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A singularity test for the existence of common quadratic Lyapunov functions for pairs of stable LTI systems

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***Abstract*— In this paper we derive algebraic conditions that are both necessary and sufficient for a pair of stable third order linear time-invariant (LTI) systems to have a common quadratic Lyapunov function (CQLF). We also show how these results can be extended to derive necessary and sufficient conditions for the existence of a CQLF for pairs of LTI systems of arbitrary order.**

I. INTRODUCTION

In this paper we consider the problem of determining whether a pair of stable linear time invariant systems,

$$\Sigma_{A_i} : \dot{x} = A_i x, \quad A_i \in \mathbb{R}^{n\times n}, \quad i \in \{1, 2\}$$

have a common quadratic Lyapunov function (CQLF); namely, whether a matrix $P = P^T > 0$, $P \in \mathbb{R}^{n\times n}$, can be found that simultaneously satisfies the Lyapunov equations $A_i^T P + PA_i = -Q_i$, $i \in \{1, 2\}$, where $Q_i > 0$. If such a P exists then $V(x) = x^T Px$ is said to be a CQLF for $\Sigma_{A_1}, \Sigma_{A_2}$. The study of such problems is closely related to recent work on the design and stability of switching and time-varying feedback systems [1]. The existence of a common quadratic Lyapunov function for two or more stable LTI systems implies uniform (w.r.t. switching) asymptotic stability of the system $\dot{x} = A(t)x$ where $A(t) \in \{A_1, A_2\}$. In this context numerous papers have appeared in the mathematics and engineering literature (see for example [2], [3], [4], [5], [6], [7]) in which sufficient conditions have been derived under which a finite number of stable dynamical systems share a common quadratic Lyapunov function. In each of these papers the authors derive, for special system classes, algebraic conditions on the system matrices that guarantee the existence of a CQLF. In this paper we obtain a complete solution to this problem for pairs of LTI systems of arbitrary order. To facilitate exposition our discussion concentrates on pairs of third order systems; the technique readily extends to pairs of systems of arbitrary finite dimension. The conditions require the non-singularity of certain block matrices constructed using the pair A_1 and A_2, and they generalize the result recently obtained in [4].

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Theorem 1: Let $\Sigma_{A_1}, \Sigma_{A_2}$ be stable LTI systems with $A_1, A_2 \in \mathbb{R}^{3\times 3}$. Then a necessary and sufficient condition for the non existence of a CQLF for Σ_{A_1} and Σ_{A_2} is that one of the following holds:

A : for some $0 < \lambda < 1$ the following 3×3 matrix is singular:

$$\lambda A_1 + (1 - \lambda)A_2 \tag{1}$$

B : for some $0 \le \lambda, \mu \le 1$, and some real number c, the following 6×6 matrix is singular:

$$\begin{pmatrix} \lambda A_1 + (1-\lambda)A_2 & cI \\ -cI & \mu A_1 + (1-\mu)A_2 \end{pmatrix} \tag{2}$$

C : for some $0 \le \lambda \le 1$, and some real numbers a, b, c, the following 9×9 matrix is singular:

$$\begin{pmatrix} \lambda A_1 + (1-\lambda)A_2 & aI & bI \\ -aI & A_1 & cI \\ -bI & -cI & A_2 \end{pmatrix} \tag{3}$$

where I denotes the 3×3 identity matrix.

The paper is organised as follows. In Section 2, we present a number of definitions and preliminary results. An outline proof of the main result of the paper is presented in Section 3; full details are given in [8]. In section 4 we indicate how our results extend in a straight forward manner to the more general problem of solving the CQLF existence problem for pairs of stable LTI systems.

II. PRELIMINARY RESULTS

The following two results are the ingredients for our singularity conditions. Lemma 1 is the key tool that enables the derivation of the singularity conditions presented in this paper.

Theorem 2: Let $\Sigma_{A_1}, \Sigma_{A_2}$, $A_1, A_2 \in \mathbb{R}^{n\times n}$ be stable LTI systems. Then a necessary and sufficient condition for the non existence of a CQLF for Σ_{A_1} and Σ_{A_2} is that there are nonzero positive semidefinite $n \times n$

matrices X_1 and X_2, with $\mathrm{rk}(X_1) < n$ and $\mathrm{rk}(X_2) < n$, such that

$$A_1X_1 + X_1A_1^{\mathrm{T}} + A_2X_2 + X_2A_2^{\mathrm{T}} = 0. \tag{4}$$

Comment : Most of Theorem 2 directly follows from well known results concerning the intersections of convex cones that have appeared in the literature [3]; the conditions on the ranks of the matrices X_1 and X_2 are new [8].

Lemma 1: Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be linearly independent vectors. The following two conditions for the vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ are equivalent:

$$(i) \qquad \sum_{i=1}^{k} \mathbf{w}_i\ \mathbf{v}_i^{\mathrm{T}} + \mathbf{v}_i\,\mathbf{w}_i^{\mathrm{T}} = 0, \tag{5}$$

$$(ii) \qquad \mathbf{w}_i = \sum_{j=1}^{k} c_{ij}\mathbf{v}_j \tag{6}$$

where $c_{ij} + c_{ji} = 0$ for all i, j.

Proof : See [8].

III. Proof of Theorem 1

Theorem 2 implies that a necessary and sufficient condition for the non-existence of a CQLF is that there are positive semidefinite matrices X_1 and X_2, each with rank either 1 or 2, such that

$$A_1X_1 + X_1A_1^{\mathrm{T}} + A_2X_2 + X_2A_2^{\mathrm{T}} = 0 \tag{7}$$

The different tests **A**, **B** and **C** arise as a consequence of the different possible ranks of the matrices X_1 and X_2, and also the dimensions of the intersections of their ranges. We will show in the sequel that the existence of X_1 and X_2 satisfying (11) implies tests **A**, **B**, and **C**. The converse statements follow simply reversing the arguments.

We will use the following elementary observation: suppose that X and Y are positive semidefinite $n \times n$ matrices with $\mathrm{Ran}(Y) \subset \mathrm{Ran}(X)$, then there is $k > 0$ such that

$$X - kY \geq 0, \qquad \text{and} \quad \mathrm{rk(X - kY) < rk(X)} \tag{8}$$

Case (i): $\mathrm{rk}(X_1) = \mathrm{rk}(X_2) = 1$

In this case

$$X_1 = \mathbf{v}_1\mathbf{v}_1^{\mathrm{T}}, \qquad X_2 = \mathbf{v}_2\mathbf{v}_2^{\mathrm{T}} \tag{9}$$

It follows from (7) that

$$A_1\mathbf{v}_1\mathbf{v}_1^{\mathrm{T}} + \mathbf{v}_1\mathbf{v}_1^{\mathrm{T}}A_1^{\mathrm{T}} + A_2\mathbf{v}_2\mathbf{v}_2^{\mathrm{T}} + \mathbf{v}_2\mathbf{v}_2^{\mathrm{T}}A_2^{\mathrm{T}} = 0 \tag{10}$$

If $\mathbf{v}_2 = \alpha\mathbf{v}_1$ for some real α, then (10) becomes

$$(A_1 + \alpha^2A_2)\,\mathbf{v}_1\mathbf{v}_1^{\mathrm{T}} + \mathbf{v}_1\mathbf{v}_1^{\mathrm{T}}\,(A_1 + \alpha^2A_2)^{\mathrm{T}} = 0 \tag{11}$$

Let $\mathbf{w}_1 = (A_1 + \alpha^2A_2)\,\mathbf{v}_1$. Then applying Lemma 1 with $k = 1$ gives

$$\mathbf{w}_1 = (A_1 + \alpha^2A_2)\,\mathbf{v}_1 = 0, \tag{12}$$

which is **Test A** with $\lambda = (1 + \alpha^2)^{-1}$.

If $\mathbf{v}_2 \neq \alpha\mathbf{v}_1$ for any α, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent. Let $\mathbf{w}_1 = A_1\mathbf{v}_1$ and $\mathbf{w}_2 = A_2\mathbf{v}_2$, then applying Lemma 1 with $k = 2$ implies that for some number c,

$$\begin{pmatrix} A_1 & cI \\ -cI & A_2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{13}$$

which is **Test B** with $\lambda = 1$ and $\mu = 0$.

The result (12) can be re-expressed as the condition that $A_2^{-1}A_1$ has a real negative eigenvalue, and (13) as the condition that A_2A_1 has a real negative eigenvalue. In this form the conditions were derived before as the conditions for non-existence of a CQLF in [4]. For second order systems Theorem 2 requires that the matrices X_1 and X_2 have rank 1, so it is clear that these are the complete conditions for non-existence of a CQLF in this case [4], [9], [10].

Case (ii): $\mathrm{rk}(X_1) = 1, \mathrm{rk}(X_2) = 2$

In this case

$$X_1 = \mathbf{v}_1\mathbf{v}_1^{\mathrm{T}} \tag{14}$$

If $\mathbf{v}_1 \in \mathrm{Ran}(X_2)$, then (8) implies that

$$X_2 = k\mathbf{v}_1\mathbf{v}_1^{\mathrm{T}} + \mathbf{v}_2\mathbf{v}_2^{\mathrm{T}} \tag{15}$$

for some $k > 0$ and some vector $\mathbf{v}_2$ independent of $\mathbf{v}_1$. Then

$$\begin{aligned} A_1X_1 + X_1A_1^{\mathrm{T}} + A_2X_2 + X_2A_2^{\mathrm{T}} &= \\ (A_1 + kA_2)\mathbf{v}_1\mathbf{v}_1^{\mathrm{T}} + \mathbf{v}_1\mathbf{v}_1^{\mathrm{T}}(A_1 + kA_2)^{\mathrm{T}} & \\ + A_2\mathbf{v}_2\mathbf{v}_2^{\mathrm{T}} + \mathbf{v}_2\mathbf{v}_2^{\mathrm{T}}A_2^{\mathrm{T}} & \end{aligned} \tag{16}$$

Let $\mathbf{w}_1 = (A_1 + kA_2)\mathbf{v}_1$ and $\mathbf{w}_2 = A_2\mathbf{v}_2$, then (16) yields

$$\mathbf{w}_1\mathbf{v}_1^{\mathrm{T}} + \mathbf{v}_1\mathbf{w}_1^{\mathrm{T}} + \mathbf{w}_2\mathbf{v}_2^{\mathrm{T}} + \mathbf{v}_2\mathbf{w}_2^{\mathrm{T}} = 0 \tag{17}$$

Applying Lemma 1 with $k = 2$ implies that for some c the following matrix is singular:

$$\begin{pmatrix} A_1 + kA_2 & cI \\ -cI & A_2 \end{pmatrix} \tag{18}$$

This is **Test B** with $\mu = 0$ and $\lambda = (1 + k)^{-1}$.

If $\mathbf{v}_1$ is not in $\mathrm{Ran}(X_2)$, then there are vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent and

$$X_2 = \mathbf{v}_2\mathbf{v}_2^{\mathrm{T}} + \mathbf{v}_3\mathbf{v}_3^{\mathrm{T}} \tag{19}$$

Let $\mathbf{w}_1 = A_1\mathbf{v}_1$, $\mathbf{w}_2 = A_2\mathbf{v}_2$ and $\mathbf{w}_3 = A_2\mathbf{v}_3$, then (7) becomes

$$\sum_{i=1}^{3} \mathbf{w}_i\mathbf{v}_i^{\mathrm{T}} + \mathbf{v}_i\mathbf{w}_i^{\mathrm{T}} = 0 \tag{20}$$

Applying Lemma 1 with $k = 3$ implies that for some numbers a, b, c the following matrix is singular:

$$\begin{pmatrix} A_1 & aI & bI \\ -aI & A_2 & cI \\ -bI & -cI & A_2 \end{pmatrix} \tag{21}$$

and this is **Test C** with $\lambda = 0$.

Case (iii): $\mathrm{rk}(X_1) = \mathrm{rk}(X_2) = 2$

There are two subcases, depending on whether $\mathrm{Ran}(X_1) = \mathrm{Ran}(X_2)$ or not. In the case $\mathrm{Ran}(X_1) = \mathrm{Ran}(X_2)$, (8) implies that there is $k' > 0$ such that

$$X_1 - k'X_2 \geq 0, \qquad \mathrm{rk}(X_1 - k'X_2) \leq 1 \tag{22}$$

Let $Y = X_1 - k'X_2$. Hence

$$\begin{aligned} A_1X_1 + X_1A_1^{\mathrm{T}} + A_2X_2 + X_2A_2^{\mathrm{T}} = \\ A_1Y + YA_1^{\mathrm{T}} + (A_2 + k'A_1)X_2 + X_2(A_2 + k'A_1)^{\mathrm{T}} \\ = 0 \end{aligned} \tag{23}$$

This reduces to Case (ii), with A_2 replaced by $A_2 + k'A_1$. Since $\mathrm{Ran}(Y) \subset \mathrm{Ran}(X_2)$, this leads to the condition that

$$\begin{pmatrix} A_1 + k(A_2 + k'A_1) & cI \\ -cI & (A_2 + k'A_1) \end{pmatrix} \tag{24}$$

is singular for some $k > 0$ and some real c, and this is **Test B** with

$$\lambda = \frac{1 + kk'}{1 + k(1 + k')}, \qquad \mu = \frac{k'}{1 + k'} \tag{25}$$

If $\mathrm{Ran}(X_1) \neq \mathrm{Ran}(X_2)$, let $\mathbf{v}_1 \in \mathrm{Ran}(X_1) \cap \mathrm{Ran}(X_2)$. Then (8) implies that there are $k_1, k_2 > 0$, and vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ independent, such that

$$\begin{aligned} X_1 &= k_1\mathbf{v}_1\mathbf{v}_1^{\mathrm{T}} + \mathbf{v}_2\mathbf{v}_2^{\mathrm{T}} \\ X_2 &= k_2\mathbf{v}_1\mathbf{v}_1^{\mathrm{T}} + \mathbf{v}_3\mathbf{v}_3^{\mathrm{T}} \end{aligned} \tag{26}$$

Let $\mathbf{w}_1 = (k_1A_1 + k_2A_2)\mathbf{v}_1$, $\mathbf{w}_2 = A_1\mathbf{v}_2$ and $\mathbf{w}_3 = A_2\mathbf{v}_3$. Then applying Lemma 1 with $k = 3$ shows that for some numbers a, b, c the following matrix is singular:

$$\begin{pmatrix} k_1A_1 + k_2A_2 & aI & bI \\ -aI & A_1 & cI \\ -bI & -cI & A_2 \end{pmatrix} \tag{27}$$

QED

IV. Extending Theorem 1

We now briefly describe the extension of Theorem 1 to pairs of n-dimensional systems.

In this case we apply the same strategy as in the proof of Theorem 1, and start with the condition (4):

$$A_1X_1 + X_1A_1^{\mathrm{T}} + A_2X_2 + X_2A_2^{\mathrm{T}} = 0 \tag{28}$$

In order to apply Lemma 1 we must rewrite the positive semidefinite matrices X_1 and X_2 as sums of projectors onto vectors. Suppose that X_1 and X_2 have ranks k_1 and k_2 respectively, and suppose also that the intersection of their ranges is a subspace V with dimension d. Then we must have $k_1, k_2 \leq d$ and $0 \leq k_1 + k_2 - d \leq n$. The first observation is that there are positive semidefinite matrices Y_1, Y_2 and Z_1, Z_2, such that $\mathrm{Ran}(Y_1) = \mathrm{Ran}(Y_2) = V$, $\mathrm{Ran}(Z_1) \cap V = \mathrm{Ran}(Z_2) \cap V = \{0\}$, and

$$X_1 = Y_1 + Z_1, \qquad X_2 = Y_2 + Z_2 \tag{29}$$

The second observation is that there are independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ in V, and positive numbers $\{r_1, \ldots, r_d\}$, $\{s_1, \ldots, s_d\}$ such that

$$Y_1 = \sum_{i=1}^{d} r_i\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}, \qquad Y_2 = \sum_{i=1}^{d} s_i\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}} \tag{30}$$

Let $d_1 = k_1 - d$ and $d_2 = k_2 - d$. Then there are d_1 vectors $\{\mathbf{u}_1, \ldots\}$ and d_2 vectors $\{\mathbf{y}_1, \ldots\}$ such that

$$Z_1 = \sum \mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}, \qquad Z_2 = \sum \mathbf{y}_j\mathbf{y}_j^{\mathrm{T}} \tag{31}$$

Combining these observations we obtain the following decompositions:

$$X_1 = \sum_{i=1}^{d} r_i\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}} + \sum_{j=1}^{d_1} \mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} \tag{32}$$

and

$$X_2 = \sum_{i=1}^{d} s_i\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}} + \sum_{j=1}^{d_2} \mathbf{y}_j\mathbf{y}_j^{\mathrm{T}} \tag{33}$$

Lemma 1 can now be applied. The result is that a certain $(d + d_1 + d_2)$-dimensional block matrix must be singular for the non-existence of a CQLF.

V. Concluding remarks

In this paper we have derived algebraic conditions for the existence of a CQLF for a pair of third order systems. We have also indicated how these conditions can be extended to provide a general solution to the CQLF existence problem for finite sets of higher dimensional systems. While the necessary and sufficient conditions for a CQLF are easy to check for pairs of systems when X_1 and X_2 in Equation (1) are rank 1 matrices (requiring only that A_1A_2 and $A_1A_2^{-1}$ have no negative eigenvalues), the general conditions for CQLF existence are difficult, if not impossible, to verify. An interesting open question is therefore to classify those system pairs for which X_1 and X_2 are rank 1 matrices. While the only currently known system class satisfying this condition is the trivial class, corresponding to pairs of second order LTI systems, evidence suggests that the assumption may apply to other more general n-dimensional system classes. We are currently investigating this conjecture and examining the applicability of our approach to the CQLF existence problem for sets of exponentially stable non-linear systems.

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