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Vector Dissipativity Theory for Discrete-Time Large-Scale Nonlinear Dynamical Systems

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***Abstract*— In analyzing large-scale systems, it is often desirable to treat the overall system as a collection of interconnected subsystems. Solution properties of the large-scale system are then deduced from the solution properties of the individual subsystems and the nature of the system interconnections. In this paper we develop an analysis framework for discrete-time large-scale dynamical systems based on *vector dissipativity* notions. Specifically, using vector storage functions and vector supply rates, dissipativity properties of the discrete-time composite large-scale system are shown to be determined from the dissipativity properties of the subsystems and their interconnections.**

I. Introduction

Modern complex dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication network constraints. The sheer size (i.e., dimensionality) and complexity of these large-scale dynamical systems often necessitates a hierarchical decentralized architecture for analyzing and controlling these systems. Specifically, in the analysis and control-system design of complex large-scale dynamical systems it is often desirable to treat the overall system as a collection of interconnected subsystems. The behavior of the aggregate or composite (i.e., large-scale) system can then be predicted from the behaviors of the individual subsystems and their interconnections. The need for decentralized analysis and control design of large-scale systems is a direct consequence of the physical size and complexity of the dynamical model. In particular, computational complexity may be too large for model analysis while severe constraints on communication links between system sensors, actuators, and processors may render centralized control architectures impractical.

An approach to analyzing large-scale dynamical systems was introduced by the pioneering work of Šiljak [1] and involves the notion of *connective stability*. In particular, the large-scale dynamical system is decomposed into a collection of subsystems with local dynamics and uncertain interactions. Then, each subsystem is considered independently so that the stability of each subsystem is combined with the interconnection constraints to obtain a *vector Lyapunov function* for the composite large-scale dynamical system guaranteeing connective stability for the overall system. Vector Lyapunov functions were first introduced by Bellman [2] and Matrosov [3] and further developed by Lakshmikantham *et al.* [4], with [1], [5]–[10] exploiting their utility for analyzing large-scale systems. The use of vector Lyapunov functions in large-scale system analysis offers a very flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single scalar Lyapunov function. Moreover, in large-scale systems several Lyapunov functions arise naturally from the stability properties of each subsystem. An alternative approach to vector Lyapunov functions for analyzing large-scale dynamical systems is an input-output approach wherein stability criteria are derived by assuming that each subsystem is either finite gain, passive, or conic [11]–[14].

Since most physical processes evolve naturally in continuous-time, it is not surprising that the bulk of large-scale dynamical system theory has been developed for continuous-time systems. Nevertheless, it is the overwhelming trend to implement controllers digitally. Hence, in this paper we extend the notions of dissipativity theory [15], [16] to develop *vector dissipativity* notions for large-scale nonlinear discrete-time dynamical systems; a notion not previously considered in the literature. In particular, we introduce a generalized definition of dissipativity for large-scale nonlinear discrete-time dynamical systems in terms of a *vector inequality* involving a *vector supply rate*, a *vector storage function*, and a nonnegative, semistable dissipation matrix. Generalized notions of vector available storage and vector required supply are also defined and shown to be element-by-element ordered, nonnegative, and finite. On the subsystem level, the proposed approach provides a discrete energy flow balance in terms of the stored subsystem energy, the supplied subsystem energy, the subsystem energy gained from all other subsystems independent of the subsystem coupling strengths, and the subsystem energy dissipated. Furthermore, for large-scale discrete-time dynamical systems decomposed into interconnected subsystems, dissipativity of the composite system is shown to be determined from the dissipativity properties of the individual subsystems and the nature of the interconnections.

II. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results needed for analyzing discrete-time large-scale nonlinear dynamical systems. Let $\mathbb{R}$ denote the set of real numbers, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, $\mathbb{R}^n$ denote the set of $n \times 1$ column vectors, $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, $\mathbb{N}^n$ (respectively, $\mathbb{P}^n$) denote the the set of $n \times n$ nonnegative (respectively, positive) definite matrices, $(\cdot)^{\mathrm{T}}$ denote transpose, and let I_n or I denote the $n \times n$ identity matrix. For $v \in \mathbb{R}^q$ we write $v \geq\geq 0$ (respectively, $v >> 0$) to indicate that every component of v is nonnegative (respectively, positive). In this case we say that v is *nonnegative* or *positive*, respectively. Let $\overline{\mathbb{R}}_+^q$ and $\mathbb{R}_+^q$ denote the nonnegative and positive orthants of $\mathbb{R}^q$; that is, if $v \in \mathbb{R}^q$, then $v \in \overline{\mathbb{R}}_+^q$ and $v \in \mathbb{R}_+^q$ are equivalent, respectively, to $v \geq\geq 0$ and $v >> 0$. Finally, we write $\|\cdot\|$ for the Euclidean vector norm,

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$\Delta V(x(k))$ for $V(x(k+1)) - V(x(k))$, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball centered at α with radius ε, and $M \geq 0$ (respectively, $M > 0$) to denote the fact that the Hermitian matrix M is nonnegative (respectively, positive) definite. The following definition introduces the notion of nonnegative matrices.

Definition 2.1 ([17]–[19]): Let $W \in \mathbb{R}^{q\times q}$. W is *nonnegative* (respectively, *positive*) if $W_{(i,j)} \geq 0$ (respectively, $W_{(i,j)} > 0$), $i, j = 1, \ldots, q$.

The following definition introduces the notion of class $\mathcal{W}$ functions involving nondecreasing functions.

Definition 2.2: A function $w = [w_1, ..., w_q]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}^q$ is of *class* $\mathcal{W}$ if $w_i(r') \leq w_i(r'')$, $i = 1, ..., q$, for all $r', r'' \in \mathbb{R}^q$ such that $r'_j \leq r''_j$, $j = 1, ..., q$, where r_j denotes the jth component of r.

Note that if $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$, then the function $w(\cdot)$ is of class $\mathcal{W}$ if and only if W is nonnegative. The following definition introduces the notion of nonnegative functions [19].

Definition 2.3: Let $w = [w_1, \cdots, w_q]^{\mathrm{T}} : \mathcal{V} \to \mathbb{R}^q$, where $\mathcal{V}$ is an open subset of $\mathbb{R}^q$ that contains $\overline{\mathbb{R}}^q_+$. Then w is *nonnegative* if $w(r) \geq\geq 0$ for all $r \in \overline{\mathbb{R}}^q_+$.

Note that if $w : \mathbb{R}^q \to \mathbb{R}^q$ is such that $w(\cdot) \in \mathcal{W}$ and $w(0) \geq\geq 0$, then w is nonnegative. Note that, if $w(r) = Wr$, then $w(\cdot)$ is nonnegative if and only if $W \in \mathbb{R}^{q\times q}$ is nonnegative.

Proposition 2.1 ([19]): Suppose $\overline{\mathbb{R}}^q_+ \subset \mathcal{V}$. Then $\overline{\mathbb{R}}^q_+$ is an invariant set with respect to

$$r(k+1) = w(r(k)), \quad r(0) = r_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{1}$$

where $r_0 \in \overline{\mathbb{R}}^q_+$, if and only if $w : \mathcal{V} \to \mathbb{R}^q$ is nonnegative.

The following definition and lemma are needed for developing several of the results in later sections.

Definition 2.4: The equilibrium solution $r(k) \equiv r_{\mathrm{e}}$ of (1) is *Lyapunov stable* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $r_0 \in \mathcal{B}_\delta(r_{\mathrm{e}}) \cap \overline{\mathbb{R}}^q_+$, then $r(k) \in \mathcal{B}_\varepsilon(r_{\mathrm{e}}) \cap \overline{\mathbb{R}}^q_+$, $k \in \overline{\mathbb{Z}}_+$. The equilibrium solution $r(k) \equiv r_{\mathrm{e}}$ of (1) is *semistable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $r_0 \in \mathcal{B}_\delta(r_{\mathrm{e}}) \cap \overline{\mathbb{R}}^q_+$, then $\lim_{k\to\infty} r(k)$ exists and converges to a Lyapunov stable equilibrium point. The equilibrium solution $r(k) \equiv r_{\mathrm{e}}$ of (1) is *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $r_0 \in \mathcal{B}_\delta(r_{\mathrm{e}}) \cap \overline{\mathbb{R}}^q_+$, then $\lim_{k\to\infty} r(k) = r_{\mathrm{e}}$. Finally, the equilibrium solution $r(k) \equiv r_{\mathrm{e}}$ of (1) is *globally asymptotically stable* if the previous statement holds for all $r_0 \in \overline{\mathbb{R}}^q_+$.

Recall that a matrix $W \in \mathbb{R}^{q\times q}$ is *semistable* if and only if $\lim_{k\to\infty} W^k$ exists [19] while W is *asymptotically stable* if and only if $\lim_{k\to\infty} W^k = 0$.

Lemma 2.1: Suppose $W \in \mathbb{R}^{q\times q}$ is nonsingular and nonnegative. If W is semistable (respectively, asymptotically stable), then there exist a scalar $\alpha \geq 1$ (respectively, $\alpha > 1$) and a nonnegative vector $p \in \overline{\mathbb{R}}^q_+$, $p \neq 0$, (respectively, positive vector $p \in \mathbb{R}^q_+$) such that

$$W^{-\mathrm{T}} p = \alpha p. \tag{2}$$

Next, we present a stability result for discrete-time large-scale nonlinear dynamical systems using vector Lyapunov functions. In particular, we consider discrete-time nonlinear dynamical systems of the form

$$x(k+1) = F(x(k)), \quad x(k_0) = x_0, \quad k \geq k_0, \tag{3}$$

where $F : \mathcal{D} \to \mathbb{R}^n$ is continuous on $\mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, and $F(0) = 0$. Here, we assume that (3) characterizes a discrete-time large-scale nonlinear dynamical system composed of q interconnected subsystems such that, for all $i = 1, ..., q$, each element of $F(x)$ is given by $F_i(x) = f_i(x_i) + \mathcal{I}_i(x)$, where $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ defines the vector field of each isolated subsystem of (3), $\mathcal{I}_i : \mathcal{D} \to \mathbb{R}^{n_i}$ defines the structure of interconnection dynamics of the ith subsystem with all other subsystems, $x_i \in \mathbb{R}^{n_i}$, $f_i(0) = 0$, $\mathcal{I}_i(0) = 0$, and $\sum_{i=1}^q n_i = n$. For the discrete-time large-scale nonlinear dynamical system (3) we note that the subsystem states $x_i(k)$, $k \geq k_0$, for all $i = 1, ..., q$, belong to $\mathbb{R}^{n_i}$ as long as $x(k) \triangleq [x_1^{\mathrm{T}}(k), ..., x_q^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathcal{D}$, $k \geq k_0$. The next theorem presents a stability result for (3) via vector Lyapunov functions by relating the stability properties of a *comparison system* to the stability properties of the discrete-time large-scale nonlinear dynamical system.

Theorem 2.1 ([4]): Consider the discrete-time large-scale nonlinear dynamical system given by (3). Suppose there exist a continuous vector function $V : \mathcal{D} \to \overline{\mathbb{R}}^q_+$ and a positive vector $p \in \mathbb{R}^q_+$ such that $V(0) = 0$, the scalar function $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ defined by $v(x) = p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is such that $v(0) = 0$, $v(x) > 0$, $x \neq 0$, and

$$V(F(x)) \leq\leq w(V(x)), \quad x \in \mathcal{D}, \tag{4}$$

where $w : \overline{\mathbb{R}}^q_+ \to \mathbb{R}^q$ is a class $\mathcal{W}$ function such that $w(0) = 0$. Then the stability properties of the zero solution $r(k) \equiv 0$ to

$$r(k+1) = w(r(k)), \quad r(k_0) = r_0, \quad k \geq k_0, \tag{5}$$

imply the corresponding stability properties of the zero solution $x(k) \equiv 0$ to (3). That is, if the zero solution $r(k) \equiv 0$ to (5) is Lyapunov (respectively, asymptotically) stable, then the zero solution $x(k) \equiv 0$ to (3) is Lyapunov (respectively, asymptotically) stable. If, in addition, $\mathcal{D} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then global asymptotic stability of the zero solution $r(k) \equiv 0$ to (5) implies global asymptotic stability of the zero solution $x(k) \equiv 0$ to (3).

If $V : \mathcal{D} \to \overline{\mathbb{R}}^q_+$ satisfies the conditions of Theorem 2.1 we say that $V(x)$, $x \in \mathcal{D}$, is a *vector Lyapunov function* for the discrete-time large-scale nonlinear dynamical system (3). Finally, we recall the notions of dissipativity [20] and geometric dissipativity [19], [21] for discrete-time nonlinear dynamical systems $\mathcal{G}$ of the form

$$\begin{aligned} x(k+1) &= f(x(k)) + G(x(k))u(k), \\ &\qquad x(k_0) = x_0, \quad k \geq k_0, \end{aligned} \tag{6}$$

$$y(k) = h(x(k)) + J(x(k))u(k), \tag{7}$$

where $x \in \mathcal{D} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^l$, $f : \mathcal{D} \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G : \mathcal{D} \to \mathbb{R}^{n\times m}$, $h : \mathcal{D} \to \mathbb{R}^l$ and satisfies $h(0) = 0$, and $J : \mathcal{D} \to \mathbb{R}^{l\times m}$. For the discrete-time nonlinear dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $u(\cdot)$ satisfies sufficient regularity conditions such that (6) has a unique solution forward in time. Note that since all input-output pairs $u \in \mathcal{U}$, $y \in \mathcal{Y}$, of the discrete-time nonlinear dynamical system $\mathcal{G}$ are defined on $\overline{\mathbb{Z}}_+$, the *supply rate* [15] satisfying $s(0,0) = 0$ is locally summable for all input-output pairs satisfying (6), (7); that is, for all input-output pairs $u \in \mathcal{U}$, $y \in \mathcal{Y}$ satisfying (6), (7), $s(\cdot,\cdot)$ satisfies $\sum_{k=k_1}^{k_2} |s(u(k), y(k))| < \infty$, $k_1, k_2 \in \overline{\mathbb{Z}}_+$.

Definition 2.5 ([20], [21]): The discrete-time nonlinear dynamical system $\mathcal{G}$ given by (6), (7) is *geometrically dissipative* (respectively, *dissipative*) with respect to the supply rate $s(u,y)$ if there exist a continuous nonnegative-definite function $v_{\rm s}:\mathbb{R}^n\to\overline{\mathbb{R}}_+$, called a *storage function*, and a scalar $\rho>1$ (respectively, $\rho=1$) such that $v_{\rm s}(0)=0$ and the *dissipation inequality*

$$\rho^{k_2}v_{\rm s}(x(k_2)) \leq \rho^{k_1}v_{\rm s}(x(k_1)) + \sum_{i=k_1}^{k_2-1}\rho^{i+1}s(u(i),y(i)),\ k_2\geq k_1, \quad (8)$$

is satisfied for all $k_2\geq k_1\geq k_0$, where $x(k), k\geq k_0$, is the solution to (6) with $u\in\mathcal{U}$. The discrete-time nonlinear dynamical system $\mathcal{G}$ given by (6), (7) is *lossless with respect to the supply rate* $s(u,y)$ if the dissipation inequality is satisfied as an equality with $\rho=1$ for all $k_2\geq k_1\geq k_0$.

III. Vector Dissipativity Theory for Discrete-Time Large-Scale Nonlinear Dynamical Systems

In this section we extend the notion of dissipative dynamical systems to develop the generalized notion of vector dissipativity for discrete-time large-scale nonlinear dynamical systems. We begin by considering discrete-time nonlinear dynamical systems $\mathcal{G}$ of the form

$$x(k+1) = F(x(k),u(k)),\ x(k_0)=x_0,\ k\geq k_0, \quad (9)$$
$$y(k) = H(x(k),u(k)), \quad (10)$$

where $x\in\mathcal{D}\subseteq\mathbb{R}^n$, $u\in\mathcal{U}\subseteq\mathbb{R}^m$, $y\in\mathcal{Y}\subseteq\mathbb{R}^l$, $F:\mathcal{D}\times\mathcal{U}\to\mathbb{R}^n$, $H:\mathcal{D}\times\mathcal{U}\to\mathcal{Y}$, $\mathcal{D}$ is an open set with $0\in\mathcal{D}$, and $F(0,0)=0$. Here, we assume that $\mathcal{G}$ represents a discrete-time large-scale dynamical system composed of q interconnected controlled subsystems $\mathcal{G}_i$ such that, for all $i=1,...,q$,

$$F_i(x,u_i) = f_i(x_i)+\mathcal{I}_i(x)+G_i(x_i)u_i, \quad (11)$$
$$H_i(x_i,u_i) = h_i(x_i)+J_i(x_i)u_i, \quad (12)$$

where $x_i\in\mathbb{R}^{n_i}$, $u_i\in\mathcal{U}_i\subseteq\mathbb{R}^{m_i}$, $y_i\triangleq H_i(x_i,u_i)\in\mathcal{Y}_i\subseteq\mathbb{R}^{l_i}$, (u_i,y_i) is the input-output pair for the ith subsystem, $f_i:\mathbb{R}^{n_i}\to\mathbb{R}^{n_i}$ and $\mathcal{I}_i:\mathcal{D}\to\mathbb{R}^{n_i}$ are continuous and satisfy $f_i(0)=0$ and $\mathcal{I}_i(0)=0$, $G_i:\mathbb{R}^{n_i}\to\mathbb{R}^{n_i\times m_i}$ is continuous, $h_i:\mathbb{R}^{n_i}\to\mathbb{R}^{l_i}$ and satisfies $h_i(0)=0$, $J_i:\mathbb{R}^{n_i}\to\mathbb{R}^{l_i\times m_i}$, $\sum_{i=1}^q n_i=n$, $\sum_{i=1}^q m_i=m$, and $\sum_{i=1}^q l_i=l$. Furthermore, for the system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied. We define the composite input and composite output for the discrete-time large-scale system $\mathcal{G}$ as $u\triangleq[u_1^{\rm T},...,u_q^{\rm T}]^{\rm T}$ and $y\triangleq[y_1^{\rm T},...,y_q^{\rm T}]^{\rm T}$, respectively. Note that in this case the set $\mathcal{U}=\mathcal{U}_1\times\cdots\times\mathcal{U}_q$ contains the set of input values and $\mathcal{Y}=\mathcal{Y}_1\times\cdots\times\mathcal{Y}_q$ contains the set of output values.

Definition 3.1: For the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10) a vector function $S=[s_1,...,s_q]^{\rm T}:\mathcal{U}\times\mathcal{Y}\to\mathbb{R}^q$ such that $S(u,y)\triangleq[s_1(u_1,y_1),...,s_q(u_q,y_q)]^{\rm T}$ and $S(0,0)=0$ is called a *vector supply rate*.

Definition 3.2: The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10) is *vector dissipative* (respectively, *geometrically vector dissipative*) *with respect to the vector supply rate* $S(u,y)$ if there exist a continuous, nonnegative definite vector function $V_{\rm s}=[v_{\rm s1},...,v_{\rm sq}]^{\rm T}:\mathcal{D}\to\overline{\mathbb{R}}_+^q$, called a *vector storage function*, and a nonsingular nonnegative *dissipation matrix* $W\in\mathbb{R}^{q\times q}$ such that $V_{\rm s}(0)=0$, W is semistable (respectively, asymptotically stable), and the *vector dissipation inequality*

$$V_{\rm s}(x(k)) \leq\leq W^{k-k_0}V_{\rm s}(x(k_0)) + \sum_{i=k_0}^{k-1}W^{k-1-i}S(u(i),y(i)),\quad k\geq k_0, \quad (13)$$

is satisfied, where $x(k)$, $k\geq k_0$, is the solution to (9) with $u\in\mathcal{U}$. The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10) is *vector lossless with respect to the vector supply rate* $S(u,y)$ if the vector dissipation inequality is satisfied as an equality with W semistable.

Note that if the subsystems $\mathcal{G}_i$ of $\mathcal{G}$ are *disconnected*; that is, $\mathcal{I}_i(x)\equiv 0$ for all $i=1,...,q$, and $W\in\mathbb{R}^{q\times q}$ is diagonal, positive definite, and semistable, then it follows from Definition 3.2 that each of isolated subsystems $\mathcal{G}_i$ is dissipative or geometrically dissipative in the sense of Definition 2.5. A similar remark holds in the case where $q=1$. Next, define the *vector available storage* of the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ by

$$V_{\rm a}(x_0) \triangleq \sup_{K\geq k_0,\,u(\cdot)}\left[-\sum_{k=k_0}^{K-1}W^{-(k+1-k_0)}S(u(k),y(k))\right], \quad (14)$$

where $x(k)$, $k\geq k_0$, is the solution to (9) with $x(k_0)=x_0$ and admissible inputs $u\in\mathcal{U}$. The supremum in (14) is taken componentwise which implies that for different elements of $V_{\rm a}(\cdot)$ the supremum is calculated separately. Note, that $V_{\rm a}(x_0)\geq\geq 0$, $x_0\in\mathcal{D}$, since $V_{\rm a}(x_0)$ is the supremum over a set of vectors containing the zero vector ($K=k_0$). To state the main results of this section the following definition is required.

Definition 3.3 ([19]): The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10) is *completely reachable* if for all $x_0\in\mathcal{D}\subseteq\mathbb{R}^n$, there exist a $k_{\rm i}<k_0$ and a square summable input $u(\cdot)$ defined on $[k_{\rm i},k_0]$ such that the state $x(k)$, $k\geq k_{\rm i}$, can be driven from $x(k_{\rm i})=0$ to $x(k_0)=x_0$. A discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ is *zero-state observable* if $u(k)\equiv 0$ and $y(k)\equiv 0$ imply $x(k)\equiv 0$.

Theorem 3.1: Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10) and assume that $\mathcal{G}$ is completely reachable. Let $W\in\mathbb{R}^{q\times q}$ be nonsingular, nonnegative, and semistable (respectively, asymptotically stable). Then

$$\sum_{k=k_0}^{K-1}W^{-(k+1-k_0)}S(u(k),y(k))\geq\geq 0,\quad K\geq k_0,\quad u\in\mathcal{U}, \quad (15)$$

for $x(k_0)=0$ if and only if $V_{\rm a}(0)=0$ and $V_{\rm a}(x)$ is finite for all $x\in\mathcal{D}$. Moreover, if (15) holds, then $V_{\rm a}(x)$, $x\in\mathcal{D}$, is a vector storage function for $\mathcal{G}$ and hence $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u,y)$.

It follows from Lemma 2.1 that if $W\in\mathbb{R}^{q\times q}$ is nonsingular, nonnegative, and semistable (respectively, asymptotically stable), then there exist a scalar $\alpha\geq 1$ (respectively, $\alpha>1$) and a nonnegative vector $p\in\overline{\mathbb{R}}_+^q$, $p\neq 0$, (respectively, $p \in \mathbb{R}^q_+$) such that (2) holds. In this case,

$$p^{\mathrm{T}} W^{-k} = \alpha p^{\mathrm{T}} W^{-(k-1)} = \cdots = \alpha^k p^{\mathrm{T}},\ k \in \overline{\mathbb{Z}}_+. \quad (16)$$

Using (16), we define the (scalar) *available storage* for the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ by

$$\begin{aligned} v_{\mathrm{a}}(x_0) &\triangleq \sup_{K \geq k_0,\, u(\cdot)} \left[-\sum_{k=k_0}^{K-1} p^{\mathrm{T}} W^{-(k+1-k_0)} S(u(k), y(k)) \right] \\ &= \sup_{K \geq k_0,\, u(\cdot)} \left[-\sum_{k=k_0}^{K-1} \alpha^{k+1-k_0} s(u(k), y(k)) \right], \quad (17) \end{aligned}$$

where $s : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ defined as $s(u,y) \triangleq p^{\mathrm{T}} S(u,y)$ is the (scalar) supply rate for the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$. Clearly, $v_{\mathrm{a}}(x) \geq 0$ for all $x \in \mathcal{D}$. As in standard dissipativity theory, the available storage $v_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, denotes the maximum amount of (scaled) energy that can be extracted from the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ at any instant K.

The following theorem relates vector storage functions and vector supply rates to scalar storage functions and scalar supply rates of discrete-time large-scale dynamical systems.

Theorem 3.2: Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10). Suppose $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}^q$ and with vector storage function $V_{\mathrm{s}} : \mathcal{D} \to \overline{\mathbb{R}}^q_+$. Then there exists $p \in \overline{\mathbb{R}}^q_+$, $p \neq 0$, (respectively, $p \in \mathbb{R}^q_+$) such that $\mathcal{G}$ is dissipative (respectively, geometrically dissipative) with respect to the scalar supply rate $s(u,y) = p^{\mathrm{T}} S(u,y)$ and with storage function $v_{\mathrm{s}}(x) \triangleq p^{\mathrm{T}} V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, satisfying

$$\begin{aligned} v_{\mathrm{s}}(x(k)) \;&\leq\; \alpha^{-(k-k_0)} v_{\mathrm{s}}(x(k_0)) \\ &\quad + \sum_{i=k_0}^{k-1} \alpha^{-(k-1-i)} s(u(i), y(i)), \\ &\qquad\qquad k \geq k_0, \quad u \in \mathcal{U}, \end{aligned} \quad (18)$$

where $\alpha \geq 1$ (respectively, $\alpha > 1$). Moreover, in this case $v_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$ and

$$0 \leq v_{\mathrm{a}}(x) \leq v_{\mathrm{s}}(x), \quad x \in \mathcal{D}. \quad (19)$$

Remark 3.1: It follows from Theorem 3.1 that if (15) holds for $x(k_0) = 0$, then the vector available storage $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$. In this case, it follows from Theorem 3.2 that there exists $p \in \overline{\mathbb{R}}^q_+$, $p \neq 0$, such that $v_{\mathrm{s}}(x) \triangleq p^{\mathrm{T}} V_{\mathrm{a}}(x)$ is a storage function for $\mathcal{G}$ that satisfies (18), and hence by (19), $v_{\mathrm{a}}(x) \leq p^{\mathrm{T}} V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$.

Remark 3.2: It is important to note that it follows from Theorem 3.2 that if $\mathcal{G}$ is vector dissipative, then $\mathcal{G}$ can either be (scalar) dissipative or (scalar) geometrically dissipative.

The following theorem provides sufficient conditions guaranteeing that all scalar storage functions defined in terms of vector storage functions; that is, $v_{\mathrm{s}}(x) = p^{\mathrm{T}} V_{\mathrm{s}}(x)$, of a given vector dissipative discrete-time large-scale nonlinear dynamical system are positive definite.

Theorem 3.3: Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10) and assume that $\mathcal{G}$ is zero-state observable. Furthermore, assume that $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u,y)$ and there exist $\alpha \geq 1$ and $p \in \mathbb{R}^q_+$ such that (2) holds. In addition, assume that there exist functions $\kappa_i : \mathcal{Y}_i \to \mathcal{U}_i$ such that $\kappa_i(0) = 0$ and $s_i(\kappa_i(y_i), y_i) < 0$, $y_i \neq 0$, for all $i = 1, ..., q$. Then for all vector storage functions $V_{\mathrm{s}} : \mathcal{D} \to \overline{\mathbb{R}}^q_+$ the storage function $v_{\mathrm{s}}(x) \triangleq p^{\mathrm{T}} V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, is positive definite; that is, $v_{\mathrm{s}}(0) = 0$ and $v_{\mathrm{s}}(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$.

Next, we introduce the concept of *vector required supply* of a discrete-time large-scale nonlinear dynamical system. Specifically, define the vector required supply of the discrete-time large-scale dynamical system $\mathcal{G}$ by

$$\begin{aligned} &V_{\mathrm{r}}(x_0) \\ &\quad \triangleq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} W^{-(k+1-k_0)} S(u(k), y(k)), \quad (20) \end{aligned}$$

where $x(k)$, $k \geq -K$, is the solution to (9) with $x(-K) = 0$ and $x(k_0) = x_0$. Note that since, with $x(k_0) = 0$, the infimum in (20) is the zero vector it follows that $V_{\mathrm{r}}(0) = 0$. Moreover, since $\mathcal{G}$ is completely reachable it follows that $V_{\mathrm{r}}(x) << \infty$, $x \in \mathcal{D}$. Using the notion of the vector required supply we present necessary and sufficient conditions for dissipativity of a large-scale dynamical system with respect to a vector supply rate.

Theorem 3.4: Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10) and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u,y)$ if and only if

$$0 \leq\leq V_{\mathrm{r}}(x) << \infty, \quad x \in \mathcal{D}. \quad (21)$$

Moreover, if (21) holds, then $V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$. Finally, if the vector available storage $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$, then

$$0 \leq\leq V_{\mathrm{a}}(x) \leq\leq V_{\mathrm{r}}(x) << \infty, \quad x \in \mathcal{D}. \quad (22)$$

The next result is a direct consequence of Theorems 3.1 and 3.4.

Proposition 3.1: Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10). Let $M = \operatorname{diag}[\mu_1, ..., \mu_q]$ be such that $0 \leq \mu_i \leq 1$, $i = 1, ..., q$. If $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, and $V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, are vector storage functions for $\mathcal{G}$, then

$$V_{\mathrm{s}}(x) = M V_{\mathrm{a}}(x) + (I_q - M) V_{\mathrm{r}}(x), \quad x \in \mathcal{D}, \quad (23)$$

is a vector storage function for $\mathcal{G}$.

Next, recall that if $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative), then there exist $p \in \overline{\mathbb{R}}^q_+$, $p \neq 0$, and $\alpha \geq 1$ (respectively, $p \in \mathbb{R}^q_+$ and $\alpha > 1$) such that (2) and (16) hold. Now, define the (scalar) *required supply* for the large-scale nonlinear dynamical system $\mathcal{G}$ by

$$\begin{aligned} &v_{\mathrm{r}}(x_0) \\ &\quad \triangleq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} p^{\mathrm{T}} W^{-(k+1-k_0)} S(u(k), y(k)) \\ &\quad = \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} \alpha^{k+1-k_0} s(u(k), y(k)), \\ &\qquad\qquad x_0 \in \mathcal{D}, \quad (24) \end{aligned}$$

where $s(u,y) = p^{\mathrm{T}} S(u,y)$ and $x(k)$, $k \geq -K$, is the solution to (9) with $x(-K) = 0$ and $x(k_0) = x_0$. It follows from (24) that the required supply of a discrete-time large-scale nonlinear dynamical system is the minimum amount of generalized energy which can be delivered to the discrete-time large-scale system in order to transfer it from an initial state $x(-K)=0$ to a given state $x(k_0)=x_0$. Using the same arguments as in case of the vector required supply, it follows that $v_{\rm r}(0)=0$ and $v_{\rm r}(x)<\infty$, $x\in\mathcal{D}$.

Next, using the notion of required supply, we show that all storage functions of the form $v_{\rm s}(x)=p^{\rm T}V_{\rm s}(x)$, where $p\in\overline{\mathbb{R}}_+^q$, $p\neq 0$, are bounded from above by the required supply and bounded from below by the available storage. Hence, a dissipative discrete-time large-scale nonlinear dynamical system can only deliver to its surroundings a fraction of all of its stored subsystem energies and can only store a fraction of the work done to all of its subsystems.

Corollary 3.1: Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10). Assume that $\mathcal{G}$ is vector dissipative with respect to a vector supply rate $S(u,y)$ and with vector storage function $V_{\rm s}:\mathcal{D}\to\overline{\mathbb{R}}_+^q$. Then $v_{\rm r}(x)$, $x\in\mathcal{D}$, is a storage function for $\mathcal{G}$. Moreover, if $v_{\rm s}(x)\triangleq p^{\rm T}V_{\rm s}(x)$, $x\in\mathcal{D}$, where $p\in\overline{\mathbb{R}}_+^q$, $p\neq 0$, then

$$0\le v_{\rm a}(x)\le v_{\rm s}(x)\le v_{\rm r}(x)<\infty,\quad x\in\mathcal{D}. \tag{25}$$

Remark 3.3: It follows from Theorem 3.4 that if $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u,y)$, then $V_{\rm r}(x)$, $x\in\mathcal{D}$, is a vector storage function for $\mathcal{G}$ and, by Theorem 3.2, there exists $p\in\overline{\mathbb{R}}_+^q$, $p\neq 0$, such that $v_{\rm s}(x)\triangleq p^{\rm T}V_{\rm r}(x)$, $x\in\mathcal{D}$, is a storage function for $\mathcal{G}$ satisfying (18). Hence, it follows from Corollary 3.1 that $p^{\rm T}V_{\rm r}(x)\le v_{\rm r}(x)$, $x\in\mathcal{D}$.

The next result relates vector (respectively, scalar) available storage and vector (respectively, scalar) required supply for vector lossless discrete-time large-scale dynamical systems.

Theorem 3.5: Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10). Assume that $\mathcal{G}$ is completely reachable to and from the origin. If $\mathcal{G}$ is vector lossless with respect to the vector supply rate $S(u,y)$ and $V_{\rm a}(x)$, $x\in\mathcal{D}$, is a vector storage function, then $V_{\rm a}(x)=V_{\rm r}(x)$, $x\in\mathcal{D}$. Moreover, if $V_{\rm s}(x)$, $x\in\mathcal{D}$, is a vector storage function, then all (scalar) storage functions of the form $v_{\rm s}(x)=p^{\rm T}V_{\rm s}(x)$, $x\in\mathcal{D}$, where $p\in\overline{\mathbb{R}}_+^q$, $p\neq 0$, are given by

$$\begin{aligned} v_{\rm s}(x_0)=v_{\rm a}(x_0)&=v_{\rm r}(x_0)\\ &=-\sum_{k=k_0}^{K-1}\alpha^{k+1-k_0}s(u(k),y(k))\\ &=\sum_{k=-K}^{k_0-1}\alpha^{k+1-k_0}s(u(k),y(k)), \end{aligned}\tag{26}$$

where $x(k)$, $k\ge k_0$, is the solution to (9) with $u\in\mathcal{U}$, $x(-K)=0$, $x(K)=0$, $x(k_0)=x_0\in\mathcal{D}$, and $s(u,y)=p^{\rm T}S(u,y)$.

The next proposition presents a characterization for vector dissipativity of discrete-time large-scale nonlinear dynamical systems.

Proposition 3.2: Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (9), (10) and assume $V_{\rm s}=[v_{\rm s1},...,v_{{\rm s}q}]^{\rm T}:\mathcal{D}\to\overline{\mathbb{R}}_+^q$ is a continuous vector storage function for $\mathcal{G}$. Then $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u,y)$ if and only if

$$V_{\rm s}(x(k+1))\le\le WV_{\rm s}(x(k))+S(u(k),y(k)),\qquad k\ge k_0,\quad u\in\mathcal{U}. \tag{27}$$

As a special case of vector dissipativity theory we can analyze the stability of discrete-time large-scale nonlinear dynamical systems. Specifically, assume that the discrete-time large-scale dynamical system $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u,y)$ and with a continuous vector storage function $V_{\rm s}:\mathcal{D}\to\overline{\mathbb{R}}_+^q$. Moreover, assume that the conditions of Theorem 3.3 are satisfied. Then it follows from Proposition 3.2, with $u(k)\equiv 0$ and $y(k)\equiv 0$, that

$$V_{\rm s}(x(k+1))\le\le WV_{\rm s}(x(k)),\quad k\ge k_0, \tag{28}$$

where $x(k)$, $k\ge k_0$, is a solution to (9) with $x(k_0)=x_0$ and $u(k)\equiv 0$. Now, it follows from Theorem 2.1, with $w(r)=Wr$, that the zero solution $x(k)\equiv 0$ to (9), with $u(k)\equiv 0$, is Lyapunov (respectively, asymptotically) stable.

More generally, the problem of control system design for discrete-time large-scale nonlinear dynamical systems can be addressed within the framework of vector dissipativity theory. In particular, suppose that there exists a continuous vector function $V_{\rm s}:\mathcal{D}\to\overline{\mathbb{R}}_+^q$ such that $V_{\rm s}(0)=0$ and

$$V_{\rm s}(x(k+1))\le\le\mathcal{F}(V_{\rm s}(x(k)),u(k)),\ k\ge k_0,\ u\in\mathcal{U}, \tag{29}$$

where $\mathcal{F}:\overline{\mathbb{R}}_+^q\times\mathbb{R}^m\to\mathbb{R}^q$ and $\mathcal{F}(0,0)=0$. Then the control system design problem for a discrete-time large-scale dynamical system reduces to constructing an *energy* feedback control law $\phi:\overline{\mathbb{R}}_+^q\to\mathcal{U}$ of the form

$$u=\phi(V_{\rm s}(x))\triangleq[\phi_1^{\rm T}(V_{\rm s}(x)),...,\phi_q^{\rm T}(V_{\rm s}(x))]^{\rm T},\ x\in\mathcal{D}, \tag{30}$$

where $\phi_i:\overline{\mathbb{R}}_+^q\to\mathcal{U}_i$, $\phi_i(0)=0$, $i=1,...,q$, such that the zero solution $r(k)\equiv 0$ to the comparison system

$$r(k+1)=w(r(k)),\quad r(k_0)=V_{\rm s}(x(k_0)),\quad k\ge k_0, \tag{31}$$

is rendered asymptotically stable, where $w(r)\triangleq\mathcal{F}(r,\phi(r))$ is of class $\mathcal{W}$. In this case, if there exists $p\in\mathbb{R}_+^q$ such that $v_{\rm s}(x)\triangleq p^{\rm T}V_{\rm s}(x)$, $x\in\mathcal{D}$, is positive definite, then it follows from Theorem 2.1 that the zero solution $x(k)\equiv 0$ to (9), with u given by (30), is asymptotically stable.

As can be seen from the above discussion, using an energy feedback control architecture and exploiting the comparison system within the control design for discrete-time large-scale nonlinear dynamical systems can significantly reduce the dimensionality of a control synthesis problem in terms of a number of states that need to be stabilized. It should be noted however that for stability analysis of discrete-time large-scale dynamical systems the comparison system need not be linear as implied by (28). A discrete-time nonlinear comparison system would still guarantee stability of a discrete-time large-scale dynamical system provided that the conditions of Theorem 2.1 are satisfied. For further details see [22].

IV. Conclusion

In this paper we have extended the notion of dissipativity theory to vector dissipativity theory. Specifically, using vector storage functions and vector supply rates, dissipativity properties of aggregate large-scale discrete-time dynamical systems are shown to be determined from the dissipativity properties of the individual subsystems and the nature of their interconnections. Detailed proofs of the results in this paper are given in [22]. In addition, [22] develops extended Kalman-Yakubovich-Popov conditions, in terms of the local subsystem dynamics and the interconnection constraints, for characterizing vector dissipativeness via vector storage functions for large-scale discrete-time dynamical systems.

Furthermore, using the concepts of vector dissipativity and vector storage functions as candidate vector Lyapunov functions, feedback interconnection stability results of large-scale discrete-time nonlinear dynamical systems are also developed in [22]. General stability criteria are given for Lyapunov and asymptotic stability of feedback interconnections of large-scale discrete-time dynamical systems. In the case of vector quadratic supply rates involving net subsystem powers and input-output subsystem energies, these results provide a positivity and small gain theorem for large-scale discrete-time systems predicated on vector Lyapunov functions.

REFERENCES

[1] D. D. Šiljak, *Large-Scale Dynamic Systems: Stability and Structure*. New York, NY: Elsevier North-Holland Inc., 1978.

[2] R. Bellman, "Vector Lyapunov functions," *SIAM Journal of Control*, vol. 1, pp. 32–34, 1962.

[3] V. M. Matrosov, "Method of vector Liapunov functions of interconnected systems with distributed parameters (Survey)," *Avtomatika i Telemekhanika*, vol. 33, pp. 63–75, 1972, (in Russian).

[4] V. Lakshmikantham, V. M. Matrosov, and S. Sivasundaram, *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*. Dordrecht, Netherlands: Kluwer Academic Publishers, 1991.

[5] A. N. Michel and R. K. Miller, *Qualitative Analysis of Large Scale Dynamical Systems*. New York, NY: Academic Press, Inc., 1977.

[6] D. D. Šiljak, "Complex dynamical systems: Dimensionality, structure and uncertainty," *Large Scale Systems*, vol. 4, pp. 279–294, 1983.

[7] L. T. Grujić, A. A. Martynyuk, and M. Ribbens-Pavella, *Large Scale Systems Stability Under Structural and Singular Perturbations*. Berlin: Springer-Verlag, 1987.

[8] J. Lunze, "Stability analysis of large-scale systems composed of strongly coupled similar subsystems," *Automatica*, vol. 25, pp. 561–570, 1989.

[9] A. A. Martynyuk, *Stability by Liapunov's Matrix Function Method with Applications*. New York, NY: Marcel Dekker, Inc., 1998.

[10] ——, *Qualitative Methods in Nonlinear Dynamics. Novel Approaches to Liapunov's Matrix Functions*. New York, NY: Marcel Dekker, Inc., 2002.

[11] M. Araki, "Input-output stability of composite feedback systems," *IEEE Trans. Autom. Contr.*, vol. AC-21, pp. 254–259, 1976.

[12] E. L. Lasley and A. N. Michel, "Input-output stability of interconnected systems," *IEEE Trans. Autom. Contr.*, vol. AC-21, pp. 84–89, 1976.

[13] ——, "L_∞- and l_∞- stability of interconnected systems," *IEEE Trans. Circuits and Systs.*, vol. CAS-23, pp. 261–270, 1976.

[14] M. Vidyasagar, *Input-Output Analysis of Large-Scale Interconnected Systems*. Berlin: Springer-Verlag, 1981.

[15] J. C. Willems, "Dissipative dynamical systems part I: General theory," *Arch. Rational Mech. Anal.*, vol. 45, pp. 321–351, 1972.

[16] ——, "Dissipative dynamical systems part II: Linear systems with quadratic supply rates," *Arch. Rational Mech. Anal.*, vol. 45, pp. 352–393, 1972.

[17] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. New York, NY: Academic Press, Inc., 1979.

[18] D. S. Bernstein and D. C. Hyland, "Compartmental modeling and second-moment analysis of state space systems," *SIAM J. Matrix Anal. Appl.*, vol. 14, pp. 880–901, 1993.

[19] W. M. Haddad, V. Chellaboina, and E. August, "Stability and dissipativity theory for discrete-time nonnegative and compartmental dynamical systems," *Int. J. Contr.*, vol. 76, pp. 1845–1861, 2003.

[20] V. Chellaboina and W. M. Haddad, "Stability margins of discrete-time nonlinear-nonquadratic optimal regulators," *Int. J. Syst. Sci.*, vol. 33, pp. 577–584, 2002.

[21] W. M. Haddad and V. Chellaboina, "Nonlinear control of Hammerstein systems with passive nonlinear dynamics," *IEEE Trans. Autom. Contr.*, vol. 46, pp. 1630–1634, 2001.

[22] W. M. Haddad, Q. Hui, V. Chellaboina, and S. Nersesov, "Vector dissipativity theory for discrete-time large-scale nonlinear dynamical systems," *Adv. Diff. Eqs.*, vol. 2004, pp. 37–66, 2004.