Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Observers for a Class of Descriptor Systems with Lipschitz Constraint

Guoping Lu, Daniel W. C. Ho and Yufan Zheng

Abstract— **This paper considers observer design for a class of continuous-time descriptor systems with Lipschitz constraint. The constraint is a function of time, state and control input variables of the system. A new and simple approach is developed to design both full-order and reduced-order observers of the system. The sufficient condition is equivalent to the solvability of a linear matrix inequality. An illustrative example is presented to show the effectiveness of the proposed approach.**

I. INTRODUCTION

State estimation or observer design has received considerable attention in the last two decades. Many approaches have been developed to design observers for descriptor systems (see [5], [6], [7], [8], [9], [10], [11], [12], etc.). In [6], [7], [8], [9], [10], observer designs for linear descriptor systems are addressed. For example, full and reduced-order observers for linear descriptor systems are given in [7], [8], where a generalized Sylvester equation is applied. In [5], local asymptotic observer is obtained for general nonlinear descriptor systems by means of a coordinate transformation. In addition, a reduced order observer design approach is developed by means of generalized Sylvester equation. [5] gives a design procedure, it is simple, but the sufficient condition (62) of Theorem 3, which is the main result of [5], may depend on the choice of matrix R and is usually hard to satisfy. [11] considers a full order observer design for a class of continuous nonlinear descriptor systems subject to unknown inputs and faults. The approach is to divide the systems into dynamic system and static system. [12] studies observer issued for continuous nonlinear descriptor systems in quasi-linear form and presents a method to construct a full order state observer. The approach is based on transforming the descriptor system as an equivalent system of (explicit) differential equations on a restricted manifold, and an observer for the descriptor system can be constructed by common state space techniques for explicit systems. But reduced order observer design is not considered in [11], [12].

The first author was supported by NSF Grant (No. BK2003034) from Jiangsu Province and NSF Grant (No. 03KJB120101) from the Education Department of Jiangsu Province. This work was also supported by Hong Kong Research Grant Council (CityU 101103), as well as Australian Research Council.

Guoping Lu is with Department of Applied Mathematics, Nantong Institute of Technology, Nantong, Jiangsu, 226007, China gplu@pub.nt.jsinfo.net

Daniel W. C. Ho is with Department of Mathematics, City University of Hong Kong, Hong Kong madaniel@cityu.edu.hk

Yufan Zheng is with Department of Electrical and Electronic Engineering, The University of Melbourne, Victoria 3010, Australia y.zheng@ee.mu.oz.au

In this paper, we will discuss observer design for continuous-time descriptor systems with Lipschitz constraint. A new approach based on linear matrix inequality (LMI) is developed to construct both full order and reduced order observers. We will show that, under the same LMI condition, the design procedure for full and reduced observers can be easily constructed. Besides, the two observers both are global asymptotic estimators. It is noticed that the approach in this paper is different from those in [5], [11], [12]. Our sufficient condition is of a simpler form and it is independent of any coordinate transformation. We emphasize that when our LMI approach in this paper is used to construct full order and reduced order observers for nonsingular Lipschitz systems, our approach is also more efficient than that developed in [13], [14].

The paper is organized as follows. Section II presents a full order observer design. Section III develops a reduced order observer design based on Section II. Finally, the concluding remarks are made in Section V.

Notation W^T: transpose of matrix $W \in \mathbf{R}^{n\times m}$; $\|W\|$: $[\lambda_{\max}(W^TW)]^{\frac{1}{2}}$, *i.e.* the square root of the maximal eigenvalue of W^TW; X^{-T}: transpose of matrix X^{-1}; I (or I_r): identity matrix of appropriate dimension(or r dimension); $\|x\| = \sqrt{(x^Tx)}$, $\|x\|_\infty = \max\{|x_i|, 1 \le i \le n\}$, where $x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T \in \mathbf{R}^n$; Throughout this note, for symmetric matrices X and Y, $X > Y (X \ge Y)$ if $X - Y$ is positive positive definite (semi-definite); $X < Y (X \le Y)$ if $X - Y$ is negative negative definite (semi-definite). In a formula matrices are assumed to have compatible dimensions if there is not explicit explanation.

II. FULL-ORDER OBSERVER DESIGN

Consider the following model described by a continuous-time singular system with Lipschitz constraint.

$$\begin{aligned} E\dot{x} &= Ax + B\Phi(t, x, u), \quad x(0) = x_0, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ is the system state, $u \in \mathbf{R}^m$ is the system input, $y \in \mathbf{R}^r$ is the system output; $A, E \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times p}$ and $C \in \mathbf{R}^{r\times n}$ are constant matrices; E may be singular. Without loss of generality, we shall assume that $0 < \operatorname{rank}(E) = s < n$; $x(0) = x_0$ is a compatible initial condition; $\Phi = \Phi(t, x, u) \in \mathbf{R}^p$ is vector-valued time-varying nonlinear function and satisfies the following Lipschitz condition for all $(t, x, u), (t, \tilde{x}, u) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m$.

$$\|\Phi(t, x, u) - \Phi(t, \tilde{x}, u)\| \le \|F(x - \tilde{x})\|, \tag{2}$$

0-7803-8335-4/04/$17.00 ©2004 AACC

where F is a constant matrix with appropriate dimension.

In this section, we consider the Luenberger-like full-order observer as follows:

$$E\dot{\hat{x}} = A\hat{x} + B\Phi(t,\hat{x},u) - L(y - C\hat{x}), \tag{3}$$

where $L \in \mathbf{R}^{n\times p}$ is the observer parameter to be determined. Let $e = x - \hat{x}$, then it follows from (1) and (3) that

$$E\dot{e} = (A + LC)e + B[\Phi(t,x,u) - \Phi(t,\hat{x},u)]. \tag{4}$$

Now we are in position to state the following theorem which presents a way to construct a full-order observer via matrix inequalities.

Theorem 2.1: There exists a full-order observer (3) for system (1) if there exist two matrices $P \in \mathbf{R}^{n\times n}$ and $Q \in \mathbf{R}^{r\times n}$ such that the following matrix inequalities are solvable.

$$\begin{aligned} &E^TP = P^TE \geq 0,\\ &\Omega := \begin{pmatrix} A^TP + P^TA + C^TQ + Q^TC + F^TF & P^TB \\ B^TP & -I \end{pmatrix}\\ &< 0. \end{aligned} \tag{5}$$

Proof: Without loss of the generality, assume that solution P from matrix inequalities (5) is non-singular. In fact, there exist two non-singular matrices $M, N \in \mathbf{R}^{n\times n}$ such that $MEN = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$, then let $M^{-T}PN = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$, where $P_1 \in \mathbf{R}^{s\times s}$, $P_2 \in \mathbf{R}^{s\times(n-s)}$, $P_3 \in \mathbf{R}^{(n-s)\times s}$, $P_4 \in \mathbf{R}^{(n-s)\times(n-s)}$. It is easy to show that $E^TP = P^TE \geq 0$ implies that $P_2 = 0$ and $P_1 \geq 0$, then P can be rewritten as follows:

$$P = M^T \begin{pmatrix} P_1 & 0 \\ P_3 & P_4 \end{pmatrix} N^{-1}. \tag{6}$$

Partition (6) implies that there exists a sufficient small scalar $\epsilon > 0$ such that

$$P_\epsilon = M^T \begin{pmatrix} P_1 + \epsilon I_s & 0 \\ P_3 & P_4 + \epsilon I_{n-s} \end{pmatrix} N^{-1}$$

is non-singular and also satisfies inequalities (5) at the same time.

If Q and non-singular matrix P are solutions of inequalities (5), then we show that (3) with gain

$$L = P^{-T}Q^T \tag{7}$$

is a full-order observer for system (1), that is, system (4) is globally asymptotically stable.

(i) Let $A_c = A + LC$. Choosing the Lyapunov function candidate as follows:

$$V = e^TE^TPe. \tag{8}$$

For convenience, let $\lambda_0 = \lambda_{\min}(-\Omega)$ and $\phi = \Phi(t,x,u) - \Phi(t,\hat{x},u)$, then $\lambda_0 > 0$ and

$$\|\phi\| \leq \|Fe\|. \tag{9}$$

The derivative of V along system (4) yields to

$$\begin{aligned} \dot{V} &= (A_ce + B\phi)^TPe + e^TP^T(A_ce + B\phi)\\ &\leq (A_ce + B\phi)^TPe + e^TP^T(A_ce + B\phi)\\ &\quad -\phi^T\phi + e^TF^TFe\\ &= \begin{pmatrix} e^T & \phi^T \end{pmatrix} \Omega \begin{pmatrix} e \\ \phi \end{pmatrix}\\ &\leq -\lambda_0\|e\|^2. \end{aligned} \tag{10}$$

Denote $\lambda_m = \lambda_{\max}(E^TP)$, then $\lambda_m > 0$ and $V = e^TE^TPe \leq \lambda_m\|e\|^2$, that is, $\|e\|^2 \geq \lambda_m^{-1}V$. Thus, $\dot{V} \leq -\lambda_0\|e\|^2 \leq -\lambda_0\lambda_m^{-1}V$, which implies that

$$V \leq \exp(-\mu t)V(0), \tag{11}$$

where $\mu = \lambda_0\lambda_m^{-1}$.

From (5), we have that $A_c^TP + P^TA_c < 0$. It follows from [2] that the pair (E, A_c) is regular and impulse free. Then there exist two non-singular matrices $M, N \in \mathbf{R}^{n\times n}$ such that the following standard decomposition holds.

$$MEN = \mathrm{diag}\{I_s, 0\}, \quad MA_cN = \mathrm{diag}\{A_1, I_{n-s}\}, \tag{12}$$

where $A_1 \in \mathbf{R}^{s\times s}$. Partition M and N in the form of $M = \begin{pmatrix} M_1^T & M_2^T \end{pmatrix}^T$ and $N = \begin{pmatrix} N_1 & N_2 \end{pmatrix}$, where $M_1 \in \mathbf{R}^{s\times n}$, $M_2 \in \mathbf{R}^{(n-s)\times n}$, $N_1 \in \mathbf{R}^{n\times s}$, $N_2 \in \mathbf{R}^{n\times(n-s)}$. Introducing a change of coordinates

$$N^{-1}e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad e_1 \in \mathbf{R}^s, \quad e_2 \in \mathbf{R}^{n-s}, \tag{13}$$

and using the same partition form of P as (6), then we have that $V = e^TE^TPe = e_1^TP_1e_1$. In addition, denoting $\lambda_1 = \lambda_{\min}(P_1)$, then it is easy to show $\lambda_1 > 0$. There it follows from (11) that we have

$$\|e_1\| \leq \sqrt{\lambda_1^{-1}V(0)}\exp(-\frac{1}{2}\mu t) \tag{14}$$

that is, e_1 is exponentially asymptotically stable.

(ii) We now show that e_2 is also exponentially asymptotically stable.

By means of the Schur Complement Lemma, the second inequality of (5) implies that

$$A_c^TP + P^TA_c + F^TF + P^TBB^TP < 0 \tag{15}$$

(15) is equivalent to

$$\begin{aligned} &(MA_cN)^T(M^{-T}PN) + (M^{-T}PN)^T(MA_cN)\\ &+N^TF^TFN + (M^{-T}PN)^TMBB^TM^T(M^{-T}PN)\\ &< 0 \end{aligned} \tag{16}$$

Then it follows from decomposition (12) that we have

$$\begin{aligned}
&\begin{pmatrix} A_1^T P_1 & 0 \\ P_3 & P_4 \end{pmatrix} + \begin{pmatrix} P_1 A_1 & P_3^T \\ 0 & P_4^T \end{pmatrix} \\
&+ \begin{pmatrix} N_1^T \\ N_2^T \end{pmatrix} F^T F \begin{pmatrix} N_1 & N_2 \end{pmatrix} \\
&+ \begin{pmatrix} P_1^T & P_3^T \\ 0 & P_4^T \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} B B^T \begin{pmatrix} M_1^T & M_2^T \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ P_3 & P_4 \end{pmatrix} \\
&< 0
\end{aligned} \tag{17}$$

The block matrix at the second block row and the second block column of the left-hand-side of (17) is negative definite, that is,

$$P_4 + P_4^T + N_2^T F^T F N_2 + P_4^T M_2 B B^T M_2^T P_4 < 0, \tag{18}$$

then there exists a sufficient small positive scalar ϵ such that

$$P_4 + P_4^T + N_2^T F^T F N_2 + P_4^T (M_2 B B^T M_2^T + \epsilon I) P_4 < 0, \tag{19}$$

which is equivalent to

$$\begin{aligned}
&\left[P_4 + (M_2 B B^T M_2^T + \epsilon I)^{-1}\right]^T \\
&\cdot (M_2 B B^T M_2^T + \epsilon I) \\
&\cdot \left[P_4 + (M_2 B B^T M_2^T + \epsilon I)^{-1}\right] \\
&-(M_2 B B^T M_2^T + \epsilon I)^{-1} + N_2^T F^T F N_2 < 0
\end{aligned} \tag{20}$$

which implies that

$$-(M_2 B B^T M_2^T + \epsilon I)^{-1} + N_2^T F^T F N_2 < 0$$

Thus, there exists a sufficiently small scalar $\epsilon_0 > 0$ such that

$$(1 + \epsilon_0) N_2^T F^T F N_2 < (M_2 B B^T M_2^T + \epsilon I)^{-1} \tag{21}$$

Without loss of generality, we only discuss the case $n - s \leq p$, the other case is similar. Using the singular value decomposition of matrix $M_2 B$ as follows.

$$M_2 B = U_1 \begin{pmatrix} \Lambda & 0 \end{pmatrix} U_2^T, \tag{22}$$

where $U_1 \in \mathbf{R}^{(n-s)\times(n-s)}$, $U_2 \in \mathbf{R}^{p\times p}$ are unitary matrices and $\Lambda \in \mathbf{R}^{(n-s)\times(n-s)}$ is a diagonal matrix with positive diagonal elements in decreasing order.

Then

$$\begin{aligned}
& B^T M_2^T (M_2 B B^T M_2^T + \epsilon I)^{-1} M_2 B \\
= \;& U_2 \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} U_1^T \left(U_1 \begin{pmatrix} \Lambda & 0 \end{pmatrix} U_2^T U_2 \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} U_1^T + \epsilon I \right)^{-1} \\
& \cdot U_1 \begin{pmatrix} \Lambda & 0 \end{pmatrix} U_2^T \\
= \;& U_2 \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} (\Lambda^2 + \epsilon I)^{-1} \begin{pmatrix} \Lambda & 0 \end{pmatrix} U_2^T \\
= \;& U_2 \begin{pmatrix} \Lambda^2 (\Lambda^2 + \epsilon I)^{-1} & 0 \\ 0 & 0 \end{pmatrix} U_2^T \\
\leq \;& U_2 U_2^T \\
= \;& I
\end{aligned} \tag{23}$$

Noticing that $e = N_1 e_1 + N_2 e_2$ and $e_2 = -M_2 B \phi$, then from (9) and (14) we have

$$\begin{aligned}
\|\phi\| &\leq \|Fe\| = \|F N_1 e_1 + F N_2 e_2\| \\
&\leq \|F N_1 e_1\| + \|F N_2 e_2\| \\
&\leq \mu_1 \exp(-\tfrac{1}{2}\mu t) + \|F N_2 e_2\|,
\end{aligned} \tag{24}$$

where $\mu_1 = \|F N_1\| \sqrt{\lambda_1^{-1} V(0)}$.

Furthermore, (21) and (23) imply

$$\begin{aligned}
& \|F N_2 e_2\| \\
= \;& \|F N_2 M_2 B \phi\| \\
= \;& \sqrt{\phi^T B^T M_2^T N_2^T F^T F N_2 M_2 B \phi} \\
\leq \;& \tfrac{1}{\sqrt{1+\epsilon_0}} \sqrt{\phi^T B^T M_2^T (M_2 B B^T M_2^T + \epsilon I)^{-1} M_2 B \phi} \\
\leq \;& \tfrac{1}{\sqrt{1+\epsilon_0}} \|\phi\|
\end{aligned} \tag{25}$$

Combining (24) and (25), we obtain

$$\|\phi\| \leq \mu_1 (1 - \frac{1}{\sqrt{1+\epsilon_0}})^{-1} \exp(-\frac{1}{2}\mu t) \tag{26}$$

Thus, $e_2 = -M_2 B \phi$ implies that e_2 is exponentially asymptotically stable. Therefore the error state $e = N_1 e_1 + N_2 e_2$ is also exponentially asymptotically stable, which completes the proof.

Since (5) is not a strict LMI, in order to use LMI Toolbox [3], (5) can be transformed into an equivalent LMI.

Let $E_\perp \in \mathbf{R}^{(n-s)\times n}$ satisfying $E_\perp E = 0$ and $\mathrm{rank}(E_\perp) = n - s$. Then we have the following lemma.

Lemma 2.1: The conditions in Theorem 2.1 are equivalent to the conditions that there exist a positive-definite matrix $X \in \mathbf{R}^{n\times n}$, two matrices $Y \in \mathbf{R}^{(n-s)\times n}$ and $Q \in \mathbf{R}^{r\times n}$ such that the following LMI is solvable.

$$\begin{pmatrix} \begin{pmatrix} A^T(XE+E_\perp^T Y) \\ +(XE+E_\perp^T Y)^T A \\ +C^T Q + Q^T C + F^T F \end{pmatrix} & (XE+E_\perp^T Y)^T B \\ B^T(XE+E_\perp^T Y) & -I \end{pmatrix} < 0 \tag{27}$$

Proof: If (27) holds for some X and Y, and choosing $P = XE + E_\perp^T Y$, then $E^T P = P^T E = E^T X E \geq 0$. In addition, it follows from the proof of Theorem 2.1 that we have

$$P = M^T \begin{pmatrix} P_1 & 0 \\ P_3 & P_4 \end{pmatrix} N^{-1}$$

where $P_1 > 0$ and M, N defined by (12). Hence, after some manipulations, we get

$$P = M^T \begin{pmatrix} P_1 & 0 \\ 0 & I \end{pmatrix} ME + M^T \begin{pmatrix} 0 \\ I \end{pmatrix} (\, P_3 \quad P_4 \,) N^{-1} \tag{28}$$

Noticing $(\, 0 \quad I \,) ME = (\, 0 \quad I \,) MENN^{-1} = 0$, then there exists a nonsingular matrix $\Gamma \in \mathbf{R}^{(n-s)\times(n-s)}$ such that $E_\perp = \Gamma (\, 0 \quad I \,) M$. Therefore $X = M^T \mathrm{diag}\{P_1, I\} M$ and $Y = \Gamma^{-T} \, (\, P_3 \quad P_4 \,) N^{-1}$ satisfy that $X > 0$ and $P = XE + E_\perp^T Y$, which completes the proof.

By means of Lemma 2.1, we have the following theorem which is direct result from Theorem 2.1.

Theorem 2.2: There exists a full-order observer (3) for system (1) if there exist a positive-definite matrix $X \in \mathbf{R}^{n\times n}$, two matrices $Y \in \mathbf{R}^{(n-s)\times n}$ and $Q \in \mathbf{R}^{r\times n}$ such that LMI (27) is solvable.

Remark 2.1: If $E = I$, then the results of Theorems 2.1 and 2.2 is similar to that in [13], but our approach based on LMI is more effective and simpler.

Remark 2.2: In [5], [11], some coordinate transformation for the original system is made in order to construct full order observer. In contrast, the full order observer design in this paper is easier to construct for it only depends on the solution of an LMI.

III. Reduced order observer design

In order to construct a reduced order observer, the separation principle will be applied. At first, we use a coordinate transformation to decompose (1) and present an estimator for the substate which can not be measured directly from the measurable output. Then we show the error between the estimator and the substate is globally asymptotically stable. To this end, we make the following assumption.

Assumption 3.1: Assume that $\mathrm{rank}(C) = r$ and $\mathrm{rank}\begin{pmatrix} E \\ C \end{pmatrix} = n$.

Remark 3.1: It is easy to show $r + s \geq n$. Using a coordinate transformation, we reconstruct a r-th order dynamics rather than a s-th order dynamics from the original system (1).

Based on the above assumption, there exists a matrix $D \in \mathbf{R}^{(n-r)\times n}$ such that $\mathrm{rank}\begin{pmatrix} C \\ D \end{pmatrix} = n$. Let $N = \begin{pmatrix} C \\ D \end{pmatrix}^{-1}$, then $CN = (\, I_r \quad 0 \,)$ and

$$\begin{aligned} n &= \mathrm{rank}\begin{pmatrix} E \\ C \end{pmatrix} = \mathrm{rank}\left(\begin{pmatrix} E \\ C \end{pmatrix} N\right) \\ &= \mathrm{rank}\begin{pmatrix} EN\begin{pmatrix} I_r \\ 0 \end{pmatrix} & EN\begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} \\ I_r & 0 \end{pmatrix} \end{aligned} \tag{29}$$

That is,

$$\mathrm{rank}\left(EN \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} \right) = n - r$$

which implies that there exists a matrix $M_0 \in \mathbf{R}^{n\times r}$ such that

$$\mathrm{rank}\begin{pmatrix} M_0 & EN\begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} \end{pmatrix} = n$$

Let $M = \begin{pmatrix} M_0 & EN\begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} \end{pmatrix}^{-1}$, then we have that MEN is of the following structure:

$$MEN = \begin{pmatrix} E_1 & 0 \\ E_2 & I_{n-r} \end{pmatrix} \tag{30}$$

where $E_1 \in \mathbf{R}^{r\times r}$ and $E_2 \in \mathbf{R}^{(n-r)\times r}$.

Denoting

$$MAN = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{31}$$

where $A_{11} \in \mathbf{R}^{r\times r}$, $A_{12} \in \mathbf{R}^{r\times(n-r)}$, $A_{21} \in \mathbf{R}^{(n-r)\times r}$, $A_{22} \in \mathbf{R}^{(n-r)\times(n-r)}$.

Introducing the following new state $v = Dx + (LE_1 + E_2)y \in \mathbf{R}^{n-r}$, where $L \in \mathbf{R}^{(n-r)\times r}$ is the observer gain to be determined. Then we have

$$\begin{aligned} x &= N\begin{pmatrix} I_r & 0 \\ -E_2 - LE_1 & I_{n-r} \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}, \\ v &= (\, L \quad I_{n-r} \,) MEx \end{aligned} \tag{32}$$

If we can obtain the estimate of v, then x can be easily estimated by (32) because y is measurable output. The dynamics of state v can be represented by

$$\begin{aligned} \dot{v} &= (\, L \quad I_{n-r} \,) ME\dot{x} \\ &= (\, L \quad I_{n-r} \,) M(Ax + B\Phi) \\ &= (\, L \quad I_{n-r} \,) MAN\begin{pmatrix} I_r & 0 \\ -E_2 - LE_1 & I_{n-r} \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} \\ &\quad + (\, L \quad I_{n-r} \,) MB\Phi \end{aligned} \tag{33}$$

That is,

$$\begin{aligned}\dot{v} &= (A_{22}+LA_{12})v \\ &+[A_{21}+LA_{11}-(A_{22}+LA_{12})(E_2+LE_1)]y \\ &+\begin{pmatrix} L & I_{n-r}\end{pmatrix} MB \\ &\cdot\ \Phi\left(t, N\begin{pmatrix} I_r & 0 \\ -E_2-LE_1 & I_{n-r}\end{pmatrix}\begin{pmatrix} y \\ v\end{pmatrix}, u\right)\end{aligned} \tag{34}$$

In view of (34), we can construct the reduced-order observer for v as follows.

$$\begin{aligned}\dot{\hat{v}} &= (A_{22}+LA_{12})\hat{v} \\ &+[A_{21}+LA_{11}-(A_{22}+LA_{12})(E_2+LE_1)]\,y \\ &+\begin{pmatrix} L & I_{n-r}\end{pmatrix} \\ &\cdot\ MB\Phi\left(t, N\begin{pmatrix} I_r & 0 \\ -E_2-LE_1 & I_{n-r}\end{pmatrix}\begin{pmatrix} y \\ \hat{v}\end{pmatrix}, u\right)\end{aligned} \tag{35}$$

Denoting $\delta = v-\hat{v}$, then from (34) and (35), we have

$$\dot{\delta} = (A_{22}+LA_{12})\delta + \begin{pmatrix} L & I_{n-r}\end{pmatrix} MB\phi_\delta \tag{36}$$

where

$$\begin{aligned}\phi_\delta &= \Phi\left(t, N\begin{pmatrix} I_r & 0 \\ -E_2-LE_1 & I_{n-r}\end{pmatrix}\begin{pmatrix} y \\ v\end{pmatrix}, u\right) \\ &-\Phi\left(t, N\begin{pmatrix} I_r & 0 \\ -E_2-LE_1 & I_{n-r}\end{pmatrix}\begin{pmatrix} y \\ \hat{v}\end{pmatrix}, u\right)\end{aligned} \tag{37}$$

Theorem 3.1: Under Assumption 3.1 there exists a reduced-order observer in the form of (35) for system (1) if the conditions in Theorem 2.1 hold.

Proof: In order to show that dynamics (35) is a reduced-order observer for system (1), we only need to show that error dynamics (36) are globally exponentially stable.

For convenience, let

$$M^{-T}PN = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4\end{pmatrix}, \tag{38}$$

where $P_1 \in \mathbf{R}^{r\times r}$, $P_2 \in \mathbf{R}^{r\times(n-r)}$, $P_3 \in \mathbf{R}^{(n-r)\times r}$, $P_4 \in \mathbf{R}^{(n-r)\times(n-r)}$.

Based on the new decomposition of matrices E and A in (30) and (31), from matrix inequalities (5) we have that $E^TP = P^TE \geq 0$ implies that $(MEN)^TM^{-T}PN = (M^{-T}PN)^TMEN \geq 0$, that is, $P_4 \geq 0$. Without loss of the generality, we assume $P_4 > 0$.

In fact, if P_4 is singular, then there exists a sufficient small positive scalar ϵ such that $P_\epsilon := P + \epsilon M^TME$ (instead of P) satisfies matrix inequalities (5). In this case, (38) will be in the following form:

$$\begin{aligned}M^{-T}P_\epsilon N &= M^{-T}PN + \epsilon MEN \\ &= \begin{pmatrix} P_1+\epsilon E_1 & P_2 \\ P_3+\epsilon E_2 & P_4+\epsilon I\end{pmatrix}\end{aligned} \tag{39}$$

then $P_4 + \epsilon I > 0$.

Choosing

$$L = P_4^{-1}P_2^T. \tag{40}$$

By pre- and post-multiplying the second LMI in (5) with $\mathrm{diag}\{N, I_{n-r}\}^T$ and its transpose, respectively, reminding (30), (31) and $CN = \begin{pmatrix} I_r & 0\end{pmatrix}$, after some manipulations, we have

$$\begin{pmatrix} \Omega_{11} & P_4\begin{pmatrix} L^T \\ I\end{pmatrix}^T MB \\ (MB)^T\begin{pmatrix} L^T \\ I\end{pmatrix}P_4 & -I\end{pmatrix} < 0 \tag{41}$$

where

$$\begin{aligned}\Omega_{11} &:= P_4(A_{22}+LA_{12}) + (A_{22}+LA_{12})^TP_4 \\ &+\begin{pmatrix} 0 \\ I_{n-r}\end{pmatrix}^T N^TF^TFN\begin{pmatrix} 0 \\ I_{n-r}\end{pmatrix}\end{aligned} \tag{42}$$

In addition, from (2) we have

$$\|\phi_\delta\| \leq \|FN\begin{pmatrix} 0 \\ I_{n-r}\end{pmatrix}\delta\| \tag{43}$$

For error dynamics (36) with (40), choosing the following Lyapunov function

$$V = \delta^TP_4\delta \tag{44}$$

then the derivative of Lyapunov function (44) along dynamics (36) yields

$$\begin{aligned}\dot{V} &= 2\delta^TP_4(A_{22}+LA_{12})\delta + 2\delta^TP_4\begin{pmatrix} L & I\end{pmatrix}MB\phi_\delta \\ &\leq 2\delta^TP_4(A_{22}+LA_{12})\delta + 2\delta^TP_4\begin{pmatrix} L & I\end{pmatrix}MB\phi_\delta \\ &\quad -\phi_\delta^T\phi_\delta + \delta^T\begin{pmatrix} 0 \\ I_{n-r}\end{pmatrix}^T N^TF^TFN\begin{pmatrix} 0 \\ I_{n-r}\end{pmatrix}\delta \\ &= \begin{pmatrix} \delta^T & \phi_\delta^T\end{pmatrix}\Omega_0\begin{pmatrix} \delta \\ \phi_\delta\end{pmatrix} \\ &< 0\end{aligned} \tag{45}$$

where Ω_0 represents the left-hand-side of inequality (41). Therefore δ is exponentially stable, which completes the proof.

Similarly to Theorem 2.2, Theorem 3.1 is equivalent to the following theorem in which the observer gain can be construct by means of the solution of an LMI.

Theorem 3.2: Under Assumption 3.1, there exists a reduced-order observer in the form of (35) for system (1) if there exist a positive-definite matrix $X \in \mathbf{R}^{n\times n}$, two matrices $Y \in \mathbf{R}^{(n-s)\times n}$ and $Q \in \mathbf{R}^{r\times n}$ such that LMI (27) is solvable.

Remark 3.2: It is an interesting fact that constructible condition (27) for both full order and reduced order observers are the same under LIM approach. Comparing with generalized Sylvester equation approach developed in [5], Theorems 3.1 and 3.2 present a new and simple method to design reduced order observer. Our sufficient condition in Theorems 3.1 and 3.2 is independent of coordinate transformation, only dependent of the solution of LMI (27). However, [5] requires the solution of a special generalized Sylvester equation. (see Theorem 3 of [5]). This constraint may depend on the choice of matrix R, that is, coordinate transformation of the discussed system.

Remark 3.3: It is easy to see that the system discussed in [14] is a special case of systems (1). If $E = I$, we can directly obtain the reduced order observer design from Theorems 3.1 and 2.2. It is interesting to find that our unified observer design for nonsingular systems by means of LMI is simpler and easier than the approach used in [14], while different cases of system matrices are discussed for design in [14].

IV. An illustrative example

Consider the system with the following system matrices:

$$\begin{aligned} E &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0.5 & 0.2 \\ 0.3 & 0.6 \end{pmatrix}, \\ C &= \begin{pmatrix} 2 & 1 \end{pmatrix},\ F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{46}$$

Choose $E_\perp = \begin{pmatrix} 0 & 1 \end{pmatrix}$, and then by solving LMI (27) we have

$$\begin{aligned} X &= \begin{pmatrix} 0.7149 & 0.0000 \\ 0.0000 & 1.1618 \end{pmatrix},\ Y = \begin{pmatrix} 0.0599 & -0.9775 \end{pmatrix}, \\ Q &= \begin{pmatrix} -1.2278 & 0.1166 \end{pmatrix} \end{aligned} \tag{47}$$

Hence the observer gain (7) is obtained as follows.

$$L = (XE + E_\perp^T Y)^{-T} Q^T = \begin{pmatrix} -1.7076 \\ -0.1193 \end{pmatrix} \tag{48}$$

Thus, a full order observer for system (1) is system (3) with the observer gain presented by (48).

Next we present the reduced order observer for system (1).

Choose $D = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $M_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$, then

$$\begin{aligned} M &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ N = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \\ MEN &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ MAN = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{49}$$

From the above solutions of LMI (27), we have

$$P = XE + E_\perp^T Y = \begin{pmatrix} 0.7149 & 0 \\ 0.0599 & -0.9775 \end{pmatrix} \tag{50}$$

which yields

$$M^{-T} P N = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} -0.9775 & 2.0149 \\ 0 & 0.7149 \end{pmatrix} \tag{51}$$

Hence an observer gain (40) is

$$L = P_4^{-1} P_2^T = 2.8186 \tag{52}$$

Therefore, a reduced order observer in the form of (35) is given as follows.

$$\begin{aligned} \dot{\hat{v}} &= -5.6372\hat{v} + 3.8186y \\ &\quad + \begin{pmatrix} 1.3456 & 1.8912 \end{pmatrix} \Phi \left[t, \begin{pmatrix} \hat{v} \\ y - 2\hat{v} \end{pmatrix}, u \right] \end{aligned} \tag{53}$$

V. Conclusion

This paper addresses the issues of full-order and reduced-order observer design for a class of descriptor systems with Lipschitz constraint. It is shown that the design of both observers can be reformulated as the same LMI. This paper presents a simple way to seek the observers. It can be easily seen that the approach in this paper can be extended to Lipschitz descriptor systems with multiple time-delays or H_∞ observer design.

References

[1] J. H. Chou, and W. H. Liao, "Stability robustness of continuous-time perturbed descriptor systems," *IEEE Trans. Circuits Syst. I,* vol. 46, pp. 1153-1155, Sept. 1999.

[2] I. Masubuchia, Y. Kamitaneb, A. Oharac, and N. Sudad, "H_∞ Control for Descriptor Systems: A Matrix Inequalities Approach," *Automatica*, vol. 33, pp. 669-673, 1997.

[3] P. Gahinet, A. Nemirovsky, A. J. Laub, and M. Chilali, *LMI Control Toolbox*, The Math Works Inc. 1995.

[4] V. A. Yakubovich, "The S-procedure in nonlinear control theory," *Vestnik Leningrad Univ. Math.*, vol. 4, pp. 73-93, 1977. (English translation)

[5] M. Boutayeb and M. Darouach, "Observers design for nonlinear descriptor systems," *Proceedings of the 34th IEEE Conference on Decision and Control,* vol. 3, pp. 2369-2374, 1995.

[6] L. Dai, *Singular Control Systems*, Berlin, Germany: Springer-Verlag, 1989.

[7] M. Darouach and M. Boutayeb, "Design of observers for descriptor systems," *IEEE Trans. Automat. Contr.,* vol. 40, pp. 1323-1327, 1995.

[8] M. Darouach, M. Zasadzinski, and M. Hayar, "Reduced-order observer design for descriptor systems with unknown inputs," *IEEE Trans. Automat. Contr.,* vol. 41, pp. 1068-1072,1996.

[9] M. Hou and P. C. Muller, "Observer design for descriptor systems," *IEEE Trans. Automat. Contr.,* vol. 44, pp. 164-169, 1999.

[10] D. Koenig and S. Mammar, "Design of proportional-integral observer for unknown input descriptor systems," *IEEE Trans. Automat. Contr.,* vol. 47, pp. 2057- 2062, 2002.

[11] D. N. Shields, "Observer design and detection for nonlinear descriptor systems," *International Journal of Control*, vol. 67(2), pp. 153-168, 1997.

[12] G. Zimmera J. Meierb, "On observing nonlinear descriptor systems," *Systems & Control Letters*, vol. 32, pp. 43-48, 1997.

[13] R. Rajamani, "Observers for Lipschitz nonlinear systems," *IEEE Trans. Automat. Contr.,* vol. 43, pp. 397-401, 1998.

[14] F. L. Zhu and Z. Z. Han, "A note on observers for Lipschitz nonlinear systems," *IEEE Trans. Automat. Contr.,* vol. 47, pp. 1751-1754, 2002.