Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Conditions for Uniform Solvability of Parameter-Dependent Lyapunov Equations with Applications †

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Abstract: We consider the problem of finding a common quadratic Lyapunov function to demonstrate stability of a family of matrices which incorporate design freedoms. Generically, this can be viewed as picking a family of controller (or observer) gains so that the family of closed-loop system matrices admit a common Lyapunov function. We provide several conditions, necessary and sufficient, for various structures of matrix families. Families of matrices containing a subset of diagonal matrices invariant under the design freedoms is also considered since this is a case that occurs in many applications. Conditions for uniform solvability of the Lyapunov equations are explicitly given and involve inequalities regarding relative magnitudes of terms in the matrices. Motivating applications of the obtained results to observer and controller designs for time-varying, switched, and nonlinear systems are highlighted.

I. Introduction and Problem Statement

We consider the following problem: Given families of matrices $\mathcal{A}(\theta) \in \mathcal{R}^{n\times n}$ and $\mathcal{C}(\theta) \in \mathcal{R}^{1\times n}$ indexed by θ ranging over some set Θ, is it possible to find a family of vectors $\mathcal{G}(\theta) \in \mathcal{R}^{n\times 1}$, a constant symmetric positive definite matrix $P \in \mathcal{R}^{n\times n}$, and a positive constant ν to satisfy, for all $\theta \in \Theta$ the Lyapunov equation

$$P[\mathcal{A}(\theta) + \mathcal{G}(\theta)\mathcal{C}(\theta)] + [\mathcal{A}(\theta) + \mathcal{G}(\theta)\mathcal{C}(\theta)]^T P \quad \leq \quad -\nu I \qquad (1)$$

with I being the identity matrix of appropriate dimensions? If yes, provide an explicit construction.

Families of Lyapunov equations of the form (1) arise in many applications (see Section V). In the literature, solvability of these equations is either taken as an assumption and numerical solutions sought in specific cases or the system class is restricted so that one of the approaches below can be applied. However, since available approaches are predominantly analysis rather than design tools, this restriction could be conservative.

The problem statement above poses essentially a design problem since $\mathcal{G}(\theta)$ is free to be picked by the designer. In applications, this function is usually an observer or a controller gain vector. The associated analysis problem requires finding a matrix $P > 0$ and a positive constant ν to satisfy, for all $\theta \in \Theta$, the Lyapunov equation $P\mathcal{A}(\theta) + \mathcal{A}^T(\theta)P \leq -\nu I$. This analysis problem has been investigated extensively and several results are available in the literature. These results are typically formulated as conditions on the eigenstructure of the matrices $\mathcal{A}(\theta)$, on stability of associated matrix pencils, or on various geometric properties [1, 2, 3, 4]. However, these results do not extend readily to the design problem above. Another class of results in the literature which is inspired by the robust control problem consider the set $\mathcal{A}(\theta)$ to have a *central* element with the radius of the set being bounded in some manner (for instance, by a bound on the singular values of the perturbation matrix, i.e., the difference between $\mathcal{A}(\theta)$ and the central element of the matrix set) so that the solution of the Lyapunov equation with the central element also works for all the matrices in the set [5, 6, 7, 8, 9]. This approach can be extended to obtain a constant vector $\mathcal{G}$ and a matrix $P > 0$ to satisfy (1). However, by the nature of the approach and the robust control application in mind, $\mathcal{G}$ is not allowed to depend on θ leading to conservative conditions. The results given here are therefore of particular interest in applications where design freedoms $\mathcal{G}(\theta)$ are available and allowed to depend on the variable θ. This is the case, for instance, in time-varying systems ($\theta = t$, the time), switched systems (θ is a switching state), and nonlinear systems (θ could be the state or some subset of the state). The obtained results provide extensions in each of these specific areas. For greater applicability, the matrix family is allowed to contain a subset which cannot be influenced by the design freedoms. Some applications of the obtained results are elaborated in Section V. The general framework of our results and some basic results are provided in Section II. Theorems giving sufficient conditions for uniform solvability of the Lyapunov equations are provided in Section III. Necessity theorems are presented in Section IV and conditions under which the imposed assumptions are necessary and sufficient are identified.

II. Definitions and a Basic Theorem

We address the design problem formulated in the introduction for particular structures of $\mathcal{A}(\theta)$ and $\mathcal{C}(\theta)$. Specifically, we consider a partition of $\Theta = \Theta_1 \cup \Theta_2$ with $\mathcal{A}(\theta) = A(\theta)$, $\mathcal{G}(\theta) = G(\theta)$, $\mathcal{C}(\theta) = C(\theta)$ if $\theta \in \Theta_1$ and $\mathcal{A}(\theta) = -D(\theta)$, $\mathcal{G}(\theta) = 0$, $\mathcal{C}(\theta) = 0$ if $\theta \in \Theta_2$. This partition of Θ corresponds to some elements of the matrix family $\mathcal{A}(\theta)$ that can not be modified by the design functions. The motivation for considering this problem is seen in the applications to scaling-based design and λ-tracking discussed in Section V. Equivalently, we consider the problem: Given families of matrices $A(\theta) \in \mathcal{R}^{n\times n}, D(\theta) \in \mathcal{R}^{n\times n}$, and $C(\theta) \in \mathcal{R}^{1\times n}$, find $G(\theta) \in \mathcal{R}^{n\times 1}$, a constant symmetric positive definite matrix $P \in \mathcal{R}^{n\times n}$, and positive constants ν_1 and ν_2 such that for all $\theta \in \Theta$

$$P[A(\theta) + G(\theta)C(\theta)] + [A(\theta) + G(\theta)C(\theta)]^T P \leq -\nu_1 I \qquad (2)$$

$$PD(\theta) + D^T(\theta)P \geq \nu_2 I. \qquad (3)$$

Definition 1: A triple (A, C, D) which satisfies (2) and (3) with some G, P, ν_1, and ν_2 is said to have property

†This work is supported in part by the NSF under grant ECS-9977693.

$\mathcal{M}$ with the $\mathcal{M}$-associated 4-tuple (G, P, ν_1, ν_2). $\Box$

The problem posed above is in the "observer design" context and the dual "controller design" problem can be formulated as follows: Given families of matrices $A(\theta) \in \mathcal{R}^{n\times n}, D(\theta) \in \mathcal{R}^{n\times n}$, and $B(\theta) \in \mathcal{R}^{n\times 1}$, find $H(\theta) \in \mathcal{R}^{1\times n}$, a constant symmetric positive definite matrix $P \in \mathcal{R}^{n\times n}$, and positive constants ν_1 and ν_2 such that for all $\theta \in \Theta$

$$P[A(\theta) + B(\theta)H(\theta)] + [A(\theta) + B(\theta)H(\theta)]^T P \leq -\nu_1 I \quad (4)$$

$$PD(\theta) + D^T(\theta)P \geq \nu_2 I. \quad (5)$$

Definition 2: A triple (A, B, D) which satisfies (4) and (5) with some H, P, ν_1, and ν_2 is said to have property $\mathcal{M}_d$ with the $\mathcal{M}_d$-associated 4-tuple (H, P, ν_1, ν_2). $\Box$

Theorem 1: Let $Q \in \mathcal{R}^{n\times n}$ be a nonsingular matrix with $\lambda_{max}(QQ^T) = \overline{Q}$. If the triple (A, C, D) has property $\mathcal{M}$ with $\mathcal{M}$-associated 4-tuple (G, P, ν_1, ν_2), then
(1.1) triple $(QAQ^{-1}, CQ^{-1}, QDQ^{-1})$ has property $\mathcal{M}$ with $\mathcal{M}$-associated 4-tuple $(QG, Q^{-T}PQ^{-1}, \frac{\nu_1}{\overline{Q}}, \frac{\nu_2}{\overline{Q}})$,
(1.2) triple (A^T, C^T, D^T) has property $\mathcal{M}_d$ with $\mathcal{M}_d$-associated 4-tuple $(G^T, P^{-1}, \frac{\nu_1}{\lambda_{max}(P^2)}, \frac{\nu_2}{\lambda_{max}(P^2)})$.

If the triple (A, B, D) has property $\mathcal{M}_d$ with $\mathcal{M}_d$-associated 4-tuple (H, P, ν_1, ν_2), then
(1.3) triple (QAQ^{-1}, QB, QDQ^{-1}) has property $\mathcal{M}_d$ with $\mathcal{M}_d$-associated 4-tuple $(HQ^{-1}, Q^{-T}PQ^{-1}, \frac{\nu_1}{\overline{Q}}, \frac{\nu_2}{\overline{Q}})$,
(1.4) triple (A^T, B^T, D^T) has property $\mathcal{M}$ with $\mathcal{M}$-associated 4-tuple $(H^T, P^{-1}, \frac{\nu_1}{\lambda_{max}(P^2)}, \frac{\nu_2}{\lambda_{max}(P^{-2})})$.

Proof of Theorem 1: By standard matrix algebra. $\diamond$

In Sections III and IV, we consider particular structures of matrix families. By Theorem 1, the obtained results are applicable if the required structures are attained under any of the transformations considered in Theorem 1. Theorem 1 also allows theorems stated in the observer design context to be mapped to the dual controller design context and vice versa. In the rest of this paper, we primarily address the observer design problem, i.e., solution of (2) and (3). Dual sufficiency results are summarized in Corollary 1. Duals of the necessity theorems can be obtained along the same lines.

III. Sufficiency Theorems

Theorem 2: Let $\phi_{(i,j)} : \Theta \mapsto \mathcal{R}$ and $D_i : \Theta \mapsto \mathcal{R}$, $i = 1, \ldots, n, j = 1, \ldots, i+1$. Let $A(\theta)$ be the $n \times n$ matrix with $(i,j)^{\text{th}}$ element

$$A_{(i,j)}(\theta) = \phi_{(i,j)}(\theta), \ i = 1, \ldots, n, j = 1, \ldots, i+1$$
$$A_{(i,j)}(\theta) = 0, \ i = 1, \ldots, n-2, j = i+2, \ldots, n. \quad (6)$$

Let $C(\theta) = C = [1, 0, \ldots, 0]$, and $D(\theta) = \text{diag}(D_1(\theta), \ldots, D_n(\theta))$. If each $\phi_{(i,i+1)}(\theta)$ assumes the same sign for all arguments θ, and positive constants σ, $\rho_{iA}, i = 2, \ldots, n-1$, $\epsilon_{(i,j)}, j = 2, \ldots, i, i = 2, \ldots, n$, $\underline{\rho}_{iD}, i = 1, \ldots, n-1$, $\overline{\rho}_{iD}, i = 1, \ldots, n-1$, and $\underline{\rho}^o_{iD}$, $i = 1, \ldots, n$, exist such that for all $\theta \in \Theta$,

$$|\phi_{(i,i+1)}(\theta)| \geq \sigma > 0 \quad, 1 \leq i \leq n-1$$
$$|\phi_{(i,i+1)}(\theta)| \leq \rho_{iA}|\phi_{(i-1,i)}(\theta)| \quad, 2 \leq i \leq n-1$$
$$|\phi_{(i,j)}(\theta)| \leq \epsilon_{(i,j)}\sqrt{|\phi_{(i,i+1)}(\theta)||\phi_{(j-1,j)}(\theta)|}, \quad 2 \leq i \leq n-1, 2 \leq j \leq i$$
$$|\phi_{(n,j)}(\theta)| \leq \epsilon_{(n,j)}\sqrt{|\phi_{(n-1,n)}(\theta)||\phi_{(j-1,j)}(\theta)|}, \quad 2 \leq j \leq n \quad (7)$$

and

$$D_i(\theta) \geq \underline{\rho}^o_{iD}, \ i = 1, \ldots, n$$
$$\underline{\rho}_{iD} D_i(\theta) \leq D_{i+1}(\theta) \leq \overline{\rho}_{iD} D_i(\theta), \ i = 1, \ldots, n-1, \quad (8)$$

then the triple (A, C, D) has property $\mathcal{M}$.

Remark 1: The conditions (7) are a weaker version of the *Cascading Upper Diagonal Dominance* (CUDD) conditions introduced in [10, 11] which require the last two equations in (7) to be strengthened to

$$|\phi_{(i,j)}(\theta)| \leq \epsilon_{(i,j)}|\phi_{(i,i+1)}(\theta)| \quad, 2 \leq i \leq n-1, 2 \leq j \leq i$$
$$|\phi_{(n,j)}(\theta)| \leq \epsilon_{(n,j)}|\phi_{(n-1,n)}(\theta)| \quad, 2 \leq j \leq n. \quad (9)$$

A matrix A of structure (6) and satisfying inequalities (7) is called a *weakly Cascading Upper Diagonal Dominant* (w-CUDD) matrix. The positive constants $\rho_{iA}, i = 2, \ldots, n-1$, and $\epsilon_{(i,j)}, i = 2, \ldots, n, j = 2, \ldots, i$, are referred to as the w-CUDD constants of A.

Theorem 2 is proved through a sequence of lemmas below. Lemma 1 identifies a family of matrices P that satisfy (3). In Lemma 2, the existence of a matrix in this family that also satisfies (2) is shown if the w-CUDD constants of A are small enough. Lemma 3 shows that a diagonal transformation can scale the w-CUDD constants. The proof of Theorem 2 is concluded by noting that D is invariant under a diagonal transformation.

Lemma 1: Let[1] $\hat{P} = I - \text{lowerdiag}([k_2, \ldots, k_n])$, $P = \hat{P}^T\hat{P}$ with $k_2, \ldots, k_n$ being constants and let D be as defined in Theorem 2. Assume that (8) is satisfied with positive constants $\underline{\rho}_{iD}$, $\overline{\rho}_{iD}$, and $\underline{\rho}^o_{iD}$. Then, a positive constant κ exists such that if $|k_i| < \kappa, i = 2, \ldots, n$, then (3) holds with some positive constant ν_2.

Proof of Lemma 1: $D = D_1(\theta)\hat{D}(\theta)$ where $\hat{D}(\theta) = \text{diag}(\hat{D}_1(\theta), \hat{D}_2(\theta), \ldots, \hat{D}_n(\theta))$ with $\hat{D}_i(\theta) = \frac{D_i(\theta)}{D_1(\theta)}$. The inequality $P\hat{D} + \hat{D}P \geq \tilde{\nu}_2 I$ holds with some $\tilde{\nu}_2 > 0$ if $|k_i| < \kappa$ with small enough κ. This is seen by continuity since this inequality holds when $k_i = 0, i = 2, \ldots, n$. Note that $\hat{D}_i(\theta)$ are bounded below and above by the positive constants $\prod_{j=1}^{i-1} \underline{\rho}_{jD}$ and $\prod_{j=1}^{i-1} \overline{\rho}_{jD}$, respectively. An estimate (possibly conservative) for κ can be written using the diagonal dominance condition as

$$\kappa = \inf_{\theta\in\Theta} \min\left\{\frac{2\hat{D}_i(\theta)}{\hat{D}_{i+1}(\theta) + 2\hat{D}_i(\theta) + \hat{D}_{i-1}(\theta)} ; 1 \leq i \leq n\right\} \quad (10)$$

where $\hat{D}_0 = -\hat{D}_1, \hat{D}_{n+1} = -\hat{D}_n$.. If $|k_i| < \kappa, i = 2, \ldots, n$, then $PD + DP \geq \nu_2 I$ with $\nu_2 = \underline{\rho}^o_{1D}\tilde{\nu}_2$. $\diamond$

Lemma 2: Let A and C be as defined in Theorem 2. Given any constant $\kappa > 0$, a constant $\overline{\rho}_A > 0$ whose choice depends only on κ and n exists such that the following property is true: If (7) holds with some $\sigma > 0$ and with w-CUDD constants ρ_{iA} and $\epsilon_{(i,j)}$ smaller than $\overline{\rho}_A$, then constants $k_2, \ldots, k_n$ smaller in magnitude than κ and $G(\theta) = [g_1(\theta), \ldots, g_n(\theta)]^T$ exist such that (2) is satisfied for some $\nu_1 > 0$ with P defined as in Lemma 1.

Proof of Lemma 2: Consider the dynamical system $\dot{\omega} = A_c\omega, \omega = [\omega_1, \ldots, \omega_n]^T \in \mathcal{R}^n, A_c = A + GC$. De-

[1] lowerdiag($[k_2, \ldots, k_n]$) denotes the $n \times n$ matrix with lower diagonal elements $k_2, \ldots, k_n$ and zeros elsewhere.

fine the change of coordinates $\Omega = \hat{P}\omega$ with the inverse transformation[2] $\omega_i = \sum_{l=1}^{i}\Omega_l \prod_{m=l+1}^{i} k_m = \sum_{l=1}^{i} p_k(l+1,i)\Omega_l$, where $p_k(n_1,n_2) \triangleq \prod_{m=n_1}^{n_2} k_m$. Note that $p_k(n_1,n_2) = 1$ if $n_1 > n_2$. For convenience, we introduce the dummy variables $\omega_{n+1} = \Omega_{n+1} = k_{n+1} = 0$ and $\phi_{(n+1,j)} = 0, 1 \le j \le n+1$. Differentiating $\Omega_i = \omega_i - k_i\omega_{i-1}$, $2 \le i \le n$,

$$\begin{aligned}\dot{\Omega}_i &= (g_i - k_i g_{i-1} + \tilde{g}_i)\Omega_1 + \phi_{(i,i+1)}\Omega_{i+1} \\ &\quad +(\phi_{(i,i)} - k_i\phi_{(i-1,i)} + k_{i+1}\phi_{(i,i+1)})\Omega_i \\ &\quad +(\phi_{(i,i)} - k_i\phi_{(i-1,i)} + k_{i+1}\phi_{(i,i+1)})\sum_{l=2}^{i-1} p_k(l+1,i)\Omega_l \\ &\quad +\sum_{j=2}^{i-1}\{(\phi_{(i,j)} - k_i\phi_{(i-1,j)})\sum_{l=2}^{j} p_k(l+1,j)\Omega_l\} \end{aligned} \quad (11)$$

$$\begin{aligned}\tilde{g}_i &= \phi_{(i,1)} - k_i\phi_{(i-1,1)} + \sum_{j=2}^{i-1}(\phi_{(i,j)} - k_i\phi_{(i-1,j)})p_k(2,j) \\ &\quad +(\phi_{(i,i)} - k_i\phi_{(i-1,i)} + k_{i+1}\phi_{(i,i+1)})p_k(2,i). \end{aligned} \quad (12)$$

Note that $|k_i|, i = 2,\ldots,n$, are required to be chosen smaller than κ if ρ_{iA} and $\epsilon_{(i,j)}$ are smaller than $\overline{\rho}_A$. The final construction of k_i and $\overline{\rho}_A$ will ensure these inequality conditions. We derive a bound below for $\sum_{i=2}^{n}\Omega_i\dot{\Omega}_i$ assuming these inequalities. Furthermore, since the positive constant $\overline{\rho}_A$ is free to be picked, assume $\overline{\rho}_A < 1$.

Using the w-CUDD conditions in (7), for $2 \le i \le n$,

$$\begin{aligned}\Omega_i\dot{\Omega}_i &\le \frac{\kappa}{4}|\phi_{(i,i+1)}|\Omega_{i+1}^2 - k_i\phi_{(i-1,i)}\Omega_i^2 \\ &\quad +[\kappa\overline{\rho}_A + \overline{\rho}_A + \frac{\overline{\rho}_A}{\kappa} + \frac{5\kappa}{8}]|\phi_{(i-1,i)}|\Omega_i^2 \\ &\quad +\sum_{l=2}^{i-1} q_{(i,l)}|\phi_{(l-1,l)}|\Omega_l^2 + \frac{2}{\kappa}\frac{(g_i - k_i g_{i-1} + \tilde{g}_i)^2}{|\phi_{(i-1,i)}|}\Omega_1^2 \end{aligned} \quad (13)$$

where, for $i = 2,\ldots,n$, and $l = 2,\ldots,i-1$,

$$\begin{aligned}q(i,l) &= \frac{(i-2)(\phi_{(i,i)} - k_i\phi_{(i-1,i)} + k_{i+1}\phi_{(i,i+1)})^2}{\kappa|\phi_{(i-1,i)}||\phi_{(l-1,l)}|}[p_k(l+1,i)]^2 \\ &\quad +\left.\sum_{j=l}^{i-1}\frac{(j-1)(i-2)(\phi_{(i,j)} - k_i\phi_{(i-1,j)})^2}{\kappa|\phi_{(i-1,i)}||\phi_{(l-1,l)}|}[p_k(l+1,j)]^2\right] \\ &\le \frac{(n-2)\overline{\rho}_A}{\kappa}\left[(1+2\kappa)^2\kappa^{2(i-l)}\right. \\ &\quad \left. +(n-1)(1+\kappa)^2\frac{\kappa^{2(i-l)} - 1}{\kappa^2 - 1}\right]. \end{aligned} \quad (14)$$

Pick any positive constant $\kappa^* \le \frac{\kappa}{8}$ and choose $\overline{\rho}_A$ to be a positive constant satisfying

$$\begin{aligned}\overline{\rho}_A &\le \left(\frac{\kappa}{8} - \kappa^*\right)\Big/\Big(\kappa + 1 + \frac{1}{\kappa} \\ &\quad +\frac{(n-2)}{\kappa}\sum_{l=i+1}^{n}\Big[(1+2\kappa)^2\kappa^{2(i-l)} \\ &\quad +(n-1)(1+\kappa)^2\frac{\kappa^{2(i-l)} - 1}{\kappa^2 - 1}\Big]\Big). \end{aligned} \quad (15)$$

Since $\overline{\rho}_A < 1$, (13) and (14) are valid. (13) reduces to

$$\begin{aligned}\sum_{i=2}^{n}\Omega_i\dot{\Omega}_i &\le -\sum_{i=2}^{n} k_i\phi_{(i-1,i)}\Omega_i^2 + \sum_{i=2}^{n}(\kappa - \kappa^*)|\phi_{(i-1,i)}|\Omega_i^2 \\ &\quad +\frac{2}{\kappa}\sum_{i=2}^{n}\frac{(g_i - k_i g_{i-1} + \tilde{g}_i)^2}{|\phi_{(i-1,i)}|}\Omega_1^2. \end{aligned} \quad (16)$$

Also,

$$\Omega_1\dot{\Omega}_1 \le (g_1 + \tilde{g}_1 + \frac{1}{\kappa^*}|\phi_{(1,2)}|)\Omega_1^2 + \frac{\kappa^*}{4}|\phi_{(1,2)}|\Omega_2^2 \quad (17)$$

$$\tilde{g}_1 = \phi_{(1,1)} + k_2\phi_{(1,2)}. \quad (18)$$

Pick[3] $k_i = (\kappa - \kappa^*/4)\mathrm{sign}(\phi_{(i-1,i)}(0))$, $i = 2,\ldots,n$. Defining G such that[4]

$$G = [g_1,\ldots,g_n]^T = -\hat{P}^{-1}\left[\frac{2}{\kappa}\sum_{i=2}^{n}\frac{\overline{g}_i^2}{|\phi_{(i-1,i)}|} + \overline{g}_1, 0,\ldots,0\right]^T \quad (19)$$

$$\begin{aligned}&\overline{g}_1 \ge |\phi_{(1,1)}| + |k_2||\phi_{(1,2)}| + \frac{1}{\kappa^*}|\phi_{(1,2)}| + \frac{\kappa^*}{2}|\phi_{(1,2)}| \\ &\overline{g}_i \ge (|\phi_{(i,i)}| + |k_i||\phi_{(i-1,i)}| + |k_{i+1}||\phi_{(i,i+1)}|)|p_k(2,i)| + |\phi_{(i,1)}| \\ &\quad +|k_i||\phi_{(i-1,1)}| + \sum_{j=2}^{i-1}(|\phi_{(i,j)}| + |k_i||\phi_{(i-1,j)}|)|p_k(2,j)|, \end{aligned} \quad (20)$$

$i = 2,\ldots,n$, we finally obtain

$$\frac{d}{dt}(\frac{1}{2}\sum_{i=1}^{n}\Omega_i^2) \le -\frac{\kappa^*}{2}\sum_{i=2}^{n}|\phi_{(i-1,i)}|\Omega_i^2 - \frac{\kappa^*}{2}|\phi_{(1,2)}|\Omega_1^2 \quad (21)$$

so that (2) is satisfied with $\nu_1 = \kappa^*\sigma\lambda_{min}(P)$. ⋄

Lemma 3: Let A be as defined in (6) and let (7) be satisfied with some positive constants σ, ρ_{iA}, and $\epsilon_{(i,j)}$. Given any positive constant $\overline{\rho}_A$, a constant diagonal matrix T exists such that the matrix $\tilde{A} = TAT^{-1}$ is a w-CUDD matrix with w-CUDD constants smaller than $\overline{\rho}_A$.

Proof of Lemma 3: If the w-CUDD constants of A are smaller than $\overline{\rho}_A$, pick $T = I$. Otherwise, let

$$\begin{aligned}\rho_0 = \frac{1}{\overline{\rho}_A}\max\Big\{&\max\{\rho_{iA}|i = 2,\ldots,n-1\}, \\ &\max\{\epsilon_{i,j}|i = 2,\ldots,n, j = 2,\ldots,i\}\Big\} \end{aligned} \quad (22)$$

$$T = \mathrm{diag}(T_1,\ldots,T_n), T_i = \rho_0^{(n-i)(n-i+1)/2}. \quad (23)$$

The $(i,j)^{\text{th}}$ element of $\tilde{A}$ is $\tilde{A}_{(i,j)} = (T_i/T_j)\phi_{(i,j)}, i = 1,\ldots,n, j = 1,\ldots,i+1$, with zeros elsewhere. Using (22), it can be seen that $\tilde{A}$ is a w-CUDD matrix with w-CUDD constants smaller than $\overline{\rho}_A$. ⋄

Proof of Theorem 2: Given $A(\theta)$ and $D(\theta)$, obtain $\kappa, \overline{\rho}_A$, and T as in Lemmas 1, 2, and 3, respectively. Defining $\tilde{A} = TAT^{-1}$, $(\tilde{A}, C, D)$ has property $\mathcal{M}$ with $\mathcal{M}$-associated 4-tuple $(\tilde{G}, \tilde{P}, \tilde{\nu}_1, \tilde{\nu}_2)$ constructed as in the proofs of Lemmas 1 and 2. Using statement (1.1) in Theorem 1 and $T^{-1}DT = D$, the triple (A, CT, D) has property $\mathcal{M}$ with $\mathcal{M}$-associated 4-tuple $(T^{-1}\tilde{G}, T\tilde{P}T, \tilde{\nu}_1\lambda_{min}(T^2), \tilde{\nu}_2\lambda_{min}(T^2))$. Since $CT = T_1C$, the triple (A, C, D) has property $\mathcal{M}$ with the $\mathcal{M}$-associated 4-tuple $(T_1T^{-1}\tilde{G}, T\tilde{P}T, \tilde{\nu}_1\lambda_{min}(T^2),$

[2] By convention, $\sum_{i=n_1}^{n_2} f_i = 0$ and $\prod_{i=n_1}^{n_2} f_i = 1$ if $n_1 > n_2$.

[3] By assumption, $\phi_{(i-1,i)}(\theta)$ assumes the same sign for all $\theta \in \Theta$.

[4] The choice of G in (19) depends only on the upper diagonal entries and upper bounds on other elements of A.

$\tilde{\nu}_2\lambda_{min}(T^2))$. $\diamond$

Theorem 3: Let A, C, and D be as defined in Theorem 2. Let $\Phi : \Theta \times \mathcal{R}^n \mapsto \mathcal{R}^n$ be a function such that[5] $|\Phi(\theta,\omega)|_e \leq_e \Gamma(\theta)|\omega|_e$ for all $\theta \in \Theta$ and $\omega \in \mathcal{R}^n$, with $\Gamma(\theta)$ being a lower triangular $n\times n$ matrix function with nonnegative entries. If A and $A+\Gamma$ are w-CUDD, then $G(\theta)$, a matrix $P>0$, and positive constants ν_1 and ν_2 exist such that for all $\theta \in \Theta$ and $\omega \in \mathcal{R}^n$,

$$\omega^T\{P[A(\theta)+G(\theta)C]+[A(\theta)+G(\theta)C]^TP\}\omega + 2\omega^TP\Phi(\theta,\omega) \leq -\nu_1|\omega|^2 \qquad (24)$$

$$PD(\theta)+D(\theta)P \geq \nu_2 I. \qquad (25)$$

Proof of Theorem 3: We have the inequality $\omega^TP\Phi \leq |\omega^TP|_e|\Phi|_e \leq |\omega^TP|_e\Gamma|\omega|_e = \omega^TPQ_1\Gamma Q_2\omega$ where Q_1 and Q_2 are appropriate diagonal matrices with 1 or -1 as each diagonal entry. Let $\mathcal{Q} : \{1,2,\ldots,2^n\} \mapsto \mathcal{R}^{n\times n}$ be an enumeration of the 2^n diagonal matrices of dimension $n\times n$ with each diagonal entry 1 or -1. Consider $\theta' = (\theta,q_1,q_2) \in \Theta\times\{1,\ldots,2^n\}\times\{1,\ldots,2^n\}$. If $\tilde{A}(\theta') = A(\theta)+\mathcal{Q}(q_1)\Gamma(\theta)\mathcal{Q}(q_2)$ is w-CUDD, then using Theorem 2, the conclusion of Theorem 3 follows. Note that G can be taken to be a function of θ and not θ' since $|Q_1\Gamma Q_2|_e = |\Gamma|_e$ and the choice of G in (19) depends only on the upper diagonal elements of $\tilde{A}$ (which are equal to the upper diagonal elements of A) and upper bounds on absolute values of the other elements. Noting that $\mathcal{Q}(q_1)$ and $\mathcal{Q}(q_2)$ vary over all the 2^n diagonal matrices with each diagonal entry 1 or -1, it is seen that the necessary and sufficient condition for $\tilde{A}$ to be w-CUDD (i.e., that $\phi_{(i,j)}+\Gamma_{(i,j)}$ and $\phi_{(i,j)}-\Gamma_{(i,j)}$ should both satisfy the w-CUDD bounds) is that A and $A+\Gamma$ are w-CUDD. $\diamond$

Corollary 1: (*A corollary to Theorems 2 and 3*) Let $A(\theta)$ be as in (6). Let $D(\theta) = \mathrm{diag}(D_1(\theta),\ldots,D_n(\theta))$ and $B(\theta) = B = [0\ldots,0,1]^T$. If positive constants σ, $\rho_{iA}, i=2,\ldots,n-1$, $\epsilon_{(i,j)}, i=1,\ldots,n-1, j=1,\ldots,i$, $\underline{\rho}_{iD}, i=1,\ldots,n-1$, $\overline{\rho}_{iD}, i=1,\ldots,n-1$, and $\underline{\rho}^o_{iD}, i=1,\ldots,n$, exist such that[6] for all $\theta\in\Theta$,

$$\begin{aligned}
|\phi_{(i,i+1)}(\theta)| &\geq \sigma>0 \quad ,1\leq i\leq n-1\\
|\phi_{(i,i+1)}(\theta)| &\geq \rho_{iA}|\phi_{(i-1,i)}(\theta)| \quad ,2\leq i\leq n-1\\
|\phi_{(i,j)}(\theta)| &\leq \epsilon_{(i,j)}\sqrt{|\phi_{(i,i+1)}(\theta)||\phi_{(j-1,j)}(\theta)|},\\
&\quad 2\leq i\leq n-1, 2\leq j\leq i\\
|\phi_{(i,1)}(\theta)| &\leq \epsilon_{(i,1)}\sqrt{|\phi_{(i,i+1)}(\theta)||\phi_{(1,2)}(\theta)|},\\
&\quad 1\leq i\leq n-1,
\end{aligned} \qquad (26)$$

and (8) are satisfied, then the triple (A,B,D) has property $\mathcal{M}_d$. Furthermore, let $\Phi(\theta,\omega)$ be as defined in Theorem 3. If A and $A+\Gamma$ are dual w-CUDD, then $H(\theta)$, a matrix $P>0$, and positive constants ν_1 and ν_2 exist such that for all $\theta\in\Theta$ and $\omega\in\mathcal{R}^n$,

$$\omega^T\{P[A(\theta)+BH(\theta)]+[A(\theta)+BH(\theta)]^TP\}\omega + 2\omega^TP\Phi(\theta,\omega) \leq -\nu_1|\omega|^2 \qquad (27)$$

$$PD(\theta)+D(\theta)P \geq \nu_2 I. \qquad (28)$$

[5] $|\beta|_e$ denotes a matrix of the same dimension as β obtained by replacing each element of β by its absolute value. The relation $\leq_e$ between two vectors denotes an element-wise inequality.

[6] Conditions (26) are dual to the w-CUDD conditions (7). A matrix of form (6) satisfying (26) is referred to as a *dual w-CUDD* matrix with the dual w-CUDD constants ρ_{iA} and $\epsilon_{(i,j)}$.

Proof of Corollary 1: Define the matrices $\tilde{A} = QA^TQ$ and $\tilde{D} = QDQ$ with Q being the $n\times n$ matrix with 1's on the anti-diagonal, i.e., $Q_{(i,n-i+1)} = 1, i = 1,\ldots,n$, with zeros elsewhere. $\tilde{A}$ is of form (6) with $(i,j)^{th}$ element $\tilde{A}_{(i,j)} = A_{(n-j+1,n-i+1)}$. Using (26), it is seen that $\tilde{A}$ is w-CUDD. $\tilde{D}$ is a diagonal matrix with $(i,i)^{th}$ element D_{n-i+1}. Using Theorem 2, the triple $(\tilde{A},C,\tilde{D})$ where $C=[1,0,\ldots,0]$ has property $\mathcal{M}$ with an $\mathcal{M}$-associated 4-tuple $(G,\tilde{P},\tilde{\nu}_1,\tilde{\nu}_2)$. Using statements (1.2) and (1.3) of Theorem 1, and noting that $Q^{-1}=Q$ and $QC^T = B$, the triple (A,B,D) has property $\mathcal{M}_d$ with $\mathcal{M}_d$-associated 4-tuple $(G^TQ, Q\tilde{P}^{-1}Q, \tilde{\nu}_1\lambda_{min}(\tilde{P}^{-2}), \tilde{\nu}_2\lambda_{min}(\tilde{P}^{-2}))$. The second assertion in Corollary 1, i.e., solvability of (27) and (28), can be proved similarly using duality and the arguments in the proof of Theorem 3. $\diamond$

IV. Necessity Theorems

In this section, the conservativeness of the assumptions in Theorem 2 is evaluated by formulating necessary conditions for solvability of the coupled Lyapunov equations (2) and (3). In Theorem 4, negative definite submatrices of $P(A+GC)+(A+GC)^TP$ which are invariant under the design freedoms G are identified. This fundamental result in Theorem 4 is applied in Theorem 5 to prove that a necessary condition for solvability of (2) is that if the elements of A off the upper diagonal are smaller than the upper diagonal elements in a sense similar to the last two equations in (7), then the upper diagonal elements must satisfy the first two equations of (7). Lemma 4 further shows that the invariance of the signs of $\phi_{(i,i+1)}(\theta)$ is also necessary if the elements of A off the upper diagonal are zero. Theorem 6 shows that the assumed conditions on D are necessary for solvability of (3). Corollary 2 combines these results to show that the imposed assumptions are necessary and sufficient if A has nonzero entries only on the upper diagonal.

Theorem 4: Let $C=[1,0,\ldots,0]$. If a matrix A of form (6) satisfies (2) with some $G(\theta)$, a constant matrix $P>0$, and a constant $\nu_1>0$, then the following holds for each $i\in\{1,\ldots,n\}$: Let A_i be the matrix obtained by deleting the first i rows and the first i columns of A. Let $C_i = [1,0,\ldots,0]^T\in\mathcal{R}^{n-i}$. Then, a constant vector $K_i = [k_{1_i},\ldots,k_{n-i_i}]^T$, a constant $(n-i)\times(n-i)$ matrix $P_i>0$, and a constant $\tilde{\nu}_i>0$ exist such that $\forall\theta\in\Theta$,

$$P_i[A_i(\theta)+K_iC_i\phi_{(i,i+1)}(\theta)] + [A_i(\theta)+K_iC_i\phi_{(i,i+1)}(\theta)]^TP_i \leq -\tilde{\nu}_iI. \qquad (29)$$

Proof of Theorem 4: Assume that (2) is satisfied with some G, P, and ν_1. Consider partitions

$$A+GC = \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_i\end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{21}^T\\ P_{21} & P_i\end{bmatrix}. \qquad (30)$$

Note that $A+GC$ differs from A only in the first column. The $(2,2)$ block of $P(A+GC)+(A+GC)^TP$ is $P_iA_i + A_i^TP_i + P_{21}A_{12} + A_{12}^TP_{21}^T$. The $i\times(n-i)$ matrix A_{12} has all elements zero except the $(i,1)$ element which is equal to $\phi_{(i,i+1)}$. Hence, $P_i^{-1}P_{21}A_{12} = K_iC_i\phi_{(i,i+1)}$ with K_i being a constant $(n-i)\times 1$ vector. Noting that $P_iA_i+A_i^TP_i+P_{21}A_{12}+A_{12}^TP_{21}^T = P_i[A_i+K_iC_i\phi_{(i,i+1)}] + [A_i+K_iC_i\phi_{(i,i+1)}]^TP_i$ and that the $(2,2)$ block being on the principal diagonal must be negative definite for (2) to hold, the theorem follows. $\diamond$

Theorem 5: Let A be of form (6) and $C = [1, 0, \dots, 0]$. Let positive constants $\epsilon_{(i,j)}, i = 2, \dots, n, j = 2, \dots, i$, and $\overline{\epsilon}_{(i,j)}, i = 3, \dots, n, j = 3, \dots, i$, exist such that $\forall \theta \in \Theta$,

$$|\phi_{(i,j)}(\theta)| \le \epsilon_{(i,j)} \sqrt{|\phi_{(i,i+1)}(\theta)||\phi_{(j-1,j)}(\theta)|}\,, 2 \le i \le n{-}1, 2 \le j \le i$$
$$|\phi_{(n,j)}(\theta)| \le \epsilon_{(n,j)} \sqrt{|\phi_{(n-1,n)}(\theta)||\phi_{(j-1,j)}(\theta)|}\,, 2 \le j \le n$$
$$|\phi_{(i,j)}(\theta)| \le \overline{\epsilon}_{(i,j)} \sqrt{|\phi_{(i,i+1)}(\theta)||\phi_{(j-2,j-1)}(\theta)|}, 3 \le i \le n{-}1, 3 \le j \le i$$
$$|\phi_{(n,j)}(\theta)| \le \overline{\epsilon}_{(n,j)} \sqrt{|\phi_{(n-1,n)}(\theta)||\phi_{(j-2,j-1)}(\theta)|}\,, 3 \le j \le n. \quad (31)$$

If (2) is satisfied with some $G(\theta)$, a constant positive definite matrix P, and a positive constant ν_1, then positive constants σ and $\rho_{iA}, i = 2, \dots, n-1$, exist such that

$$\begin{aligned} |\phi_{(i,i+1)}(\theta)| &\ge \sigma > 0 \quad ,1 \le i \le n-1 \\ |\phi_{(i,i+1)}(\theta)| &\le \rho_{iA} |\phi_{(i-1,i)}(\theta)| \quad ,2 \le i \le n-1. \end{aligned} \quad (32)$$

Proof of Theorem 5: Applying Theorem 4 with $i = n{-}1$, the existence of positive constants P_{n-1} and $\tilde{\nu}_{n-1}$ and a constant $k_{1_{n-1}}$ are inferred such that $2P_{n-1}[\phi_{(n,n)} + k_{1_{n-1}} \phi_{(n-1,n)}] \le -\tilde{\nu}_{n-1}$. Noting that, by assumption, $|\phi_{(n,n)}| \le \epsilon_{(n,n)} |\phi_{(n-1,n)}|$, it is seen that a positive constant σ_n exists such that $|\phi_{(n-1,n)}| \ge \sigma_n$. Applying Theorem 4 with $i = n-2$, a 2×2 matrix $P_{n-2} > 0$, a 2×1 vector K_{n-2}, and a positive constant $\tilde{\nu}_{n-2}$ are obtained such that $P_{n-2}[A_{n-2} + K_{n-2} C_{n-2} \phi_{(n-2,n-1)}] + [A_{n-2} + K_{n-2} C_{n-2} \phi_{(n-2,n-1)}]^T P_{n-2} \le -\tilde{\nu}_{n-2} I$. Denote the $(j,k)^{th}$ element of P_i by $P_{(j,k)_i}$ and the j^{th} element of vector K_i by k_{j_i}. Since the determinant of $P_{n-2}[A_{n-2} + K_{n-2} C_{n-2} \phi_{(n-2,n-1)}] + [A_{n-2} + K_{n-2} C_{n-2} \phi_{(n-2,n-1)}]^T P_{n-2}$ must be positive, we have

$$\begin{aligned} 0 \ < \ & -[P_{(1,2)_{n-2}} \phi_{(n-1,n-1)} + P_{(2,2)_{n-2}} \phi_{(n,n-1)} \\ & +(P_{(1,2)_{n-2}} k_{1_{n-2}} + P_{(2,2)_{n-2}} k_{2_{n-2}}) \phi_{(n-2,n-1)} \\ & +P_{(1,1)_{n-2}} \phi_{(n-1,n)} + P_{(1,2)_{n-2}} \phi_{(n,n)}]^2 \\ & +4[P_{(1,1)_{n-2}} \phi_{(n-1,n-1)} + P_{(1,2)_{n-2}} \phi_{(n,n-1)} \\ & +(P_{(1,1)_{n-2}} k_{1_{n-2}} + P_{(1,2)_{n-2}} k_{2_{n-2}}) \phi_{(n-2,n-1)}] \times \\ & \times [P_{(1,2)_{n-2}} \phi_{(n-1,n)} + P_{(2,2)_{n-2}} \phi_{(n,n)}]. \end{aligned} \quad (33)$$

Using (31) and (33), positive constants a_1, a_2, and a_3 exist such that

$$\begin{aligned} P^2_{(1,1)_{n-2}} \phi^2_{(n-1,n)} < |\phi_{(n-1,n)}| [a_1 \sqrt{|\phi_{(n-1,n)}||\phi_{(n-2,n-1)}|} \\ +a_2 |\phi_{(n-2,n-1)}|] + a_3 \phi^2_{(n-2,n-1)}. \end{aligned} \quad (34)$$

Since $|\phi_{(n-1,n)}(\theta)| > \sigma_n > 0$, $\phi_{(n-2,n-1)}(\theta)$ cannot be zero for any $\theta \in \Theta$. Dividing both sides of (34) by $\phi^2_{(n-2,n-1)}$, it is inferred that $\sup_{\theta \in \Theta} \frac{|\phi_{(n-1,n)}(\theta)|}{|\phi_{(n-2,n-1)}(\theta)|} < \infty$. Hence, a positive constant σ_{n-1} exists such that $|\phi_{(n-2,n-1)}(\theta)| \ge \sigma_{n-1} \ \forall \theta \in \Theta$.

We now proceed by induction. Assume that $\sup_{\theta \in \Theta} \frac{|\phi_{(j,j+1)}(\theta)|}{|\phi_{(j-1,j)}(\theta)|} < \infty$ and $\sigma_{j+1} \triangleq \inf_{\theta \in \Theta} |\phi_{(j,j+1)}| > 0, j = i{+}2, \dots, n{-}1$. Using Theorem 4, a matrix $P_i > 0$, a vector K_i, and a positive constant $\tilde{\nu}_i$ are obtained to satisfy (29). Noting that the leading 2×2 minor of the left hand side of (29) must have positive determinant

$$0 < 4 \sum_{j=1}^{n-i} P_{(1,j)_i} (\phi_{(i+j,i+1)} + k_{j_i} \phi_{(i,i+1)}) \sum_{j=1}^{n-i} P_{(2,j)_i} \phi_{(i+j,i+2)}$$
$$-[\sum_{j=1}^{n-i} P_{(1,j)_i} \phi_{(i+j,i+2)} + \sum_{j=1}^{n-i} P_{(2,j)_i} (\phi_{(i+j,i+1)} + k_{j_i} \phi_{(i,i+1)})]^2. (35)$$

Using (31) and (35), positive constants $a_{1j}, a_{2j}, a_{3j}, i{+}1 \le j \le n{-}1$, a_4, and a_5 exist such that

$$\begin{aligned} P^2_{(1,1)_i} \phi^2_{(i+1,i+2)} < \left[\sum_{j=i+1}^{n-1} a_{2j} \sqrt{|\phi_{(j,j+1)}||\phi_{(i+1,i+2)}|} \right] \times \\ \times \left[a_4 |\phi_{(i,i+1)}| + \sum_{j=i+1}^{n-1} a_{1j} \sqrt{|\phi_{(j,j+1)}||\phi_{(i,i+1)}|} \right] \\ + \left[a_5 |\phi_{(i,i+1)}| + \sum_{j=i+1}^{n-1} a_{3j} \sqrt{|\phi_{(j,j+1)}||\phi_{(i,i+1)}|} \right]^2. \end{aligned} \quad (36)$$

By the induction hypothesis, $\sup_{\theta \in \Theta} \frac{|\phi_{(j,j+1)}(\theta)|}{|\phi_{(j-1,j)}(\theta)|} < \infty$ for $j = i+2, \dots, n-1$. Using (36), $\sup_{\theta \in \Theta} \frac{|\phi_{(i+1,i+2)}(\theta)|}{|\phi_{(i,i+1)}(\theta)|} < \infty$ and $\sigma_{i+1} \triangleq \inf_{\theta \in \Theta} |\phi_{(i,i+1)}(\theta)| > 0$. The proof is complete by induction. ⋄

Lemma 4: Let A be of form (6) and $C = [1, 0, \dots, 0]$. For each $i \in \{2, \dots, n\}$, let $\inf_{\theta \in \Theta} \sum_{j=i}^n |\phi_{(j,i)}(\theta)| = 0$. Let (2) be satisfied with some $G(\theta)$, a constant matrix $P > 0$, and a constant $\nu_1 > 0$. Then, the elements of P on the upper and lower diagonals, i.e., $P_{(i,i+1)}, i = 1, \dots, n-1$, are nonzero. Furthermore, if $\phi_{(i,j)} \equiv 0, i = 2, \dots, n, j = 2, \dots, i$, then each $\phi_{(i,i+1)}(\theta), i = 1, \dots, n-1$, assumes the same sign for all arguments.

Proof of Lemma 4: If (2) holds, the diagonal entries of $A_P \triangleq P[A + GC] + [A + GC]^T P$ must be bounded above by a negative constant. The i^{th} diagonal entry, $i \ge 2$, is given by $2 \sum_{j=i-1}^n P_{(i,j)} \phi_{(j,i)}$. By assumption, $\inf_{\theta \in \Theta} \sum_{j=i}^n |\phi_{(j,i)}(\theta)| = 0$. Therefore, the i^{th} diagonal entry of A_P can be bounded above by a negative constant only if $P_{(i,i-1)} = P_{(i-1,i)} \ne 0$. If the entries of A off the upper diagonal are zero, then the i^{th} diagonal entry of A_P is $2P_{(i,i-1)} \phi_{(i-1,i)}$ implying that $\phi_{(i-1,i)}(\theta)$ must assume the same sign for all arguments. ⋄

Theorem 6: Let $D(\theta) = \mathrm{diag}(D_1(\theta), \dots, D_n(\theta))$. If a matrix $P > 0$ exists such that (3) is satisfied with $\nu_2 > 0$, then $\inf_{\theta \in \Theta} D_i(\theta) > 0$. Furthermore, if the entries of P on the upper and lower diagonals, i.e., $P_{(i,i+1)}, i = 1, \dots, n-1$, are nonzero, then positive constants $\underline{\rho}_{iD}$ and $\overline{\rho}_{iD}, i = 1, \dots, n-1$, exist such that $\underline{\rho}_{iD} D_i(\theta) \le D_{i+1}(\theta) \le \overline{\rho}_{iD} D_i(\theta), i = 1, \dots, n-1$.

Proof of Theorem 6: The i^{th} diagonal element of $PD + DP$ given by $2P_{(i,i)} D_i$ must be bounded below by a positive constant to satisfy (3). Hence, D_i must be bounded below by a positive constant. Furthermore, to satisfy (3), the 2×2 matrices on the principal diagonal of $PD + DP$ must have positive determinant, i.e., for $i = 1, \dots, n-1$,

$$4P_{(i,i)} P_{(i+1,i+1)} D_i D_{i+1} \ > \ P^2_{(i,i+1)} (D_i + D_{i+1})^2, \quad (37)$$

implying that $\frac{D_{i+1}(\theta)}{D_i(\theta)}$ and $\frac{D_i(\theta)}{D_{i+1}(\theta)}$ are upper and lower bounded by positive constants. ⋄

Corollary 2: Let $A(\theta)$ be a matrix of form (6) with $\phi_{(i,j)} = 0, i = 2, \dots, n, j = 2, \dots, i$, i.e., A contains nonzero terms only on the upper diagonal and the first column. Let $C = [1, 0, \ldots, 0]$ and $D(\theta) = \mathrm{diag}(D_1(\theta), \ldots, D_n(\theta))$. A matrix $P > 0$, positive constants ν_1 and ν_2, and a $G(\theta)$ exist to satisfy (2) and (3) if and only if A is w-CUDD, each upper diagonal entry of A takes the same sign for all arguments, and $D_i(\theta)$ satisfy (8) with some positive constants $\underline{\rho}_{iD}^o, i = 1, \ldots, n$, $\underline{\rho}_{iD}$, and $\overline{\rho}_{iD}$, $i = 1, \ldots, n-1$. Furthermore, a matrix $P > 0$, a constant $\nu_1 > 0$, and a $G(\theta)$ exist to satisfy (2) if and only if A is w-CUDD and each upper diagonal entry of A assumes the same sign for all arguments.

Proof of Corollary 2: Evident from Lemma 4 and Theorems 2, 5, and 6. ⋄

V. Applications to Controller and Observer Designs for Nonlinear, Time-Varying, and Switched Systems

1) State-feedback: Consider a system

$$\dot{x} = A_{l(t)}(x,t)x + B_{l(t)}(x,t)u + \Phi_{l(t)}(x,z,t) \quad (38)$$

with state $x \in \mathcal{R}^n$, input $u \in \mathcal{R}$, and disturbance $z \in \mathcal{R}^{n_z}$. $A_1(x,t), \ldots, A_N(x,t)$ are $n \times n$ matrices. $B_1(x,t), \ldots, B_N(x,t), \Phi_1(x,z,t), \ldots, \Phi_N(x,z,t)$ are $n \times 1$ vectors. $l(t)$ is a positive integer function of time taking values in the set $\{1, 2, \ldots, N\}$. Thus, (38) is a nonlinear time-varying switched system. If a matrix $P > 0$, a positive constant ν, and functions $K_i(x,t) \in \mathcal{R}^{1\times n}, i = 1, \ldots, N$, can be found to satisfy

$$2x^T P\Phi_i(x,z,t) + x^T \Big\{ P[A_i(x,t) + B_i(x,t)K_i(x,t)] + [A_i(x,t) + B_i(x,t)K_i(x,t)]^T P \Big\} x \leq -\nu|x|^2 \quad (39)$$

for all $x \in \mathcal{R}^n, z \in \mathcal{R}^{n_z}, i \in \{1, \ldots, N\}$ then, the state-feedback control law $u = K_{l(t)}(x,t)x$ makes the origin $x = 0$ a globally asymptotically stable equilibrium of system (38). The inequality (39) is of the form (27).

2) Observer design (Dual to state-feedback): Given a system $\dot{x} = A_{l(t)}(y,t)x + B_{l(t)}(y,t)u$ with output $y = C_{l(t)}(t)x$, a full-order state observer is given by

$$\dot{\hat{x}} = A_{l(t)}(y,t)\hat{x} + B_{l(t)}(y,t)u + G_{l(t)}(y,t)[C_{l(t)}(t)\hat{x} - y] \quad (40)$$

if a matrix $P > 0$, a positive constant ν, and functions $G_i \in \mathcal{R}^{n\times 1}, i = 1, \ldots, N$, can be found such that[10]

$$P[A_i(y,t) + G_i(y,t)C_i(t)] + [A_i(y,t) + G_i(y,t)C_i(t)]^T P \leq -\nu I, i = 1, \ldots, N. \quad (41)$$

3) Output-feedback: A dynamic output-feedback controller can be constructed by combining the state-feedback and observer designs above as long as (39) and (41) can be satisfied (not necessarily with the same P).

4) Dynamic high-gain scaling based state-feedback/observer design/output-feedback: Dynamic high-gain scaling [12, 13, 14, 15, 16] provides a method for triangular systems to attenuate part of the system dynamics so that the system is approximated by a chain of nonlinear integrators, i.e., the system matrix is approximated by $A(x)$ with A having the form shown in (6) with only the upper diagonal entries being nonzero. The coupled equations (2) and (3) appear in observer design and their dual, (4) and (5), appear in controller design based on this technique. The approach is applicable to systems in both lower triangular (strict-feedback) [13, 14, 15] and upper triangular (feedforward) [16] forms.

5) λ-tracking: Equations (2) and (3), and their dual (4) and (5), also appear in the λ-tracking problem for nonlinear systems [17] where the control objective is to regulate the tracking error to a λ-neighbourhood of zero.

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