Robust state estimation with q-Integral observers

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***Abstract*— A notion of q-integral (qI) observers for multiple-input single output linear systems is introduced. Then the theory is extended to nonlinear systems with single output. We show that the q-integral observer guarantees robustness against both measurement errors and unmodeled dynamics. An example is given to show the efficacy of the proposed robust observers.**

***Index Terms*— Lyapunov theory; Nonlinear observers; Robustness; LMI design.**

I. Introduction

Numerous control strategies are based on the assumption that all internal states of the control object are available for feedback. In most cases, however, only a few of the states or some functions of the states can be measured. This circumstance raises the need for techniques, which makes it possible not only to estimate states, but also to derive control laws that guarantee stability when using the estimated states instead of the true ones. By observer we mean a deterministic dynamical system which uses observed information to compute an estimate of the state of the control system in such as way that the error decays to zero. State reconstruction and estimation are used in numerous different types of applications and play a fundamental role in modern control theory, signal processing, telecommunications, and fault detection. Diagnosis and supervision of critical processes are of major importance for reliability and safety in industry today. The application of observers in fault detection and isolation provide one means to these problems.

High-gain observers continue to be efficient tools to estimate unmeasured states from the knowledge of the inputs and the outputs of the system being observed. In such observer design, the high-gain output injection is conceived to defeat the inherent nonlinearities, however, this proportional injection arises two serious drawbacks: noise amplification and peaking phenomenon. In this paper we plan to reformulate the high-gain observation scheme by replacing the proportional P injection term with a multiple integral injection term that involves the qth integral of the output. As a matter of fact, the notion of adding an integral path is not quite new. The first idea of proportional integral PI observers has been proposed by Wojciechwski [12] and further developed by Beale and Shafai [2], and Niemann et al [7].

The proposed observers differ from the conventional P and PI observers proposed in [2], [7], [3]. Our goal is to cancel the proportional term P from the observer dynamics and replace it by a novel injection term that depends upon the qth integral of the measured output. First, we begin by the development of qI observers for MISO linear systems. Subsequently, we exploit the new structure of the qI observer to build robust observers for Lipschitzian nonlinear systems. We show that the qI term permits to decouple the effect of uncertainties from the state estimates and makes the filtering operation internally incorporated in the dynamics of the observer. Cancelling the proportional term from the q-integral observer permits to filter the estimates whatever the Lipschitz constant is.

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Throughout this paper, we note $\mathbb{R}$ the set of real numbers. $|f(t)|$ is the absolute value of the function $f(t)$. A' is the matrix transpose of A. $A^{\frac{1}{2}}$ is the square root of A. $\|A\| = \max\left\{\sqrt{\lambda} : \lambda \text{ is the eigenvalue of } A^T A\right\}$. $\lambda_{\min}(A)$: is the smallest eigenvalue of A. $\lambda_{\max}(A)$: is the largest eigenvalue of A. $\mathscr{S}^+(n, \mathbb{R})$ denotes the set of positive definite matrices of order n. I is the identity matrix with appropriate dimension. We note $\Re(\lambda)$ the real part of the complex λ. $\dot{x}$ denotes the first derivative of x with respect to time. $\int_q y(s)ds = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{q} y(s)ds$ is the q-th integral of the scalar function $y(t)$.

II. A robust linear observer

Consider the uncertain MISO linear system

$$\begin{aligned} \dot{x} &= A\,x + B\,u + v, \\ y &= C\,x + d, \end{aligned} \tag{1}$$

where the state $x = x(t)$ and the input $u = u(t)$ are n and m dimensional vectors, respectively. $y = y(t)$ is a scalar output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{1\times n}$ are the nominal matrices, and the pair (A, C) is assumed to be rank observable. $v = v(x(t), t) \in \mathbb{R}^n$ represents the bounded unmodeled dynamics and $d = d(t) \in \mathbb{R}$ is a bounded high-frequency disturbance which comes corrupting the measurements. A classical Luenberger observer is readily constructed as

$$\dot{\hat{x}} = A\,\hat{x} + L\,(y - C\hat{x}) + B\,u. \tag{2}$$

From equations (1) and (2), the dynamics of the observer error $e = \hat{x} - x$ is

$$\dot{e} = (A - LC)\,e + L\,d - v. \tag{3}$$

It is clear that the perturbation d is amplified if the observer gain L is large. Consequently, system (2) could not be a potential candidate observer if we want to decouple the effect of noise from the dynamics of the observer.

A. Robust observer design

Our goal is to decouple the effect of uncertainties from the observer gain, for this purpose, we define the q-integral observer as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 - k_{\xi_1}\,\xi_1, \\ \dot{\xi}_2 &= \xi_3 - k_{\xi_2}\,\xi_1, \\ &\vdots \\ \dot{\xi}_q &= y - C\hat{z} - k_{\xi_q}\,\xi_1, \\ \dot{\hat{z}} &= A\,\hat{z} - K_I\,\xi_1 + B\,u, \end{aligned} \tag{4}$$

where $\xi \in \mathbb{R}^{q\times 1}$ is the state of the ξ-subsystem, $\hat{z}$ is the observer state vector, $K_I \in \mathbb{R}^{n\times 1}$ is the integral gain of the observer and $K_\xi = \begin{bmatrix} k_{\xi_1} & k_{\xi_2} & \cdots & k_{\xi_q} \end{bmatrix}' \in \mathbb{R}^{q\times 1}$ is the ξ-subsystem gain. System (4) can be rewritten as follows

$$\begin{aligned} \dot{\xi} &= \left(A_\xi - \widetilde{K}_\xi\right)\xi + B_\xi\,(y - C\hat{z}), \\ \dot{\hat{z}} &= A\,\hat{z} - \widetilde{K}_I\,\xi + B\,u, \end{aligned} \tag{5}$$

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where

$$A_\xi = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{q\times q}, B_\xi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{q\times 1}, \quad (6)$$

$$\widetilde{K}_I = \begin{bmatrix} K_I & 0_{n\times q-1} \end{bmatrix}_{n\times q}, \widetilde{K}_\xi = \begin{bmatrix} K_\xi & 0_{q\times q-1} \end{bmatrix}_{q\times q}. \quad (7)$$

Notice that the main difference between (4) and (2) is that the P injection term $L(y - C\hat{x})$ in (2) is replaced by $K_I \xi_1$ in (4). The relationship between ξ_1 and $(y - C\hat{x})$ is given by the ξ-subsystem (4). ξ_1 can be seen as the output of $q-$th order filter whose input is $(y - C\hat{x})$. By increasing the order q, ξ_1 is filtered more and more and a great amount of noise shall be eliminated from the observer dynamics. Let $\tilde{e} = \hat{z} - x$, then from equation (5) and (1), we have

$$\begin{bmatrix} \dot{\xi} \\ \dot{\tilde{e}} \end{bmatrix} = \begin{bmatrix} A_\xi - \widetilde{K}_\xi & -B_\xi C \\ -\widetilde{K}_I & A \end{bmatrix} \begin{bmatrix} \xi \\ \tilde{e} \end{bmatrix} + \begin{bmatrix} B_\xi \\ 0_{n\times 1} \end{bmatrix} d - \begin{bmatrix} 0_{q\times 1} \\ v \end{bmatrix}. \quad (8)$$

Observer (5) is internally stable if and only if the gains $\widetilde{K}_I$ and $\widetilde{K}_\xi$ are chosen such that the eigenvalues of the matrix

$$\begin{bmatrix} A_\xi - \widetilde{K}_\xi & -B_\xi C \\ -\widetilde{K}_I & A \end{bmatrix}_{(n+q)\times(n+q)} \quad (9)$$

have negative real parts. If we compare (8) and (3), we conclude that the q-integral observer decouples the disturbance d and permits to have a good filtering quality. Notice that the design of $\widetilde{K}_I$, and $\widetilde{K}_\xi$ cannot be done independently. We will give a direct method to choose the $n + q$ parameters for any observable pair (A, C). The whole design is summarized in the following theorem.

Theorem 1: Consider system (1) and let

$$F = \begin{bmatrix} A_\xi & -B_\xi C \\ 0_{n\times q} & A \end{bmatrix}_{(n+q)\times(n+q)}, G = \begin{bmatrix} B_\xi \\ 0_{n\times 1} \end{bmatrix}_{(n+q)\times 1},$$

$$\bar{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times(n+q)}, \widetilde{w} = \begin{bmatrix} 0_{q\times 1} \\ v \end{bmatrix}_{(n+q)\times 1}.$$

If the positive parameter μ is selected such that

$$\Re\left(\lambda_i\left(F\right)\right) > -\mu/2 \quad (10)$$

for every eigenvalue λ_i of F, then

$$\begin{bmatrix} \dot{\xi} \\ \dot{\hat{z}} \end{bmatrix} = \left(F - P^{-1}\bar{C}'\bar{C}\right)\begin{bmatrix} \xi \\ \hat{z} \end{bmatrix} + G\,y, \quad -\mu P - F'P - PF + \bar{C}'\bar{C} = 0. \quad (11)$$

is a robust observer of (1) which decouples both the noise effect and the unmodeled dynamics from the observer states. Furthermore, if $d = 0$ and $v = 0$, then

$$\lim_{t\to\infty}\left(x - \hat{z}\right) = 0. \quad (12)$$

Proof. The eigenvalue condition is met if and only if the matrix $-\left(\frac{\mu}{2}I + F\right)$ is Hurwitz. This in turn, is equivalent to the existence of a positive definite matrix $P \in \mathscr{S}^+(n+q, \mathbb{R})$ that satisfies the Lyapunov matrix equation

$$-\left(\frac{\mu}{2}I + F\right)'P - P\left(\frac{\mu}{2}I + F\right) = -\bar{C}'\bar{C}. \quad (13)$$

Here, the integral gain of the observer is given by

$$P^{-1}\bar{C}' = \begin{bmatrix} K_\xi \\ K_I \end{bmatrix}. \quad (14)$$

Put $w = \hat{z} - x$, where x is the state vector of system (1), and define $\rho = \begin{bmatrix} \xi \\ w \end{bmatrix}$. By taking $V = \rho' P \rho$ as a Lyapunov function candidate associated to the following system

$$\dot{\rho} = \left(F - P^{-1}\bar{C}'\bar{C}\right)\rho + G\,d - \widetilde{w}, \quad (15)$$

one could easily show that

$$\begin{aligned} \dot{V} &\le -\mu V + 2\rho' P\,G\,d - 2\rho' P\,\widetilde{w} \\ &\le -\mu V + 2\left\|\rho' P^{\frac{1}{2}}\right\|\left\|P^{\frac{1}{2}}G\right\||d| \\ &+ 2\left\|\rho' P^{\frac{1}{2}}\right\|\left\|P^{\frac{1}{2}}\right\|\|\widetilde{w}\| \end{aligned}$$

which implies that

$$\begin{aligned} \dot{V}^{\frac{1}{2}} &\le -\tfrac{\mu}{2}V^{\frac{1}{2}} + \left\|P^{\frac{1}{2}}G\right\||d| \\ &\le -\frac{\mu}{2}V^{\frac{1}{2}} + \frac{|d|}{\lambda_{\min}(P^{-\frac{1}{2}})} + \frac{\|w\|}{\lambda_{\min}(P^{-\frac{1}{2}})} \end{aligned} \quad (16)$$

Now we shall prove that uncertainties can be reduced by increasing the value of μ. For this purpose, we introduce the following lemma.

Lemma 1: Let μ_1 and μ_2 be two positive real constants such that

$$\Re\left(\lambda_i\left(F\right)\right) > -\frac{\mu_1}{2},\ \Re\left(\lambda_i\left(F\right)\right) > -\frac{\mu_2}{2}, \quad (17)$$

for every eigenvalue λ_i of F, and let P_1 and P_2 be the solutions of the following Lyapunov-like matrix equations

$$-\mu_1 P_1 - F'P_1 - P_1 F + \bar{C}'\bar{C} = 0, \quad (18)$$

$$-\mu_2 P_2 - F'P_2 - P_2 F + \bar{C}'\bar{C} = 0. \quad (19)$$

Then for any $\mu_1 < \mu_2$, $P_1^{-1} < P_2^{-1}$.

Proof. The difference between (18) and (19) gives

$$-\mu_1 P_1 + \mu_2 P_2 - F'\left(P_1 - P_2\right) - \left(P_1 - P_2\right)F = 0. \quad (20)$$

The last equation can be rewritten as

$$-\left(F + \frac{\mu_1}{2}I\right)'\left(P_1 - P_2\right) - \left(P_1 - P_2\right)\left(F + \frac{\mu_1}{2}I\right) = -(\mu_2 - \mu_1)P_2. \quad (21)$$

Since $-\left(F + \frac{\mu_1}{2}I\right)$ and $-\left(F + \frac{\mu_2}{2}I\right)$ are Hurwitz by the eigenvalue condition (17), then P_1 and P_2 are positive definite. Using the fact that $(\mu_2 - \mu_1)P_2 > 0$ and the matrix $-\left(F + \frac{\mu_1}{2}I\right)$ is Hurwitz, then the solution $P_1 - P_2$ of the Lyapunov equation (21) is positive definite, or $P_1^{-1} - P_2^{-1} < 0$.

From inequality (16) and using results of lemma 1, we conclude that when the observer gain $P^{-1}\bar{C}'$ is high (i.e., $\left\|P^{-\frac{1}{2}}\right\|$ is high), the amount of noise is reduced since the norm of the perturbation is multiplied by $1/\lambda_{\min}(P^{-\frac{1}{2}})$. In addition, the observer is also able to reduce the effects of model uncertainties by increasing the value of μ, see (16).

Remark 1: The proposed observer design has a relationship with H_∞-filtering. The main difference between the two approaches is that the proposed solution of the observer gain is given by the solution of a Lyapunov-like equation which always exists for a suitable choice of μ. Furtheremore, if for a certain $\mu > \mu^\star$, the matrix P is positive definite, then we realize that choosing μ

large does not affect the existence of P and hence, the uncertain term along with noise disturbance are significantly reduced.

Remark 2: Observer (4) is not the only possible scheme to reduce the effect of uncertainties. The ξ-subsystem can be rewritten in controllable canonical form, which gives the following observer scheme

$$\begin{aligned} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \xi_3, \\ &\vdots \\ \dot{\xi}_q &= y - C\hat{z} - k_{\xi_1}\xi_1 - k_{\xi_2}\xi_2 - \cdots - k_{\xi_q}\xi_n, \\ \dot{\hat{z}} &= A\hat{z} - K_I\xi_1 + Bu. \end{aligned} \tag{22}$$

In matrix notation, observer (22) takes the form

$$\begin{aligned} \dot{\xi} &= \left(A_\xi - B_\xi K'_\xi\right)\xi + B_\xi\left(y - C\hat{z}\right), \\ \dot{\hat{z}} &= A\hat{z} - K_I\xi_1 + Bu. \end{aligned} \tag{23}$$

By forming the observation error $\hat{e} = \hat{z} - x$ where x is the state vector of (1) and $\hat{z}$ is the $\hat{z}$-state vector of (22), we obtain

$$\begin{bmatrix} \dot{\xi} \\ \dot{\hat{e}} \end{bmatrix} = \begin{bmatrix} A_\xi - B_\xi K'_\xi & -B_\xi C \\ -\widetilde{K}_I & A \end{bmatrix} \begin{bmatrix} \xi \\ \hat{e} \end{bmatrix} + \begin{bmatrix} B_\xi \\ 0_{n\times 1} \end{bmatrix} d - \begin{bmatrix} 0_{q\times 1} \\ v \end{bmatrix}. \tag{24}$$

Evidently, the observation error is stable if and only if the eigenvalues of the matrix

$$\begin{bmatrix} A_\xi - B_\xi K'_\xi & -B_\xi C \\ -\widetilde{K}_I & A \end{bmatrix}, \tag{25}$$

are stable.

B. Other scheme of robust observers

The aim of this subsection is to present another scheme of robust observers that behave more resistant to measurement errors of high levels. The basic idea is to augment the original system with q integrators and feed back the observer dynamics with the exact q-th integral of the noisy output. The amount of noise that may contain the system output will be enfeebled with the presence of the successive q integrators.

Consider the linear system (1) augmented with the q-chain of integrators

$$\begin{aligned} &\dot{x} = Ax + Bu + v, \\ &\dot{\xi}_1 = \xi_2, \\ &\dot{\xi}_2 = \xi_3, \\ &\vdots \\ &\dot{\xi}_q = y, \end{aligned} \tag{26}$$

where $\xi(0) = 0$ and $y = Cx + d$ is the system noisy output. Here, the ξ-subsystem is not a part of the observer dynamics but just an augmentation of the original system that permits us to extract the q-th integral of the noisy output. The corresponding observer is given by

$$\begin{aligned} &\dot{\hat{\xi}}_1 = \hat{\xi}_2 - k_{\xi_1}\left(\hat{\xi}_1 - \xi_1\right), \\ &\dot{\hat{\xi}}_2 = \hat{\xi}_3 - k_{\xi_2}\left(\hat{\xi}_1 - \xi_1\right), \\ &\vdots \\ &\dot{\hat{\xi}}_q = C\hat{x} - k_{\xi_q}\left(\hat{\xi}_1 - \xi_1\right), \\ &\dot{\hat{x}} = A\hat{x} + Bu - K_I\left(\hat{\xi}_1 - \xi_1\right), \end{aligned} \tag{27}$$

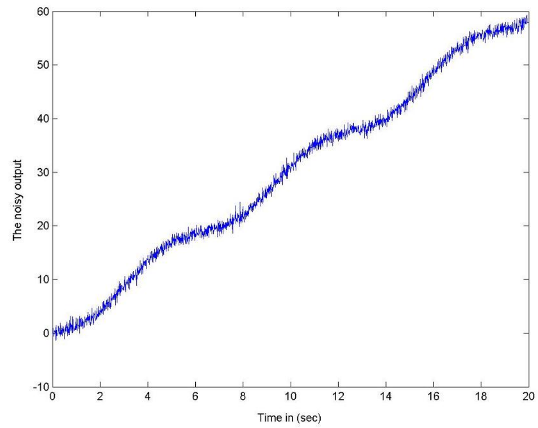

Fig. 1. The noisy output.

where the observer gain K_ξ and K_I are defined as in section II. The last system can be rewritten as

$$\begin{aligned} &\dot{\hat{\xi}}_i = \hat{\xi}_{i+1} - k_{\xi_i}\left(\hat{\xi}_1 - \int_q y(s)ds\right), \qquad 1 \le i \le q-1 \\ &\dot{\hat{\xi}}_q = C\hat{x} - k_{\xi_q}\left(\hat{\xi}_1 - \int_q y(s)ds\right), \\ &\dot{\hat{x}} = A\hat{x} + Bu - K_I\left(\hat{\xi}_1 - \int_q y(s)ds\right). \end{aligned}$$

It is clear, in this representation, that the observer is alimented with the q-th integral of y, but in the meantime, the order of the observer is augmented by q supplementary dynamical equations. If we note

$$e = \begin{bmatrix} \hat{\xi} - \xi \\ \hat{x} - x \end{bmatrix}, \tag{28}$$

then

$$\dot{e} = \begin{bmatrix} A_\xi - \widetilde{K}_\xi & B_\xi C \\ -\widetilde{K}_I & A \end{bmatrix} e - \begin{bmatrix} B_\xi \\ 0_{n\times 1} \end{bmatrix} d - \begin{bmatrix} 0_{q\times 1} \\ v \end{bmatrix}, \tag{29}$$

where $\widetilde{K}_\xi$, $\widetilde{K}_I$, A_ξ, B_ξ are defined as in section II. With an appropriate choice of $\widetilde{K}_I$ and $\widetilde{K}_\xi$, the observer error dynamics (29) can be made stable.

Remark 3: Observer (27) is in the ideal case to apply result of theorem 1. The determination of $\widetilde{K}_\xi$, $\widetilde{K}_I$ can be obtained from the Lyapunov matrix equation (11) by replacing the matrix F by

$$\begin{bmatrix} A_\xi & B_\xi C \\ 0_{n\times q} & A \end{bmatrix}. \tag{30}$$

C. An example

Consider the linear system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin(t), \\ y &= x_1 + d, \end{aligned} \tag{31}$$

where d is a norm-bounded noise, and $\sin(t)$ is considered as a known input. The objective is to show the effectiveness of observer (27) for $q = 2$. In figure 1, we give the noisy output y and in figure 2 and 3, we show the performances of observer (27). The simulation is made for $k_{\xi_1} = 4$, $k_{\xi_2} = 6$, $k_1 = 4$, $k_2 = 1$.

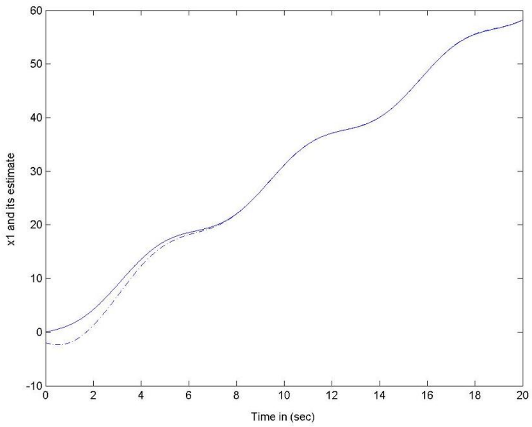

Fig. 2. The ideal x_1 and its estimate.

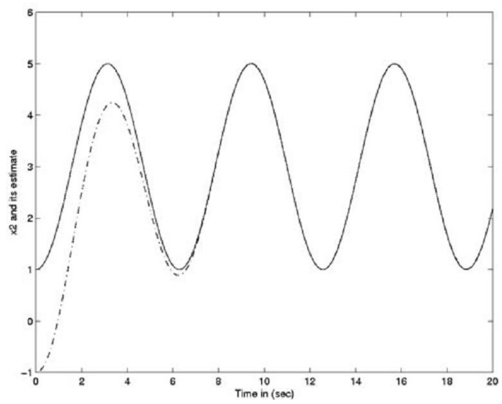

Fig. 3. The state x_2 and its estimate.

III. EXTENSION TO NONLINEAR SYSTEMS

A. On observer design for nonlinear systems

For linear systems the observability condition implies existence of exponentially converging observers. For general nonlinear systems the different definitions and properties on observability described in the literature are fundamental, but the relation to observers and the observer design is far more complex then for the linear case. A standard approach to solve the state reconstruction problem is to use a copy of the observed system and to add some correction terms attenuating the difference of the outputs [6], [11], [5], [4], [9], [10]. Many standard nonlinearities, as for instance trigonometric functions, or terms as x^2 can be bounded by linear functions satisfying Lipschitz conditions. This property has been exploited by Thau [11] to construct a nonlinear observer to systems of the form

$$\begin{aligned} \dot{x} &= Ax + f(x,\, u,\, t) + \phi(y,\, u,\, t), \\ y &= Cx. \end{aligned} \tag{32}$$

where $f(x,\, u,\, t)$ is Lipschitz with respect to the state x with a Lipschitz constant γ. Thau proposed the model-based observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + f(\hat{x},\, u,\, t) + L\,(y - \hat{y}) + \phi\,(y,\, u,\, t)\,, \\ \hat{y} &= C\hat{x}, \end{aligned} \tag{33}$$

and proved that if the Lyapunov equation

$$(A - LC)'\,P + P\,(A - LC) = -Q,\ P > 0,\ Q > 0, \tag{34}$$

is satisfied with

$$\gamma \leq \frac{\lambda_{\min}\,(Q)}{2\lambda_{\max}\,(P)}, \tag{35}$$

then the error $e = x - \hat{x}$ decays exponentially to zero. The result of Thau ensures the stability of the observer estimates, but unfortunately, equations (33), (35) provides very little insight how the observer gain L can be found. The eigenvalues of the matrix $(A - LC)$ can be placed arbitrarily, but the crucial part is the relation between these eigenvalues and the spectral radius of the matrix P. In article [8], the authors showed that the ratio (35) can be maximized for $Q = I$. Raghavan and Hedrick have proposed a method to construct the observer gain L. The design strategy was based on theory for quadratic stabilization of uncertain systems [9]. Recently, Rajamani [10] studied extensively the conditions of existence of the observer gain L and has proposed an algorithm for its computation. However, the structure of the nonlinearities were not fully utilized witch makes the results somewhat conservative as the observer gain, if found, will give an asymptotically observer for all nonlinearities satisfying the Lipschitz conditions. Arcak and Kokotović [1] has considered locally Lipschitz nonlinear systems and the observer design decomposes the error dynamics into a linear system in feedback with a multivariable sector nonlinearity. Linear matrix inequalities (LMIs) are used to state the conditions for the existence of a stable observer error dynamics with respect to the imposed observer structure. As we have showed latter, the presence of the P term $L\,(y - C\hat{x})$ in the proposed observers will amplifies enormously the noise that contains the output y, especially when the constant Lipschitz is high.

As it was mentioned in reference [9], linear transformation can be used to reduce the value of the Lipschitz constant. In their design the observer gain is calculated through an algebraic Riccati equation (ARE), which depends on the Lipschitz constant of nonlinearities. The authors have proposed an algorithm how can one design progressively the observer gain by testing the solution of the ARE. In this subsection, we develop an efficient LMI-based algorithm that can inform as about the allowed maximum value of the Lipschitz constant and compute the maximum observer gain L if it exists. We summarize the design in the following statement.

Theorem 2: Consider the nonlinear system

$$\dot{x} = A\,x + f\,(x,\, u) + g(y,\, u), \tag{36}$$

$$y = C\,x, \tag{37}$$

where $x \in M \subset \mathbb{R}^n$ and $f : M \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a Lipschitz nonlinearity of Lipschitz constant γ and $f(0,\, 0) = 0$. The nominal matrices $A \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{p\times n}$ are assumed to be detectable. If there exist a positive definite matrix P, and a matrix $Y \in \mathbb{R}^{n\times p}$ such that the optimization problem

$$\min_{P,\,Y}\ \rho \tag{38}$$

subject to

$$\begin{bmatrix} A'P + PA - C'Y' - YC + I & P \\ P & -\rho\, I \end{bmatrix} < 0 \tag{39}$$

is feasible, then system

$$\dot{\hat{x}} = A\,\hat{x} + f\,(\hat{x},\, u) + g(y,\, u) + P^{-1}Y\,(y - C\hat{x})\,, \tag{40}$$

is an asymptotic observer if $\frac{1}{\gamma^2} \geq \rho$.

Proof. By forming the dynamics of the observer error $e = \hat{x} - x$ from Eq. (36), we have

$$\dot{e} = \left(A - P^{-1}YC\right)e + f\,(\hat{x},\, u) - f\,(x,\, u)\,. \tag{41}$$

By taking the quadratic Lyapunov function $e'Pe$, its first time derivative is

$$\begin{array}{rcl}\dot{V} &=& \dot{e}'Pe + e'P\dot{e} \\ &=& e'\left(A'P + PA - C'Y' - YC\right)e \\ &+& 2e'P\left(f\left(\hat{x},\,u\right) - f\left(x,\,u\right)\right)\end{array}$$

Using

$$\begin{array}{l}2e'P\left(f\left(\hat{x},\,u\right) - f\left(x,\,u\right)\right) \\ \quad \leq 2\|e\|\|P\left(f\left(\hat{x},\,u\right) - f\left(x,\,u\right)\right)\| \\ \quad \leq \gamma^2 e'PPe + e'e\end{array}$$

then we conclude that the observer error is stable if and only if

$$A'P + PA - C'Y' - YC + \gamma^2 PP + I < 0. \tag{42}$$

By the Schur complement lemma, the last inequality is equivalent to (39) with $\rho = \frac{1}{\gamma^2}$. This ends the proof.

We see that the observer gain $L = P^{-1}Y$ depends on the maximum value γ which makes (39) satisfied. If the optimization problem (38) and (39) fails, then based on the obtained minimum value ρ, one can then choose an appropriate linear transformation which can reduce the value of the existing Lipschitz constant γ, at least, to $\frac{1}{\sqrt{\rho}}$.

B. Practical observer design

We have seen in the previous subsection that the existence of the observer gain depends on the solvability of an LMI problem. As we have mentionned before, if the Lipschitz constant is high then the solution of the LMI problem may fail. In this subsection, we plan to generalize the results of section I to the nonlinear case where the nonlinearity $f(x,\,u)$ is upper bounded by a certain constant $\frac{\gamma}{2}$ for all admissible u. Here, γ is not the Lipschitz constant of $f(\cdot,\,\cdot)$ but its maximum over a compact set $M \subset \mathbb{R}^n$.

Theorem 3: Consider the nonlinear system

$$\begin{array}{l}\dot{x} = A\,x + f(x,\,u) + g(y,\,u) + v, \\ \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_2 = \xi_3, \\ \vdots \\ \dot{\xi}_q = Cx + d,\end{array} \tag{43}$$

where $x = x(t) \in M \subset \mathbb{R}^n$ is the state vector, $u = u(t) \in \mathscr{U}$ is the control input that belongs to the set of admissible bounded inputs $\mathscr{U}$, and f is a globally Lipschitz function that verifies $\sup_{x\in M,\,u\in\mathscr{U}} \|f(x,\,u)\| \leq \frac{\gamma}{2}$. The disturbances $v = v(x(t), t)$ and $d = d(t)$ are defined as in section II. Let

$$\widetilde{F} = \left[\begin{array}{cc} A_\xi & B_\xi\,C \\ 0_{n\times q} & A \end{array}\right]_{(n+q)\times(n+q)}, \tag{44}$$

and let $P \in \mathbb{R}^{(n+q)\times(n+q)}$ be the solution of the Lyapunov matrix equation

$$-\mu P - \widetilde{F}'P - P\widetilde{F} = -\bar{C}'\bar{C}. \tag{45}$$

Then for any μ such that

$$\Re\left(\lambda_i(\widetilde{F})\right) > -\mu/2, \tag{46}$$

for every eigenvalue of $\widetilde{F}$, the system

$$\left[\begin{array}{c}\dot{\hat{\xi}} \\ \dot{\hat{x}}\end{array}\right] = \widetilde{F}\left[\begin{array}{c}\hat{\xi} \\ \hat{x}\end{array}\right] - P^{-1}\bar{C}'\bar{C}\left[\begin{array}{c}\hat{\xi}-\xi \\ \hat{x}\end{array}\right] + \left[\begin{array}{c}0_{q\times1} \\ f(\hat{x},\,u)\end{array}\right] + \left[\begin{array}{c}0_{q\times1} \\ g(y,\,u)\end{array}\right] - \chi,$$

is a robust observer of (43) that decouples the effects of uncertainties from the observer states and guarantees the exponential decay of the observation error to the origin if $d = v = 0$. χ is a nonlinear term defined as

$$\chi = \left\{\begin{array}{ll}\gamma^2\frac{\|P\|}{\|\bar{C}e\|^2}P^{-1}\bar{C}'\left(\hat{\xi}_1 - \xi_1\right) & \text{if } \left\|\bar{C}e\right\| \neq 0, \\ 0, & \text{otherwise,}\end{array}\right. \tag{47}$$

and A_ξ, B_ξ, $\bar{C}$ are defined as in section II.

The formula of χ is just a conceptual rule to guarantee the convergence of the observer error. In practice we can fix χ as

$$\chi = \left\{\begin{array}{ll}\gamma^2\frac{\|P\|}{\|\bar{C}e\|^2}P^{-1}\bar{C}'\left(\hat{\xi}_1 - \xi_1\right) & \text{if } \left\|\bar{C}e\right\| \geq \epsilon, \\ 0, & \text{if } \left\|\bar{C}e\right\| < \epsilon,\end{array}\right. \tag{48}$$

where ϵ is any desired error. Remark also that the high-gain term given by χ can not deteriorate the quality of estimation since ξ_1 and $\hat{\xi}_1$ represent the qth integral of the noisy output and its estimate, respectively.

Proof of theorem 3. Define the observation error as $e = \left[\begin{array}{c}\hat{\xi}-\xi \\ \hat{x}-x\end{array}\right]$. Then, we have

$$\dot{e} = \left(\widetilde{F} - P^{-1}\bar{C}'\bar{C}\right)e + \Delta f - G\,d - \widetilde{w} - \chi, \tag{49}$$

where G, $\widetilde{w}$ are defined as in section II, and

$$\Delta f = \left[\begin{array}{c}0_{q\times1} \\ f(\hat{x},\,u) - f(x,\,u)\end{array}\right]. \tag{50}$$

Taking $V = e'Pe$ as a Laypunov function candidate for (49), we have

$$\begin{array}{rcl}\dot{V} &=& e'\left(\widetilde{F}'P + P\widetilde{F} - 2\bar{C}'\bar{C}\right)e + 2e'P\Delta f - 2e'P\chi \\ &-& 2e'P\,G\,d - 2e'P\,\widetilde{w} \\ &=& e'\left(-\mu P - \bar{C}'\bar{C}\right)e + 2e'P\Delta f - 2e'P\,\chi \\ &-& 2e'P\,G\,d - 2e'P\,\widetilde{w}\end{array}$$

We have for $P > 0$, the matrix

$$\left[\begin{array}{cc}P & -I \\ -I & 2P\end{array}\right] > 0. \tag{51}$$

This comes from the fact that for any $P > 0$

$$\left[\begin{array}{cc}I & P\end{array}\right]\left[\begin{array}{cc}P & -I \\ -I & 2P\end{array}\right]\left[\begin{array}{c}I \\ P\end{array}\right] = P > 0. \tag{52}$$

This implies that for given vectors $\alpha \in \mathbb{R}^{q+n}$, $\beta \in \mathbb{R}^{q+n}$

$$\left[\begin{array}{cc}\alpha' & \beta'\end{array}\right]\left[\begin{array}{cc}P & -I \\ -I & 2P\end{array}\right]\left[\begin{array}{c}\alpha \\ \beta\end{array}\right] > 0. \tag{53}$$

This gives

$$2\alpha'\beta \leq \alpha'P\alpha + 2\beta'P^{-1}\beta. \tag{54}$$

Let $e = \alpha$, $P\Delta f = \beta$, then

$$2e'P\Delta f \leq e'Pe + 2\Delta f'P\Delta f. \tag{55}$$

Substituting the last inequality in (51), we obtain

$$\begin{array}{rcl}\dot{V} &\leq& -(\mu-1)e'Pe + 2\Delta f'P\Delta f - 2e'P\,\chi \\ &+& 2\left\|e'P^{\frac{1}{2}}\right\|\left\|P^{\frac{1}{2}}\,G\right\|\|d\| + 2\left\|e'P^{\frac{1}{2}}\right\|\left\|P^{\frac{1}{2}}\right\|\|\widetilde{w}\| \\ &\leq& -(\mu-1)V + 2\Delta f'P\Delta f - 2e'P\,\chi \\ &+& 2\sqrt{V}\left\|P^{\frac{1}{2}}\,G\right\|\|d\| + 2\sqrt{V}\left\|P^{\frac{1}{2}}\right\|\|\widetilde{w}\|.\end{array}$$

From the definition of χ, we have for $\left\|\bar{C}e\right\| \neq 0$

$$\begin{aligned} 2\Delta f' P \Delta f - 2e' P\,\chi &\leq 2\left\|P\right\|\left\|\Delta f\right\|^2 - 2e'P\,\chi \\ &= 2\gamma^2\left\|P\right\| - 2\frac{\gamma^2\left\|P\right\|}{\left\|\bar{C}e\right\|^2} e'\bar{C}'\bar{C}e \\ &= 0. \end{aligned} \quad (56)$$

Let $W = \sqrt{V}$, then

$$\dot{W} \leq -\frac{(\mu-1)}{2}W + \left\|P^{\frac{1}{2}}\,G\right\|\left\|d\right\| + \left\|P^{\frac{1}{2}}\right\|\left\|\widetilde{w}\right\| \quad (57)$$

Then we conclude that the observer error is stable. Furthermore, the norm $\left\|P^{\frac{1}{2}}\right\|$ can be made as small as possible by increasing the parameter μ, see the proof of lemma 1.

IV. Conclusions

In this paper we have examined the problem of robust observer design for both MISO linear and nonlinear systems. The observer strategy is based on Lyapunov theory and linear matrix inequalities. Under certain conditions, we showed that unmodeled dynamics and measurement errors can be enfeebled by injection of the qth integral of the measured output instead of the usual proportional injection term. For nonlinear systems subject to bounded nonlinearities, the problem of high-gain observer design with guaranteed robustness against measurement errors is considered. The extension of the present work to MIMO systems is under investigation.

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