stabilizing controllers.

The LMI approach for synthesizing a Static Output Feedback (SOF) controller is also explored in [17], [22], [18]. Other approaches to this problem can be found in [19], [34], [21], [31]

Gradient based techniques for the synthesis of stabilizing SOF controllers is presented in [26]. A gradient update scheme for the controller parameters is proposed based on the minimization of the spectral radius of the characteristic polynomial. Since the spectral radius is not necessarily a convex function of the controller parameters, the gradient scheme may yield local optima.

In the doctoral dissertation of [10], two approaches for synthesizing a low order controller are considered: an optimization technique based on D-decomposition and a recursive method for reducing the order. The problem of approximating the set of stabilizing controllers of a fixed order using linear programming techniques is also initiated in [10].

This paper differs from the contributions in the literature on this problem in its exploitation of the Interlacing Property of Hurwitz polynomials for synthesis through the construction of linear programs in controller parameters.

The results of this paper are similar in spirit to those in [14], where the interior of the monotone increasing (convex) non-negative cone of $n$ frequencies is bijectively mapped into the set of all Hurwitz polynomials of degree $n$. The parametrization in [14] is a convex parametrization. However, in the present case, the underlying set is, in general, neither convex nor connected. Nevertheless, by working in the space of frequencies, we produce approximations to the stabilizing set which are unions of convex sets.

This paper is organized as follows: In Section II, we deal with the real stabilization problem. Using the interlacing property of Hurwitz polynomials, we provide an inner approximation $\mathcal{S}_i$ and an outer approximation, $\mathcal{S}_o$ of the set, $\mathcal{S}$ of *all* stabilizing controllers of the given structure, where $\mathcal{S}_i \subset \mathcal{S} \subset \mathcal{S}_o$. We provide relevant examples to illustrate the techniques developed in this section. In section III, we provide concluding remarks.

## II. Synthesis of sets of stabilizing controllers

In this section, we seek to exploit the Interlacing Property (IP) of Hurwitz polynomials to systematically generate sets of controllers contained in $\mathcal{S}$. This approach leads to sets of Linear Programs(LPs). The procedure proposed here for generating the set of all fixed order controllers using the feasible sets of LPs can be applied to discrete-time LTI plants also; in the interest of saving space, we shall not describe them here.

We will present our proposed approach to find the set, $\mathcal{S}$ of all fixed order stabilizing controllers in this section.

### *A. On characterizing the set of all stabilizing controllers using a Linear Programming Approach*

Let $P(s,K)$ be a *real* closed loop characteristic polynomial whose coefficients are affinely dependent on the design parameters $K$; one can define the even and odd parts of $P(s,K)$ through $P_e(w^2,K)+jwP_o(w^2,K) := P(jw,K)$. The degrees of polynomials $P_e$ and $P_o$ are $n_e$ and $n_o$ respectively in $w^2$; specifically, if $n$ is odd, $n_e = n_o = \frac{n-1}{2}$ and if $n$ is even, $n_e = \frac{n}{2}$ and $n_o = n_e - 1$. Let $w_{e,i}$, $w_{o,i}$ denote the $i^{th}$ positive real roots of $P_e$ and $P_o$ respectively.

The Hermite-Biehler theorem for real polynomials may be stated as follows; for the sake of clarity, and for the general case, the dependence on $K$ is suppressed.

**Hermite-Biehler Theorem for real polynomials:** *A real polynomial $P(s)$ is Hurwitz iff*

1) *The constant coefficients of $P_e(w^2)$ and $P_o(w^2)$ are of the same sign,*
2) *All roots of $P_e(w^2)$ and $P_o(w^2)$ are real and distinct; the positive roots interlace according to the following:*
   - *if $n$ is even:*

$$0 < w_{e,1} < w_{o,1} < \cdots < w_{o,n_e-1} < w_{e,n_e};$$

- *if $n$ is odd:*

$$0 < w_{e,1} < w_{o,1} < \cdots < w_{e,n_e} < w_{o,n_e}.$$

*Proof:* A proof of the Hermite-Biehler theorem is found in [5]. ■

Therefore, $P(s,K)$ is Hurwitz *iff*:

1) the constant coefficients of $P_e(w^2,K)$ and $P_o(w^2,K)$ are of the same sign,
2) All roots of $P_e(w^2,K)$ and $P_o(w^2,K)$ are real and distinct; furthermore, the positive roots of $P_e(w^2,K)$ and $P_o(w^2,K)$ interlace.

The set $\mathcal{S}$ is, therefore, the set of all controllers, $K$, that simultaneously satisfy conditions (1) and (2) of the Hermite-Biehler theorem. The following version of the Hermite-Biehler theorem poses the problem of rendering $P(s,K)$ Hurwitz through a search for $n-1$ frequencies. By way of notation, we represent the polynomials $P_e$ and $P_o$ compactly in the following form:

$$P_e(w^2,K) = \begin{bmatrix} 1 & w^2 & \cdots & w^{2n_e} \end{bmatrix} \Delta_r \begin{bmatrix} 1 \\ K \end{bmatrix}, \quad (2)$$

$$P_o(w^2,K) = \begin{bmatrix} 1 & w^2 & \cdots & w^{2n_o} \end{bmatrix} \Delta_i \begin{bmatrix} 1 \\ K \end{bmatrix}. \quad (3)$$

In (2) and (3), $\Delta_r$ and $\Delta_i$ are real constant matrices that depend on the plant data and the structure of the controller sought; they are respectively of dimensions $(n_e+1)\times(l+1)$ and $(n_o+1)\times(l+1)$, where, for $n$ odd, we have $n_e = n_o = \frac{n-1}{2}$, and for even $n$, we have $n_e = \frac{n}{2}, n_o = n_e - 1$; $l$ is the size of the controller parameter vector. For $i = 1,2,3,4$, let $C_i$ and $S_i$ be diagonal matrices of size $n$; for an integer $m$, the $(m+1)^{st}$ diagonal entry of $C_i$ is $\cos(\frac{(2i-1)\pi}{4} + \frac{m\pi}{2})$ and the corresponding entry for $S_i$ is $\sin(\frac{(2i-1)\pi}{4} + \frac{m\pi}{2})$. For any given set of $n$ distinct frequencies, $w_0 < w_1 < \cdots < w_{n-1}$, and for any integer $m$ define a Vandermonde-like matrix, $V(w_0, w_1, \ldots, w_{n-1}, m)$, as:

$$V(w_0, w_1, \ldots, w_{n-1}, m) := \begin{bmatrix} 1 & w_0^2 & \ldots & w_0^{2m} \\ 1 & w_1^2 & \ldots & w_1^{2m} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & w_{n-1}^2 & \ldots & w_{n-1}^{2m} \end{bmatrix}.$$

We are now ready to characterize the set of stabilizing controllers $K$ in terms of $(n-1)$ frequencies:

**Theorem 1** *There exists a real control parameter vector $K = (k_1,\ k_2,\ \cdots,\ k_l)$ so that the real polynomial*

$$\begin{aligned} P(s,K) &:= P_0(s) + k_1 P_1(s) + \ldots + k_l P_l(s) \\ &= p_n(K)s^n + p_{n-1}(K)s^{n-1} + \cdots + p_0(K), \end{aligned}$$

*is Hurwitz iff there exists a set of $n-1$ frequencies, $0 = w_0 < w_1 < w_2 < w_3 < \cdots < w_{n-1}$, so that one of the following two Linear Programs (LPs) is feasible:*

**LP 1:**

$$C_1 V(w_0, w_1, \ldots, w_{n-1}, n_e)\Delta_r \begin{bmatrix} 1 \\ K \end{bmatrix} > 0,$$

$$S_1 V(w_0, w_1, \ldots, w_{n-1}, n_o)\Delta_i \begin{bmatrix} 1 \\ K \end{bmatrix} > 0. \quad (4)$$

**LP 2:**

$$C_3 V(w_0, w_1, \ldots, w_{n-1}, n_e)\Delta_r \begin{bmatrix} 1 \\ K \end{bmatrix} > 0, \quad (5)$$

$$S_3 V(w_0, w_1, \ldots, w_{n-1}, n_o)\Delta_i \begin{bmatrix} 1 \\ K \end{bmatrix} > \quad . \quad (6)$$

*Moreover, the union of the feasible sets of the above LPs corresponding to all such sets of frequencies ($0 < w_1 < w_2 < \ldots < w_{n-1}$) is the set of all stabilizing controllers.*

*Proof:* We provide a sketch of the proof. If a $K$ renders the polynomial $P(s,K)$ Hurwitz, its Mikhailov plot will go through $n$ quadrants in the counterclockwise direction. In such a case, if the Mikhailov starts on the positive real axis, then for some $0 < w_1 < w_2 < \ldots < w_{n-1}$, the complex numbers $P(jw_i)$ will lie in the $(i+1)^{th}$ quadrant and this is equivalent to the LP1 described above. If the Mikhailov plot starts on the negative real axis, one gets the LP2. The sufficiency part may be inferred from the fact that any controller $K$ which lies in the union of feasible sets of LPs 1 and 2 is such that $P_e(0,K)$ and $P(0,K)$ are of the same sign and the positive roots of $P_e(w^2,K)$ and $P_o(w^2,K)$ lie alternately in the intervals

$$(0,w_1),(w_1,w_2),\ldots,(w_{n-2},w_{n-1}),(w_{n-1},\infty).$$

This implies that all conditions of the Hermite-Biehler theorem for $P(s,K)$ are met and hence, $P(s,K)$ is Hurwitz. ■

**Remark 1** *With the above result, the problem of determining the set of all stabilizing controllers can therefore be posed as the search for all possible $n$ tuples (($n-1$) tuples of frequencies and the binary number indicating the sign of the coefficients), whose corresponding LP is feasible.*

As can be seen from the LPs given by (4) and (6), one can associate with every linear program an $n-1$ tuple of frequencies and a binary number which indicates the sign of the coefficients of the characteristic polynomial. The frequency information is used in reconstructing the Vandermonde matrix $V$ and the sign information is used in the choice of $C_1, S_1$ or $C_3, S_3$. Hence, storing the $n-1$ tuple of frequencies and the sign of the coefficients corresponding to feasible LPs is sufficient to reconstruct the entire set.

**Remark 2** If the characteristic polynomial is monic, then only the first LP needs to be considered for checking the feasibility since the coefficients $P_e(0,K)$ and $P_o(0,K)$ must also be positive.

**Remark 3** Recent solutions to the PID controller design [11] problem requires the even and odd parts of a polynomial to have certain patterns of root separation. In [11], the authors carry out a search for the separating frequencies by exploiting the structure of the PID control problem, wherein the only proportional gain, $K_p$, appears in the even or the odd part of a related polynomial. For a fixed value of $K_p$, one can find the roots of the even (odd) part of the polynomial. Then, one can find linear inequality constraints on the integral and derivative gains, $K_i$ and $K_d$, so that desired patterns of root separation are feasible.

To aid the systematic search for these $n-1$ tuples of frequencies and to construct an outer approximation of the set of stabilizing controllers, we propose to use Descartes' rule of signs and its generalization due to Poincare. We state them without proofs below:

**Descartes' Rule of Signs:** *Let*

$$P(s) = p_0 + p_1 s + \ldots + p_n s^n,$$

*be a $n^{th}$ degree real polynomial, $p_n \neq 0$. Then:*

1) *The number of positive, real roots of $P(s)$ is at most equal to the number of the variations in sign of the sequence of its coefficients $p_0, p_1, \ldots, p_n$; moreover, if the number of positive roots is less than the number of variations, the difference is an even number.*
2) *If $P(s)$ has all real roots, then the number of variations in sign of the sequence of coefficients equals the number of real, positive roots of $P(s)$.*

**Poincare's Generalization:***The number of sign changes in the coefficients of $Q_k(s) := (s+1)^k P(s)$ is a non-increasing function of $k$; for a sufficiently large $k$, the number of sign changes in the coefficients exactly equals the number of real, positive roots of $Q(s)$.*

The proof of the generalization due to Poincare is given in [27]. The essential idea of the proof provided in [27] is that, for a sufficiently large $k$, the number of sign changes of the coefficients of $P(s)$ is the same as the number of sign changes of the sequence of values an associated polynomial $R(u)$ takes at $u = \frac{l}{k},\ l = 1, 2, \ldots, k-1$; the associated polynomial $R(u)$ is defined as follows:

$$R(u) = (1-u)^n P(\frac{u}{1-u}).$$

In other words, Poincare's scheme samples the polynomial $R(u)$ uniformly with a grid width of $\frac{1}{k}$ and examines the variations in the value of the polynomial at these points. Clearly, for a sufficiently fine grid, the number of sign variations in the sampled values of $R(u)$ exactly equals the number of roots of $R(u)$ in $(0,1)$, and this is equal to the number of real, positive roots of $P(s)$.

This idea will be exploited in finding the sets of fixed structure stabilizing controllers and the bounds for the set of all fixed structure stabilizing controllers.

### *B. Constructing the set of stabilizing controllers*

Motivated by Poincare's generalization of the rule of signs, we define the polynomials, $R_e(u,K) = (1-u)^{n_e} P_e(\frac{u}{1-u}, K)$ and $R_o(u,K) = (1-u)^{n_o} P_o(\frac{u}{1-u}, K)$. Let $D_e$ and $D_o$ be diagonal matrices whose $i^{th}$ diagonal entries are $(1-u_i)^{n_e}$ and $(1-u_i)^{n_o}$ respectively. We now state Theorem 1 in an alternate useful manner omitting the proof:

**Theorem 2** *There exists a stabilizing $K$ iff there exists a $n-1$ tuple, $0 (= u_0) < u_1 < u_2 < \ldots < u_{n-1} < 1$, such that at least one of the following two LPs is feasible:*

**LP 1:**

$$\begin{aligned} C_1 D_e V(\frac{u_0}{1-u_0}, \ldots, \frac{u_{n-1}}{1-u_{n-1}}, n_e)\Delta_r \begin{bmatrix} 1 \\ K \end{bmatrix} &> 0, \\ S_1 D_o V(\frac{u_0}{1-u_0}, \ldots, \frac{u_{n-1}}{1-u_{n-1}}, n_o)\Delta_i \begin{bmatrix} 1 \\ K \end{bmatrix} &> 0. \end{aligned}$$

**LP 2:**

$$\begin{aligned} C_3 D_e V(\frac{u_0}{1-u_0}, \ldots, \frac{u_{n-1}}{1-u_{n-1}}, n_e)\Delta_r \begin{bmatrix} 1 \\ K \end{bmatrix} &> 0, \\ S_3 D_o V(\frac{u_0}{1-u_0}, \ldots, \frac{u_{n-1}}{1-u_{n-1}}, n_o)\Delta_i \begin{bmatrix} 1 \\ K \end{bmatrix} &> 0. \end{aligned}$$

*Moreover, the union of feasible sets of the above LPs corresponding to all possible $(n-1)$ tuples, $0 < u_1 < u_2 < \cdots < u_{n-1} < 1$, is the set of all stabilizing controllers, $\mathcal{S}$.*

### *C. Outer Approximation*

In the previous subsection, we outlined a procedure to construct LPs whose feasible set is contained in $\mathcal{S}$. Their union $\mathcal{S}_i$ is an inner approximation to $\mathcal{S}$. For computation, it is useful to develop an outer approximation, $\mathcal{S}_o$ that contains $\mathcal{S}$. In this subsection, we will present how to construct an arbitrarily tight outer approximation $\mathcal{S}_o$ as a union of the feasible sets of LPs.

As an example of an outer approximation, consider the scheme presented in [6]. One may ask the following question: Exactly how does the requirement, that the coefficients of the characteristic polynomial be of the same sign, relate to the conditions of Hermite-Biehler theorem? An answer to this question can provide the gap between the set of the stabilizing controllers, $\mathcal{S}$ and $\bigcup_{i=1}^{2} \mathcal{S}_{outer,i}$. It can also provide directions to tightening the outer approximation.

For the sake of a discussion on outer approximation, we will treat the polynomials, $P_e(w^2, K)$ and $P_o(w^2, K)$, as polynomials in $w^2$. Let $\lambda = w^2$ and let the $i^{th}$ roots of $P_e(\lambda, K)$ and $P_o(\lambda, K)$ be represented as $\lambda_{e,i}$ and $\lambda_{o,i}$ respectively. Since the polynomials $P_e$ and $P_o$ must have respectively $n_e$ and $n_o$ real, positive roots, an application of Poincare's result to the polynomials $P_e$ and $P_o$ yields the following:

**Lemma 1** *If $K$ is a stabilizing control vector, then $(\lambda+1)^{k-1} P_e(\lambda, K)$ and $(\lambda+1)^{k-1} P_o(\lambda, K)$ have exactly $n_e$ and $n_o$ sign changes in their coefficients respectively for every $k \geq 1$.*

The procedure in [6] corresponds to $k = 1$ of the above lemma.

The following lemma takes care of the interlacing of the roots of two polynomials.:

**Lemma 2** *Let $K$ render an odd degree polynomial $P(s,K)$ Hurwitz. Then the polynomial $\tilde{Q}(\lambda,K,\eta) = \lambda P_o(\lambda,K) - \eta P_e(\lambda,K)$ has exactly $n_o+1$ real positive roots for $\eta > 0$ and $n_o$ real positive roots for $\eta < 0$.*

*Proof:* In the interest of saving space, we only provide a sketch of the proof. To prove sufficiency, we consider the graph of the rational function $y := \frac{\lambda P_o(\lambda)}{P_e(\lambda)}$ and consider the intersections with $y = \eta$. To prove necessity, we argue, via a root locus, that if the interlacing or real roots condition is violated, for some value of $\eta \in \Re$, polynomial $\tilde{Q}(\lambda,K,\eta)$ will have atleast a pair of complex conjugate roots. ■

Lemmas 1 and 2 can be put together to show that an arbitarily tight outer approximation can be constructed.

**Proposition 1** *Let $P(s,K)$ be of odd degree and $K$ be such that $P(s,K)$ is not Hurwitz. Then, one of the following must hold:*

1) *All coefficients of $P(s,K)$ are not of the same sign.*
2) *For some $l > 1$, the number of sign changes in the coefficients of $(\lambda+1)^{l-1}P_e(\lambda,K)$ is fewer than $n_e$.*
3) *For some $l > 1$, the number of sign changes in the coefficients of $(\lambda+1)^{l-1}P_o(\lambda,K)$ is fewer than $n_o$.*
4) *For some $l > 1$ and for some $\eta > 0$, the number of sign changes in the coefficients of $(\lambda+1)^{l-1}(\lambda P_o(\lambda,K) - \eta P_e(\lambda,K))$ is fewer than $n_o+1$.*
5) *For some $l > 1$ and for some $\eta < 0$, the number of sign changes in the coefficients of $(\lambda+1)^{l-1}(\lambda P_o(\lambda,K) - \eta P_e(\lambda,K))$ is fewer than $n_o$.*

One can get an outer approximation of the set of stabilizing controllers as follows:

- **Step 1:** Construct and check the feasibility of the two LPs corresponding to the cases when all coefficients of $P(s,K)$ are positive and when they are negative.
- **Step 2:** Choose $l > 1$. Construct and check the feasibility of LPs corresponding to the case that the coefficients of the polynomial $(\lambda+1)^{l-1}P_e(\lambda,K)$ have exactly $n_e$ sign changes. Suppose $p_e$ of them are feasible.
- **Step 3:** Similarly, construct and check the feasibility of LPs corresponding to the case that the coefficients of the polynomial $(\lambda+1)^{l-1}P_o(\lambda,K)$ have exactly $n_o$ sign changes. Suppose $p_o$ of them are feasible.
- **Step 4:** By picking an LP each from the steps 1, 2 and 3, check the simultaneous feasibility. There will be atmost $2p_ep_o$ of such augmented LPs; of these, let $p_{eo}$ be feasible.
- **Step 5:** For this step, consider the degree of $P(s,K)$. If it is even, consider $(s+1)P(s,K)$ and construct its odd and even parts, $P_e(\lambda,K)$ and $P_o(\lambda,K)$. Pick some values of $\eta \in \Re$. For each $\eta \in \Re$, construct and check the feasibility of LPs corresponding to the case that the number of sign changes in the coefficients of $(\lambda+1)^{l-1}(\lambda P_o(\lambda,K,\eta) - \eta P_e(\lambda,K))$ is equal to $n_o+1$ if $\eta > 0$ and to $n_o$ if $\eta < 0$. Let $p^*$ of them be feasible.
- **Step 6:** Check the simultaneous feasibility of LPs constructed by taking one LP from Step 4 and one from Step 5. There will be atmost $p^*p_{eo}$ LPs to be checked.
- **Step 7:** Update $l$ to any number greater than $l$ for refinement and go to Step 2.

*1) Search for the $n-1$ tuple of frequencies and the computation of an inner approximation:* Using Theorem 2, the problem of finding stabilizing controllers may be posed as the problem of searching for *all* such sets of $n-1$ points, $u_1,\ldots,u_{n-1} \in (0,1)$. We propose the following scheme: First, we partition $(0,1)$ using more than $(n-1)$ points. For example, a uniform partition may be employed or a partition using the positive roots of an appropriate Chebychev polynomial may be used. Then, we systematically pick each of the $n-1$ tuples in the partition from all possible $n-1$ tuples. Corresponding to each choice of $(n-1)$ tuples, $0 < u_1 < u_2 < \ldots < u_{n-1}$, we can construct two LPs as described in Theorem 2; if any of the LPs is feasible, it yields a set of stabilizing controllers. The union of the feasible sets (each of which is polyhedral) corresponding to all possible $(n-1)$ tuples, therefore, provides an approximation of the set of stabilizing controllers. It is an *inner approximation* - every element of the approximate set is a stabilizing controller. The approximation may be made more accurate by refining the partition of $(0,1)$, because if $K$ is a stabilizing controller, a partition fine enough to separate the roots of $R_e(u,K)$ and $R_o(u,K)$ will always capture $K$.

The general procedure for the inner approximation of the set of stabilizing controllers can be used to construct a computer program, whose inputs are: (i) the desired number of feasible LPs, (ii) minimum allowed width of the partition, and (iii) plant data and the controller order from which LPs are generated. By the width of a partition, we mean the maximum difference between successive points of a partition. The program starts with a crude partition of the interval $(0,1)$. A counter in the program keeps track of the number of feasible LPs obtained thus far. The partition is refined further if the number of feasible LPs obtained is less than the desired number of feasible LPs. The program stops if the number of feasible LPs obtained is equal to the desired number of feasible LPs or if the width of the generated partition falls below a prespecified threshold.

The minimum separation between the roots of even and odd polynomials of $P(s,K)$ is a measure of the coprimeness of the the two polynomials, and hence, a measure of the distance of the polynomial $P(s,K)$ to a polynomial with imaginary axis roots. For this reason, a lower bound on the minimum width of the partition is reasonable, from both a computational as well as a robust stability point of view.

To illustrate the proposed approach, consider the following examples:

**Example 1** This example is from [2]. Consider a LTI plant described by the following equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix} x.$$

We will illustrate how to find the set of all static stabilizing controllers, i.e., $u = Ky$ where $K = [K_1 \quad K_2]$, for this system using the method proposed here.

The characteristic polynomial of the closed loop system is

$$P(s,K) = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -13 & -5 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}$$

For this example, $n_e = 1$ and $n_o = 1$. The real and imaginary parts of the characteristic polynomial, when evaluated at $jw$ are given by:

$$P_e(w^2,K) = \begin{bmatrix} 1 & w^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}$$
$$P_o(w^2,K) = \begin{bmatrix} 1 & w^2 \end{bmatrix} \begin{bmatrix} -13 & -5 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}$$

After using the transformation $w^2 = \frac{u}{1-u}$,

$$R_e(u,K) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}$$
$$R_o(u,K) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} -13 & -5 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}$$

**Construction of the sets of fixed order stabilizing controllers for this example problem:** For the closed loop system to be stable, there must exist a set of frequencies $0 = u_0 < u_1 < u_2 < 1$ such that:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} U_o \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix} > 0,$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} U_o \begin{bmatrix} -13 & -5 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix} > 0.$$

where,

$$U_o = \begin{bmatrix} 1-u_0 & u_0 \\ 1-u_1 & u_1 \\ 1-u_2 & u_2 \end{bmatrix}$$

Figures 1 and 2 show the results of inner approximation. In figure 1, the number of LPs is chosen to be 5, and the number of LPs chosen is 74 in figure 2. The red and blue sets (in the PDF file) indicate that they are from different LPs. It can be noticed that, as we increase the number of LPs considered, the approximation is better. In these figures, the black curve indicates the boundary of the actual stabilizing set of controllers obtained using the Routh Criterion, and the boundary of the outer approximation of the stabilizing controllers is shown with a dot-dash line. The outer approximation was obtained by requiring the coefficients of the characteristic polynomial to be of the same sign.

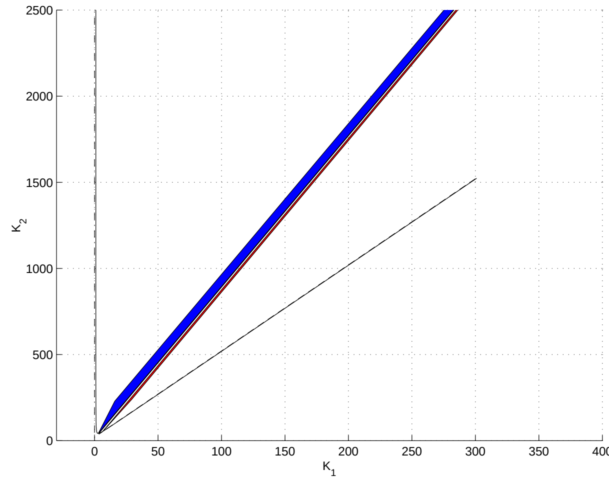


Fig. 1. Inner and Outer Approximation of the set of stabilizing controllers for Example 1 with 5 LPs.

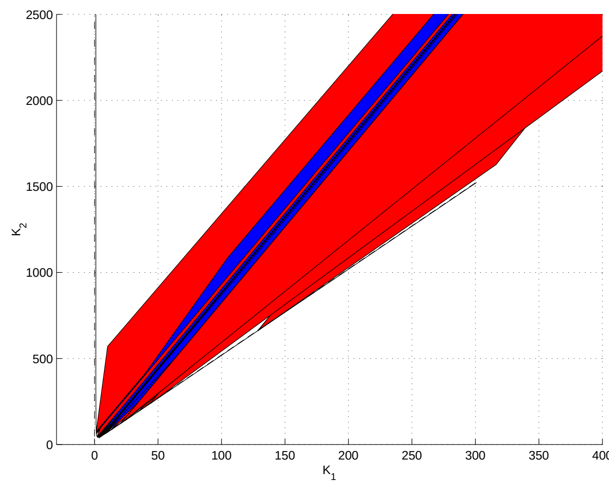


Fig. 2. Inner and Outer Approximation of the set of stabilizing controllers for Example 1 with 75 LPs.

**Example 2** The following example is from [30]. The example is that of a Saturn V booster and its model can be described by a single input, two output seventh order model:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.65 & -0.002 & 2.6 & 0 \\ -0.014 & 1 & -0.041 & 0.0002 & -0.015 & -0.033 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -45 & -0.13 & 255 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -50 & -10 \end{bmatrix},$$

$$B = \left[ \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]'$$

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= \left[ \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right] x. \end{aligned}$$

Figure 3 illustrates the inner and outer approximations of the set of **static output feedback** stabilizing controllers. Requiring all coefficients of the characteristic polynomial to be of the same sign yields an outer approximation (set in blue in the PDF file). Tightening the set using Lemmas 1 and 2 yields the improved outer approximation (sets in yellow and red). The set in yellow corresponds to a lower value of $l$ in Lemma 2 than the set in the red. The exact set of stabilizing controllers found by parameter sweeping techniques of [11] yields the approximately oval shaped (green) region. The inner approximation with a Chebychev partitioning is shown in white and is the union of the feasible sets of 150 LPs.

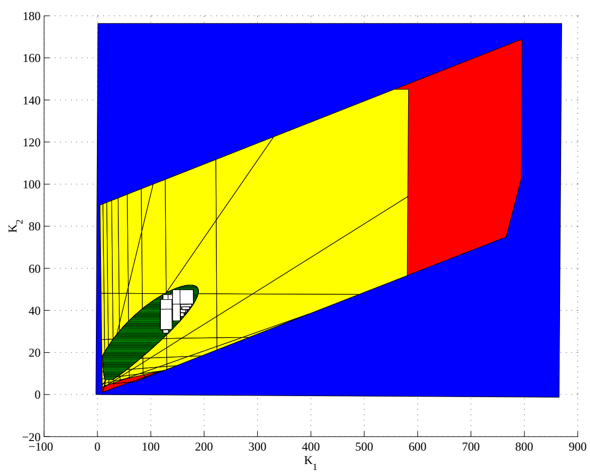


Fig. 3. Approximation of the set of static stabilizing controllers for the seventh order system described in Example 2

An example of approximating a disconnected set of stabilizing controllers is the following [11]:

**Example 3** The plant to be controlled by a PID controller has the transfer function:

$$H(s) = \frac{s^3 + 3s^2 + 9}{s^4 + 2s^3 + 3s^2 + 7s + 14}.$$

An inner approximation of the set of PID gains is shown below:

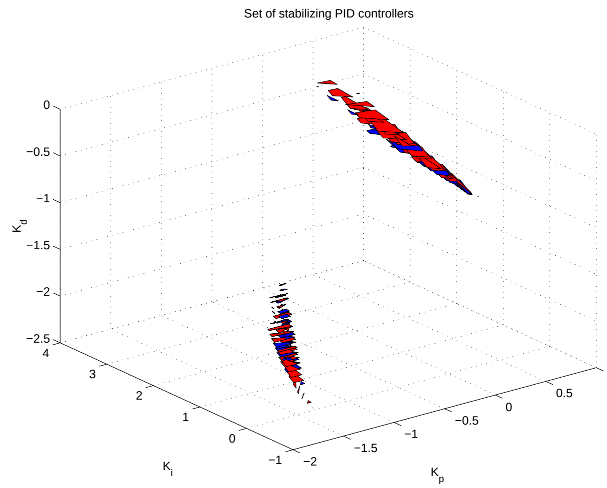


Fig. 4. Set of stabilizing PID controllers - An inner approximation

## III. Conclusions

In this paper, we consider the problem of the synthesis of fixed order and structure controllers, where the coefficients of the closed loop polynomial are linear in the parameters of the controller. A novel feature of this paper is the systematic exploitation of the interlacing property of Hurwitz polynomials and the use of Descartes' rule of signs to generate LPs in the parameters of a fixed order controller. For real stabilization, the feasible set of any LP generated for an inner approximation of the set of all stabilizing controllers, can be indexed by a set of $n-1$ increasing frequencies, $0 = w_0 < w_1 < w_2 < \cdots < w_n$; in particular, any controller in the feasible set of LPs places the roots of the even and odd polynomials of $P(s, K)$ alternately in the intervals $(w_i, w_{i+1})$, $i = 0, \ldots, n-1$. The problem of inner approximation of the set of stabilizing controllers is then posed as the search for all sets of ordered $n-1$-tuples of frequencies for which the associated LP is feasible; the union of all feasible LPs is an inner approximation for the set of all stabilizing controllers. A necessary condition for a polynomial to be Hurwitz is that the roots of even and odd parts of the polynomial have all real, positive and interlacing roots; the Descartes' rule of signs and its generalization due to Poincare is used to generate the LPs. The proposed methodology naturally extends to the computation of the set of simultaneously stabilizing controllers. We provide examples to illustrate some of the results derived in this paper.

The method proposed here is promising although preliminary. A significant advantage of the presented method is that robust stability and performance specifications such as gain and phase margins, and specifications of simultaneous stabilization can be naturally accommodated by imposing further linear inequality constraints. Further research is needed to deal with the existence, performance and robustness of stabilizing controllers of fixed order. It is worth pointing out that the method proposed here directly controls the separation of roots of even and odd parts and therefore has built-in robustness properties.

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## REFERENCES

[1] Ackermann, J., "Robust Control Systems with Uncertain Physical Parameters," *Springer Verlag*, Berlin, 1993.
[2] Anderson, B. D. O., Bose, N.K., and Jury, E. I.,"Output feedback stabilization and related problems - solution via decision methods," *IEEE Transactions on Automatic Control*, AC-20, pp. 53-65, 1975.
[3] Bernstein, D., "Some open problems in matrix theory arising in linear systems and control," *Linear Algebra and Applications*, No.162-164, pp. 409-432, 1992.
[4] Bengtsson, G., and Lindahl, S., "A design scheme for incomplete state or output feedback with applications to boiler and power system control," *Automatica*, no. 10, pp. 15-30, 1974.
[5] Bhattacharyya, S. P., Chapellat, H., and Keel, L. H., "Robust Control: The Parametric Approach," *Prentice-Hall*, Upper Saddle River, New Jersey, 1995.
[6] Bhattacharyya, S. P., Keel, L. H., and Howze, J.,"Stabilizability conditions using linear programming," *IEEE Transactions on Automatic Control*, vol. 33, no.5, pp. 460-463, 1988.
[7] Blondel, V., Gevers, M., and Lindquist, A., "Survey on the state of systems and control," *European Journal of Control*, Vol. 1, pp. 5-23, 1995.
[8] Brasch, F. M., and Pearson, J. B., "Pole placement using dynamic compensator," *IEEE Transactions on Automatic Control*, AC-15, pp. 34-43, February 1970.
[9] Buckley, A., "Hubble Telescope pointing control system design improvement study," *Journal of Guidance, Control and Dynamics*, vol. 18, pp. 194-199, 1995.
[10] Choi, Woosuk, "On the Synthesis of Low Order Stabilizing Controllers," Ph. D. dissertation, Department of Mechanical Engineering, Texas A&M University, College Station, May 2003.
[11] Datta, A., Ho, M. T., and Bhattacharyya, S. P., "Structure and Synthesis of PID Controllers," Springer-Verlag, 2000.
[12] Davison, E., "An automatic way of finding optimal control systems for large multivariable plants," *Proceedings of IFAC Tokyo Symposium on Control*, pp. 357-373, 1965.
[13] Davison, E. J., and Chattarjee, R., "A note on pole assignment in linear systems with incomplete state feedback," *IEEE Transactions on Automatic Control*, AC-16, pp. 98-99, February 1971.
[14] Djaferis, T.E., D.L. Pepyne, and D.M. Cushing, "A New Parameterization of Stable Polynomials," *IEEE Transactions on Automatic Control*, Vol. 47, No. 9, pp. 1546-1550, September 2002.
[15] Dorato, P., "Analytic Feedback System Design: An Interpolation Approach," Brooks Cole Publshing, New York, 2000.
[16] Dorato, P., "Quantified multivariable polynomial inequalities: The mathematics of (almost) all practical design problems," *Sixth IEEE Mediterranean Conference on Control and Systems*, Alghero, Italy, June 9-11, 1998.
[17] El Ghaoui, L., Oustry, F., and AitRami, M., "A Cone Complementarity Linearization Algorithm for Static Output Feedback and Related Problems," *IEEE Transactions on Automatic Control*, vol. 42, no. 8, pp. 1171-1176, August 1997.
[18] Geromel, J. C., de Souza, C. C., and Skelton, R. E., "LMI numerical solution for output feedback stabilization," *Proceedings of IEEE Conference on Decision and Control*, pp. 40-44, December 1995.
[19] Geromel, J. C., and Peres, P. L. D., "Decentralized load-frequency control," *Proceedings of IEE, part D*, vol. 132, pp. 225-230, 1985.
[20] Henrion, D., Sebek, M., and Kucera, V., "Positive Polynomials and Robust Stabilization with Fixed-Order Controllers," *IEEE Transactions on Automatic Control*, Vol. 48, No. 7, pp. 1178-1186, July 2003.
[21] Hyland, D. C., and Bernstein, D. S., "The Optimal Projection Equations for Fixed-Order Dynamic Compensation," *IEEE Transactions on Automatic Control*, AC-29, no. 11, pp. 1034-1037, November 1984.
[22] Iwasaki, T., and Skelton, R. E., "The XY-centering algorithm for the dual LMI problem: A new approach to fixed order control design," *International Journal of Control*, Vol. 62, no. 6, pp. 1257-1272, 1995.
[23] Kimura, H., "Pole Placement by gain output feedback," *IEEE Transactions on Automatic Control*, vol. 20, pp. 509-516, 1975.
[24] Kucera, V., and de Souza, C. E., "A necessary and sufficient condition for output feedback stabilizability," *Automatica*, vol. 31, no. 9, pp. 1357-1359, 1995.
[25] Laguerre, E., Stewart A. Levin (translator),"On the Theory of Numeric Equations," *Journal de Mathematiques pures et appliquees*, $3^e$ serie, t. IX, 1883. (Available at the website of Stewart A. Levin at Stanford University).
[26] Miller, L. F., Cochran, R. G., and Howze, J. W., "Output feedback stabilization by minimization of spectral radius functional," *International Journal of Control*, vol. 27, pp. 455-462, 1978.
[27] Polya, G., and Szego, G., "Problems and Theorems in Analysis II- Theory of Functions, Zeros, Polynomials, Determinants, Number Theory, Geometry," Classics in Mathematics, *Springer Verlag*, Berlin-Heidelberg, 1998.
[28] Rosenthal, J., and Wang, X. A., "Output feedback pole placement with dynamic compensators," *IEEE Transactions on Automatic Control*, vol. 41, pp. 830-843, June 1996.
[29] Siljak, D. D., "Nonlinear Systems: Parameter Analysis and design," Wiley, New York, 1969.
[30] Sirisena, H. R., and Choi, S. S., "Pole Placement in Prescribed Regions of the Complex Plane Using Output Feedback," *IEEE Transactions on Automatic Control*, vol. 20, pg. 810, 1975.
[31] Skelton, R. E., and Iwasaki, T., "Liapunov and Covariance Controllers," *International Journal of Control*, Vol. 57, pp. 519-536, 1993.
[32] Stilwell, D. J., and Bishop, B., "Platoons of Underwater Vehicles - Communication, Feedback and Decentralized Control," *IEEE Control Systems Magazine*, pp. 45-52, December 2000.
[33] Syrmos, V. L., Abdullah, C. T., Dorato, P., Grigoriadis, K., "Static Output Feedback - A Survey," *Automatica*, Vol. 33, no. 2, pp. 125-137, 1997.
[34] Trofino Neto, A., and Kucera, V., "Stabilization via Static Output Feedback," *IEEE Transactions on Automatic Control*, AC-38, pp. 764-765, 1993.
[35] Zhu, G., Grigoriadis, K., and Skelton, R., "Covariance Control Design for the Hubble Space Telescope," *Journal of Guidance, Control and Dynamics*, vol. 18, pp. 230-236, 1995.