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Controller Design of Switched Linear Systems

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***Abstract*— This paper is about controller design for switched linear systems with input signals. The synthesis problem how to design the individual feedback gains and how to switch among the individual closed-loop feedback systems such that the switched system becomes (exponentially) stable, is constructively formulated as a bilinear matrix inequality problem. By using multiple quadratic Lyapunov functions, one for each closed-loop linear vector field, a stabilizing controller is synthesized for a broader class of switched systems than earlier proposed in the literature. One example is given to illustrate the synthesis procedure.**

I. Introduction

A large class of systems is reasonable modelled by a family of continuous-time subsystems and logic rules that govern the switchings between them. Such *switched systems* have gained much attention the last decade, mostly due to the growing use of computers in the control of physical plants (making it easy to switch between controllers) but also as a result of the switched nature of many physical processes, see for instance [1], [2] for examples.

Specifically in this paper, we are focused on *switched linear systems with input signals* u

$$\dot{x} = A_i x + B_i u, \quad i \in I_m = \{1, \ldots, m\}, \tag{1}$$

where $x \in \Re^n$, $u \in \Re^p$ and the index function $i : [0\ \infty) \to I_m$ decides which one of the linear vector fields that is active at a certain time instant. The problem treated in this paper is how to design the feedback gains

$$u = K_i x, \quad K_i K_i^T \leq \kappa^2 I, \quad i \in I_m, \tag{2}$$

(I is the $p \times p$ identity matrix) and the index function i such that (the origin of) the *closed-loop switched linear system*

$$\dot{x} = (A_i + B_i K_i)x \equiv A_i^{cl} x, \quad i \in I_m, \tag{3}$$

becomes (exponentially) stable. Since both the gains K_i and the index function i is crucial for the stability of (1), the design of both has to be performed simultaneously.

The constraints $K_i K_i^T \leq \kappa^2 I$, where κ is a design parameter, are introduced to restrict the control signal from being too large. Without constraints, the eigenvalues of each of the individual closed-loop linear system in (3) can be placed arbitrarily far away on the negative real axis in the complex plane, if controllable, leading to arbitrarily fast (exponential) convergence of the switched linear system.

Except designing an exponentially stable controller, the goal of this paper is to find the gains K_i and the index function i that bests estimate the exponential convergence rate. This means that even if the constraints on the gains K_i is about so generous that at least one of the closed-loop subsystems become individual stable, the trivial solution to switch to that subsystem is not necessarily the best one.

The result in this paper relies on the existence of multiple Lyapunov functions. There are approaches in the literature suggesting conditions guaranteeing stability of closed-loop switched and hybrid systems using multiple Lyapunov functions, see [3], [4], [5], [6], [7]. In general, however, these results are not constructive in the sense that they explicitly determine the switching rule but merely give conditions that possible stabilizing switching rules have to satisfy.

There are several constructive propositions in the literature how to stabilize *classes* of switched system without input signals, see for instance [8]. The methods suggested in [3], [9], [10] are restricted to switched closed-loop linear systems consisting of (unstable) linear vector fields for which there exists a stable convex combination of the corresponding A-matrices. These methods guarantee stability by using a common quadratic Lyapunov function. The work in [11] is a first attempt to propose constructive synthesis results for switched systems consisting of *two* linear vector fields by using multiple Lyapunov functions.

Recently, in an earlier paper [12], a constructive synthesis method has been suggested that is applicable to a broader class of closed-loop switched linear systems than earlier proposed in the literature. In this paper, the method is applied to switched linear systems with input signals. The method utilizes multiple Lyapunov functions, and it is shown how the synthesize problem can be formulated as a Bilinear Matrix Inequality (BMI) problem [13], a bilinear optimization problem, consisting of unknown scalars and gains multiplied by unknown matrices. By gridding the unknown scalars and gains, the verification of the stability conditions becomes a Linear Matrix Inequality (LMI) problem [14]; a convex optimization problem for which there exists efficient numerical software.

The outline of this paper is: we start by giving the synthesis result for switched linear systems with input signals. It is explained how the number of unknowns in the obtained bilinear matrix inequality problem can be reduced, in order to reduce the complexity of the optimization problem. Section III relates the proposed method by other ones found in the literature. Finally, the method is applied to an example.

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II. SWITCHING SYNTHESIS

The intention of this section is to present the synthesis method for switched linear systems with input signals. The method is an application of the synthesis method originally proposed in [12] for closed-loop switched linear systems. The section begins with some preliminaries needed to understand the method.

A. Preliminaries

The origin of the closed-loop switched linear system (3) is exponentially stable if all trajectories satisfy

$$||x(t)|| \leq k_1 e^{-k_2 t} ||x_0||, \tag{4}$$

for some $k_1 > 0$ and $k_2 > 0$. The goal of this paper is to design the feedback gains K_i in (2) and the index function i such that the closed-loop switched linear system (3) fulfills this definition.

B. Potential activation of local subsystems

The specific closed-loop linear subsystem $\dot{x} = A_i^{cl} x$ in (3) may *potentially* be activated in parts of the state space, specified by a region Ω_i. Switches of the closed-loop linear subsystem $\dot{x} = A_i^{cl} x$ to $\dot{x} = A_j^{cl} x$ *potentially* occur in parts of the state space, specified by a switch region $\Omega_{i,j}$.

Whether a specific linear subsystem is activated or not inside an associated region is decided by the switch strategy in the next subsection, which indirectly also specify the switch regions.

Specifically in this paper, we assume that the regions Ω_i and $\Omega_{i,j}$ have the structure of quadratic forms

$$\begin{aligned} \Omega_i &= \{x \in \Re^n \mid x^T Q_i x \geq 0\}, & i \in I_m, \\ \Omega_{i,j} &= \{x \in \Re^n \mid x^T (Q_j - Q_i) x = 0\}, & (i,j) \in I_m \times I_m, \end{aligned} \tag{5}$$

where $Q_i \in \Re^{n \times n}$, $i \in I_m$, are *symmetric* matrices that will be determined by the synthesis method given later on.

C. Largest region function strategy

We define the index function in this paper according to:

$$i(x) = \arg\big(\max_{i \in I_m} x^T Q_i x\big), \tag{6}$$

which is denoted the *largest region function strategy* due to the selection of closed-loop linear subsystem (at state x) corresponding to the largest value of the region functions $x^T Q_i x$.

D. Multiple Lyapunov functions

The synthesis method for closed-loop switched linear systems proposed in [12] is based on the use of multiple quadratic Lyapunov functions. For every region Ω_i, there is a local quadratic Lyapunov function associated, which means that the overall energy of the closed-loop switched linear system (3) is measured by $x^T P_i x$ when the specific linear subsystem $\dot{x} = A_i^{cl} x$ is activated. The synthesis method in [12], which is given later on in subsection II-F, is essentially conditions that mean that the overall abstract energy, measured by the different local Lyapunov functions, must decrease for all times.

E. Equal energy when switching

By restricting the different local quadratic Lyapunov functions according to

$$P_i = P_j + \eta_{i,j}(Q_j - Q_i), \quad i \in I_m,\ j \in I_m, \tag{7}$$

where $\eta_{i,j} = \eta_{j,i}$, $i \in I_m$, $j \in I_m$, are arbitrary scalars, it means that $x^T P_i x = x^T P_j x$ for all states satisfying $x^T Q_i x = x^T Q_j x$, which are the potential switching states according to (5).

The requirement that the local energy functions have to be equal for *all* potential switchings is introduced since it, in general, is impossible to *a priory* know which switchings that actually will occur in the controlled system, since this is a result of the synthesis method.

F. Synthesis conditions closed-loop switched linear system

The following synthesis method (in form of a theorem) for switched linear systems with input signals, recently suggested in [12] for closed-loop switched linear systems (in which a formal proof is given), guarantees (exponential) stability using the *largest region function strategy*.

Theorem 1 (Closed-loop synthesis) *If there exist symmetric matrices P_i and Q_i, gains K_i, and scalars α, μ_i, ν_i, ϑ_i, θ_i and $\eta_{i,j}$, solving the constrained optimization problem:*

$\min \beta$ *subject to*

0. $\alpha > 0,\ \mu_i \geq 0,\ \nu_i \geq 0,\ \vartheta_i \geq 0,\ \theta_i \geq 0, \quad i \in I_m$
1. $\alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i, \quad i \in I_m$
2. $(A_i^{cl})^T P_i + P_i A_i^{cl} + \vartheta_i Q_i \leq -I, \quad i \in I_m$
3. $P_i = P_j + \eta_{i,j}(Q_j - Q_i), \quad i \in I_m,\ j \in I_m$
4. $\theta_1 Q_1 + \ldots + \theta_m Q_m \geq 0$
5. $K_i K_i^T \leq \kappa^2 I, \quad i \in I_m$

then the largest region function strategy *implies that the origin of the closed-loop switched linear system (3) is exponentially stable according to (4), where*

$$k_1 = \sqrt{\frac{\beta}{\alpha}}, \quad k_2 = \frac{1}{2\beta},$$

if no sliding motion occurs.

The largest region function strategy is one of the key ingredients in the proposed synthesis result (together with the equal energy requirement (7) when switching). Proposed strategies in the literature are based on switchings due to the smallest or largest local Lyapunov function, see for instance [11], [15]. There are examples that cannot be stabilized based on switchings due to the smallest local Lyapunov function and vice versa (see for instance Example IV in which case a strategy to activate the subsystem corresponding to the smallest local Lyapunov function would not lead to any solution satisfying the stability conditions due to the vector field direction of the two subsystems). However, this problem is avoided applying the largest region function strategy, since this method indirectly is coupled to the local Lyapunov functions through the local regions.

The conditions in the theorem are essentially requirements that the energy, measured by $V_i(x) = x^T P_i x$ in region Ω_i, has to be positive (Condition 1 together with $\alpha > 0$), and decrease, i.e. $\dot{V}_i = x^T(A_i^T P_i + P_i A_i)x \leq -I$ (where I is the identity matrix of dimension $n \times n$) when inside the region (Condition 2), and equal $V_j(x) = x^T P_j x = x^T P_i x = V_i(x)$ when changing region (Condition 3), which occurs for the set of states fulfilling $x^T q_i x = x^T Q_j x$. These conditions are all constrained to be valid in specific regions (Ω_i and $\Omega_{i,j}$ respectively) but can be replaced by the unconstrained conditions in the theorem by using the so called $\mathcal{S}$-procedure, see [14], which introduces the additional variables μ_i, ν_i, ϑ_i and $\eta_{i,j}$, satisfying Condition 0. Condition 4 is introduced to guarantee that for all states $x \in \Re^n$, at least one subsystem can be activated fulfilling the energy decrease conditions, which together with the largest region function strategy means that only subsystems fulfilling the energy decrease conditions are activated.

G. Parameters in the bilinear matrix inequality problem

Defining $P_m = P$ and $\eta_i = \eta_{i,m}$, $i \in I_{m-1}$, it follows from Condition 3 in Theorem 1 that

$$P_i = P + \eta_i(Q_m - Q_i), \quad i \in I_{m-1}.$$

This means that Conditions 1 and 2 become:

1. $\alpha I + \mu_i Q_i \leq P + \eta_i(Q_m - Q_i) \leq \beta I - \nu_i Q_i,\ i \in I_{m-1}$
   $\alpha I + \mu_m Q_m \leq P \leq \beta I - \nu_m Q_m$
2. $(A_i^{cl})^T(P + \eta_i(Q_m - Q_i)) + (P + \eta_i(Q_m - Q_i))A_i^{cl} + \vartheta_i Q_i \leq -I, \quad i \in I_{m-1}$
   $(A_m^{cl})^T P + P A_m^{cl} + \vartheta_m Q_m \leq -I$

Hence, the number of unknown variables in Theorem 1 is $(m+1)\frac{n(n+1)}{2} + mnp + 5m$ (P and Q_i, $i \in I_m$, all with $\frac{n(n+1)}{2}$ unknown elements, the m gains with np elements, the positive scalars μ_i, ν_i, ϑ_i and θ_i, $i \in I_m$, and the arbitrary scalars η_i, $i \in I_{m-1}$ and α), β not included.

The problem verifying the existence of the unknown variables satisfying the conditions in Theorem 1 is a Bilinear Matrix Inequality (BMI) problem, due to unknown scalars and gains multiplied by matrices. BMI problems are NP-hard, which means that its solution cannot be computed in polynomial time in the worst case [13]. However, this does not imply that practical algorithms are not possible; practical algorithms for NP-hard problems exist and typically involve approximations, heuristics, branch-and-bound, or local search. Algorithms for solving optimization problems over BMIs are currently limited to problems of modest size.

One way to compute the BMI problem in Theorem 1 is to grid up the gains and the unknown scalars. For fixed values of these parameters, the verification of the remaining unknown variables becomes a Linear Matrix Inequality (LMI) problem [14], which is a convex optimization problem that can be solved efficiently by existing numerical software, for instance [16] which is the one used in this paper.

Since any β satisfying the conditions in Theorem 1 is a solution guaranteeing (exponential) stability using the *largest region function strategy*, it is not crucial to find the optimal value of β. Hence, the gridding of the unknown scalars can be made quite sparsely. If the constraints on the gains K_i is about so generous that at least one of the closed-loop subsystems become individual stable, this solution may serve as a valid starting point in the search for the optimal solution. The number of unknown variables in Theorem 1 can be reduced, which simplifies the search for the optimal solution, which is described next.

If there is a solution to Theorem 1, the same solution is valid also in the case when the scalars ν_i, $i \in I_m$, are put to zero, with the difference that the optimal value of β might be higher, leading to a worse estimate of the exponential convergence rate $k_2 = 1/(2\beta)$ in (4). This is true since the optimal value value satisfies $\lambda_{max}(P_i) = \beta - \nu_i \lambda_{min}(Q_i)$, $i \in I_m$, where $\lambda_{max}(P_i)$ is the largest eigenvalue of P_i and $\lambda_{min}(Q_i)$ is the smallest eigenvalue of Q_i. In the case when an eigenvalue of Q_i is strictly negative and ν_i strictly positive (leading to a positive value of $-\nu_i \lambda_{min}(Q_i)$), the optimal value of β would be increased to $\beta - \nu_i \lambda_{min}(Q_i)$ if ν_i instead was put to zero. Note that all Q_i's with only positive eigenvalues (which means that the corresponding matrices are positive definite) leads to an optimal solution where the corresponding $\nu_i = 0$; otherwise β could be decreased further by decreasing ν_i, which means that it is not optimal. Hence, it is advantageous to first put the unknown scalars ν_i, $i \in I_m$, to zero, reducing the number of unknown scalars by m, find a solution and then either accept this worse estimate of the exponential convergence k_2 or improve it by afterwards find the smallest β satisfying $P_i \leq \beta I - \nu_i Q_i$, $i \in I_m$, where now P_i and Q_i are known matrices and the $\nu_i \geq 0$, $i \in I_m$, are unknown. This search for the smallest β is an LMI problem.

In a similar way, the left hand side of Condition 1 in Theorem 1 can be changed to the condition $0 < P_i$, reducing the number of unknowns by $m+1$ (the scalars μ_i, $i \in I_m$ and α), with a worse estimate $k_1 = \sqrt{\beta/\alpha}$ in (4). The estimate of k_1 can then be accepted or improved by finding the largest α satisfying $\alpha + \mu_i Q_i \leq P_i$, $i \in I_m$, where P_i and Q_i are known matrices and the $\mu_i \geq 0$, $i \in I_m$, are unknown. As above, this is an LMI problem.

One of the θ_i's in Condition 4 of Theorem 1 can be scaled to 1 without loss of generality, reducing the unknowns by one. Furthermore, the optimum value of Theorem 1 means that Condition 4 and Condition 5 are satisfied with equality implying that the number of unknowns can be further reduced by one parameter.

To conclude, the number of unknown scalars can be reduced by $2(m+1)+1$ parameters resulting in less BMI conditions. However, BMI conditions will still remain, and needs to be solved by some method, for instance by gridding.

H. Check for sliding motions

Stability of Theorem 1 is only guaranteed if no sliding motions occur applying the *largest region function strategy*.

Sliding motions may occur at the surface of states satisfying $\max_{i\in I_m} x^T Q_i x = \max_{j\in I_m} x^T Q_j x$ ($i \neq j$), which are states where the subsystem changes occur. If sliding motions occur, the trajectory moves along the surface (with dynamics defined for instance according to Filippov's convex combination [17]), which may lead to either a stable or an unstable equilibrium point. Therefore, it must be verified that no sliding motion occurs in the application of Theorem 1.

Assume that a solution to Theorem 1 is obtained with two neighboring regions $x^T Q_i x \geq 0$ and $x^T Q_j x \geq 0$, that both are the largest region functions among all region functions. Let the boundary between these regions be defined by $x^T Q x = x^T Q_i x - x^T Q_j x = 0$. This means that the gradient $\frac{\partial x^T Q x}{\partial x} = 2x^T Q$ points into the region $x^T Q_i x \geq 0$, since $x^T Q_i x - x^T Q_j x \geq 0$ in region $x^T Q_i x \geq 0$ and $x^T Q_i x - x^T Q_j x \leq 0$ in region $x^T Q_j x \geq 0$.

Sliding motions do not occur at the surface $x^T Q x = 0$ either if the two vector fields $A_i^{cl} x$ and $A_j^{cl} x$ both points into region $x^T Q_i x \geq 0$ or $x^T Q_j x \geq 0$, or if vector field $A_i^{cl} x$ points into region $x^T Q_i x \geq 0$ and vector field $A_j^{cl} x$ points into region $x^T Q_j x \geq 0$, see [17]. By using the gradient $\frac{\partial x^T Q x}{\partial x} = 2x^T Q$ of the surface $x^T Q x = 0$, this implies that no sliding motion occurs if either the condition

$$x^T (A_i^{cl})^T Q x + x^T Q A_i^{cl} x \cdot x^T (A_j^{cl})^T Q x + x^T Q A_j^{cl} x \geq 0,$$

or the condition

$$x^T (A_i^{cl})^T Q x + x^T Q A_i^{cl} x \geq 0 \text{ and } x^T (A_j^{cl})^T Q x + x^T Q A_j^{cl} x \leq 0,$$

is satisfied for all points fulfilling $x^T Q x = 0$.

On the other hand, if vector field $A_i^{cl} x$ points into region $x^T Q_j x \geq 0$ and vector field $A_i^{cl} x$ points into region $x^T Q_j x \geq 0$, sliding motion occurs along the surface $x^T Q x = 0$, see [17]. This implies that sliding motion occurs if the condition

$$x^T (A_i^{cl})^T Q x + x^T Q A_i^{cl} x \leq 0 \text{ and } x^T (A_j^{cl})^T Q x + x^T Q A_j^{cl} x \geq 0,$$

is satisfied for some state $x \in \Re^n$ satisfying $x^T Q x = 0$.

Since Q is symmetric, it can be factorized into $Q = S \Lambda S^T$, with the orthonormal eigenvectors in S and the eigenvalues (on the diagonal) in Λ [18] ($S^T = S^{-1}$). By coordinate transformation $y = S^T x$ ($x = Sy$), and the note that neither $x^T Q x = x^T S \Lambda S^T x = y^T \Lambda y = 0$ nor the inequalities are affected by a scaling, we have the following lemma:

Lemma 1 *No sliding motion occurs if and only if either*

$$y^T (S^T (A_i^{cl})^T S \Lambda + \Lambda S^T A_i^{cl} S) y \cdot y^T (S^T (A_j^{cl})^T S \Lambda + \Lambda S^T A_j^{cl} S) y \geq 0$$

or

$$y^T (S^T (A_i^{cl})^T S \Lambda + \Lambda S^T A_i^{cl} S) y \geq 0$$
$$\text{and } y^T (S^T (A_j^{cl})^T S \Lambda + \Lambda S^T A_j^{cl} S) y \leq 0$$

is satisfied for all $y \in \Re^n$ fulfilling $y^T \Lambda y = 0$ and $y^T y = 1$.

In the case when $y \in \Re^2$, the verification of either of these lemma is easily performed since $y^T \Lambda y = 0$ is equal to $y_2 = \pm\sqrt{-\frac{\lambda_1}{\lambda_2}} y_1$ (i.e. a linear relation), where λ_1 and λ_2 are the two eigenvalues and $y = [y_1 \;\; y_2]^T$. Hence, since $y^T \Lambda y = 0$ is equal to $y_1^2 + y_2^2 = 1$, the solution is $y_1 = \pm\sqrt{-\frac{\lambda_2}{\lambda_1 - \lambda_2}}$ and $y_2 = \pm\sqrt{\frac{\lambda_1}{\lambda_1 - \lambda_2}}$. Therefore, the verification of the inequality in the lemma has to be performed only for the two different points $y = [\sqrt{-\frac{\lambda_2}{\lambda_1 - \lambda_2}} \;\; \pm\sqrt{\frac{\lambda_1}{\lambda_1 - \lambda_2}}]^T$ ($-y$ gives the same result as y).

In higher dimensions than two, the inequality in Lemma 1 has to be verified for a continuum of points satisfying $y^T \Lambda y = 0$ and $y^T y = 1$.

I. Special case: two linear closed-loop subsystems

In case of switching between only two linear subsystems, we can set $Q_1 = Q$ and $Q_2 = -Q$ (without loss of generality, we can scale $\theta_1 = \theta_2 = 1$) implying that Condition 4 in Theorem 1 is satisfied. Using the variables P_1 and P_2, Theorem 1 becomes:

Corollary 1 (Closed-loop synthesis two subsystems) *If there exist symmetric matrices P_i, gains K_i, and scalars α, μ_i, ν_i, ϑ_i and η solving the constrained optimization problem:*

$\min \beta$ *subject to*

0. $\alpha > 0,\ \mu_i \geq 0,\ \nu_i \geq 0,\ \vartheta_i \geq 0,\ i \in I_2$
1. $\alpha I + \frac{\mu_1}{2\eta}(P_2 - P_1) \leq P_1 \leq \beta I - \frac{\nu_1}{2\eta}(P_2 - P_1)$
   $\alpha I - \frac{\mu_2}{2\eta}(P_2 - P_1) \leq P_2 \leq \beta I + \frac{\nu_2}{2\eta}(P_2 - P_1)$
2. $(A_1^{cl})^T P_1 + P_1 A_1^{cl} + \frac{\vartheta_1}{2\eta}(P_2 - P_1) \leq -I$
   $(A_2^{cl})^T P_2 + P_2 A_2^{cl} - \frac{\vartheta_2}{2\eta}(P_2 - P_1) \leq -I$
3. $K_i K_i^T \leq \kappa^2 I, \quad i \in I_2$

then the largest region function strategy *implies that the origin of the closed-loop switched linear system (3) is exponentially stable according to (4), where*

$$k_1 = \sqrt{\frac{\beta}{\alpha}}, \quad k_2 = \frac{1}{2\beta},$$

if no sliding motion occurs.

III. Relation existing methods

In Theorem 1, if all $\eta_i = 0$, $i \in I_{m-1}$, it means that a common P matrix has to satisfy the conditions (in this case, a solution to Theorem 1 (for designed K_i's) implies that sliding motions also are allowed to occur, if they are defined according to Filippov's convex combination definition [17]). Multiplying Condition 2 by θ_i/ϑ_i and summing up gives

$$\sum_{i=1}^{m} \frac{\theta_i}{\vartheta_i}((A_i^{cl})^T P + P(A_i^{cl})) + \theta_1 Q_1 + \ldots + \theta_m Q_m \leq -I,$$

which implies that

$$\sum_{i=1}^{m} \frac{\theta_i}{\vartheta_i}((A_i^{cl})^T P + P A_i^{cl}) \leq -I,$$

due to Condition 4. This condition together with Condition 1 where $\mu_i = \nu_i = 0$, are the synthesis conditions proposed in [10] applying the *min-projection strategy*, and a solution exists only if there exists a stable convex combination of the A-matrices. Hence, this BMI problem can be solved by first searching for a stable convex combination, and then solving an LMI to find the P-matrix.

In the *min-switch strategy* proposed in [15] it is assumed that each closed-loop subsystem has an associated Lyapunov function, decided *a priori*, for which the energy decreases in a certain region, and these regions together cover the state space. A *min-switch strategy* is proposed meaning that the subsystem corresponding to the smallest Lyapunov function is selected. The problem is, however, that there is no guarantee that a trajectory cannot move outside a region for where it is valid, since the states where the closed-loop subsystem are switched are not necessarily inside the regions where the energy for two consecutive activated subsystems are decreasing. Hence, the approach is not constructive. In the approach suggested in this paper (and originally in [12]), it is guaranteed that the closed-loop subsystems are switched at states at the boundary of the regions where the energy for two consecutive activated subsystems are decreasing, due to the restriction (7) and the application of the *Largest region function strategy* in (6).

Switching between two closed-loop linear subsystems and defining $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, Corollary 1 becomes ($\beta$ neglected)

0. $\vartheta_i \geq 0$, $i \in I_2$
1. $\alpha I \leq P_i \leq \beta I$, $i \in I_2$
2. $(A_1^{cl})^T P_1 + P_1 A_1^{cl} + \frac{\vartheta_1}{2\eta}(P_2 - P_1) \leq -I$
   $(A_2^{cl})^T P_2 + P_2 A_2^{cl} - \frac{\vartheta_2}{2\eta}(P_2 - P_1) \leq -I$

which in case of $\eta = -1/2$ ($\eta = 1/2$), are essentially the same conditions as the one proposed in [11], in which the strategy is to select the closed-loop linear subsystem corresponding to the largest (smallest) local quadratic Lyapunov function. A negative value of η means that the subsystem corresponding to the largest local quadratic Lyapunov function is activated and a positive value of η means that the subsystem corresponding to the smallest local quadratic Lyapunov function is activated. Hence, when a specific strategy is decided *a priori*, it means that the sign of η implicitly is decided. The specific choice of strategy is crucial, since a wrong strategy may lead to the non-existence of a solution satisfying the stability conditions (see for instance Example IV in which case a strategy to activate the subsystem corresponding to the smallest local Lyapunov function would not lead to any solution satisfying the stability conditions due to the vector field direction of the two subsystems).

IV. EXAMPLE

We will now illustrate the synthesis procedure in this paper in the case of two unstable subsystems given by

$$A_1 = \begin{bmatrix} 1 & -5 \\ 0 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

for different values of κ. Due to symmetry in this example, it can be seen that the optimal value of ϑ_1 is equal to ϑ_2, and the gains $K_1 = [k_{11}\ k_{12}]$ and $K_2 = [k_{21}\ k_{22}]$ are related according to $k_{21} = k_{12}$ and $k_{22} = -k_{11}$, reducing the number of unknowns by two. Furthermore, the optimal gains are obtained with equalities, reducing the unknowns by one more. By scaling $\eta = -1$ (implying that the switched linear subsystem corresponding to the largest local Lyapunov function is chosen, according to the discussion in the previous section), only $\vartheta = \vartheta_1 = \vartheta_2$ and one gain parameter remain to be gridded.

a) $\kappa = 0$: In this case, it is not possible to stabilize the system by any switching at all, so there is of course no solution to Corollary 1.

b) $\kappa = 2$: Gridding up the unknown parameters, and solving the corresponding LMI problem in Corollary 1, results in a solution

$$\beta = 3.3993\ (\nu_1 = \nu_2 = 0), \quad \alpha = 2.0077\ (\mu_1 = \mu_2 = 1),$$

and

$$\begin{aligned}
K_1 &= \begin{bmatrix} 0 & -2 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -2 & 0 \end{bmatrix}, \\
P_1 &= = \begin{bmatrix} 1.0432 & 1.0031 \\ 1.0031 & 2.9722 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 2.9722 & -1.0031 \\ -1.0031 & 1.0432 \end{bmatrix}, \\
Q_1 &= -Q_2 = \tfrac{1}{2\eta}(P_2 - P_1) = \begin{bmatrix} -0.9645 & 1.0031 \\ 1.0031 & 0.9645 \end{bmatrix}.
\end{aligned}$$

Since the conditions in Lemma 1 are fulfilled, no sliding motions occur. Hence, applying the *largest region function strategy* results in an exponentially stable switched system, where the estimate of the exponential convergence becomes

$$||x(t)|| \leq 1.3012 e^{-0.1471t} ||x_0||.$$

Fig. 1 shows a trajectory simulation and the contour curves of the two quadratic Lyapunov functions $x^T P_1 x$ and $x^T P_2 x$. The shaded region corresponds to the region $x^T Q_1 x \geq 0$ where the closed-loop subsystem 1 is active and the energy is measured by $x^T P_1 x$, and the non-shaded region corresponds to $x^T Q_2 x \geq 0$ where the closed-loop subsystem 2 is active and the energy is measured by $x^T P_2 x$. The lines between the two regions indicate where the subsystem changes occur, and are given by the solution to $x^T Q x = x^T (Q_1 - Q_2) x = 0$.

c) $\kappa = 4$: The solution now becomes

$$\beta = 0.5910\ (\nu_1 = \nu_2 = 0), \quad \alpha = 0.3554\ (\mu_1 = \mu_2 = 1),$$

and

$$\begin{aligned}
K_1 &= \begin{bmatrix} 0 & -4 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -4 & 0 \end{bmatrix}, \\
Q_1 &= -Q_2 = Q = \begin{bmatrix} -0.2037 & 0.1185 \\ 0.1185 & 0.2037 \end{bmatrix},
\end{aligned}$$

which implies that the *largest region function strategy* results in the estimate

$$||x(t)|| \leq 1.2896 e^{-0.8460t} ||x_0||.$$

A trajectory simulation and the contour curves of the two quadratic Lyapunov functions can be seen in Fig. 2.

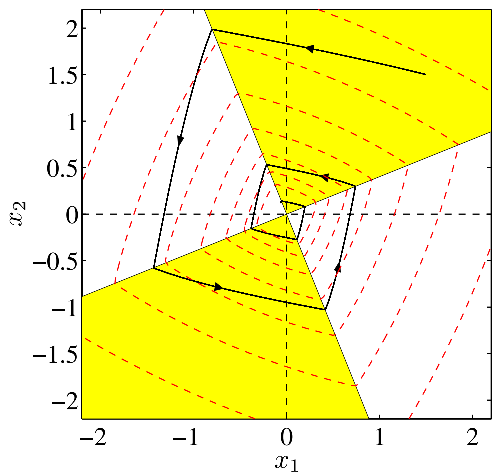

Fig. 1. In the case when $\kappa = 2$, the application of the synthesis method in this paper results in a switched system that is exponentially stable, indicated by the trajectory simulation. The dashed lines are the contour curves of the two quadratic Lyapunov functions $x^T P_1 x$ and $x^T P_2 x$, that measure the switched system's (abstract) energy in the regions $x^T Q_1 x \geq 0$ (shaded region) and $x^T Q_2 x \geq 0$ respectively.

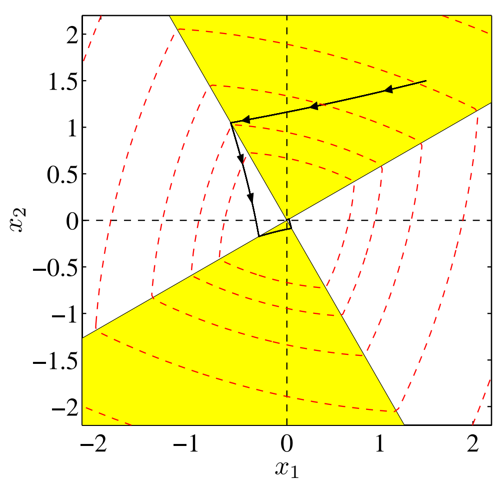

Fig. 2. In the case when $\kappa = 4$, the application of the synthesis method in this paper results in a switched system that is exponentially stable, with faster convergence than in the case $\kappa = 2$, indicated by the trajectory simulation. The dashed lines are the contour curves of the two quadratic Lyapunov functions $x^T P_1 x$ and $x^T P_2 x$, that measure the switched system's (abstract) energy in the regions $x^T Q_1 x \geq 0$ (shaded region) and $x^T Q_2 x \geq 0$ respectively.

V. Conclusions

A *constructive synthesis method*, applicable to a broader class of switched linear systems with input signals than earlier proposed in the literature, has been presented in this paper. The method is based on the use of multiple quadratic Lyapunov functions, one for each linear subsystem, and conditions are introduced guaranteeing the energy to decrease at all time. The synthesis problem is formulated as a bilinear matrix inequality (BMI) problem, where unknown scalars are multiplied by unknown matrices. A solution to the problem gives regions where the different subsystems are activated, resulting in a switched linear system that is exponentially stable. An example is given to illustrate the success of the method.

By exploiting the specific structure of the resulting BMI problem, more efficient algorithms (than gridding the unknown scalars) might be developed. Furthermore, in case of third-order systems or higher, there is a need of efficient methods to verify the non-existence of sliding motions. However, finding efficient methods for solving these type of problems is beyond the scope of this paper but are interesting future research problems. Up to this point, the main attention has been given to the synthesis problem formulation, since there have been a lack of constructively design results in the literature.

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