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Analysis, design, and performance limitations of H_2 optimal filtering in the presence of an additional input with known frequency

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***Abstract*— The standard H_2 optimal filtering problem considers the estimation of a certain output based on the measured output when the input is a zero mean white noise stochastic process of known intensity. In this paper, the inputs are considered to be of two types. The first type of input, as in standard H_2 optimal filtering, is a zero mean wide sense stationary white noise, while the second type is a linear combination of sinusoidal signals each of which has an unknown amplitude and phase but known frequency. Under such inputs, a generalized H_2 optimal filtering problem is formulated here. As in the standard H_2 optimal filtering problem, the generalized H_2 optimal filtering problem seeks to find a *linear stable unbiased* filter (called the generalized H_2 optimal filter) that estimates a desired output while utilizing the measured output such that the H_2 norm of the transfer matrix from the white noise input to the estimation error is minimized. The analysis, design, and performance limitations of generalized H_2 optimal filters are presented here.**

I. Introduction

In filtering theory, a well known problem is the Kalman filtering problem [1], also known as the H_2 optimal filtering problem. In such a problem, one assumes that the input to the given system is white noise of unit intensity. In this paper, however, we assume that the inputs are of two types. The first type of input, as in H_2 optimal filtering, is a zero mean wide sense stationary white noise, while the second type is a linear combination of sinusoidal signals each of which has an unknown amplitude and phase but known frequency. Under such inputs, we seek here a stable '*unbiased*' filter that renders the RMS norm of the error signal as small as *possible* (an '*unbiased*' filter is a filter which, in the absence of the first type of input, renders the steady state error zero). We call such filtering problems generalized H_2 optimal filtering problems. The corresponding filters are of course termed as generalized H_2 optimal filters. If the infimum cannot be attained then we will look at a problem of finding families of unbiased filters which get arbitrarily close to the infimum of the RMS norm of the error signal. We will refer to such problems as generalized H_2 suboptimal filtering problems. After formulating such generalized H_2 optimal (suboptimal) filtering problems, we show that these problems can be reduced to standard H_2 optimal (suboptimal) filtering problems for an expanded system constructed from the data of the given system. We will then study the cost incurred by the additional requirement of rejecting a sinusoidal signal of known frequency but unknown amplitude and phase. We will show that the infimum of the RMS norm is not affected by the additional requirement. In general, the solvability conditions of the generalized H_2 optimal filtering problem might be stronger than the solvability conditions of the H_2 optimal filtering problem but for a large class of problems the solvability conditions are identical.

The cost of unbiasedness can be expressed in terms of the energy of the error signal. On the other hand, the cost of rejecting a white noise input is expressed in terms of the RMS norm of the error signal. We will see that both the cost of rejecting a white noise input and the cost of unbiased filters are related to the locations of the non-minimum-phase zeros of the system. Moreover, we will uncover a peculiar property that the cost of rejecting a white noise input *reduces* when the non-minimum-phase zeros are moved closer to the boundary of the stability domain (imaginary axis in continuous-time and unit circle in discrete-time) while the cost of unbiased filters *increases to infinity* when the non-minimum-phase zeros are moved closer to the boundary of the stability domain.

In what follows, the entire complex plane, the open left-half complex plane, the imaginary axis, and the open right-half complex plane are respectively denoted by $\mathbb{C}$, $\mathbb{C}^-$, $\mathbb{C}^0$, and $\mathbb{C}^+$. Similarly, $\mathbb{C}^\ominus$, $\mathbb{C}^0$, and $\mathbb{C}^\oplus$, denote respectively the set of complex numbers inside the unit circle, on the unit circle, and outside the unit circle.

All proofs are omitted in this conference version of the paper.

II. Preliminaries and Problem statement

Let us consider a system model,

$$\Sigma : \begin{cases} \sigma x &= Ax + B_1 u_1 + B_2 u_2, \\ y &= Cx + D_1 u_1 + D_2 u_2, \\ z &= Ex + F_1 u_1 + F_2 u_2, \end{cases} \tag{1}$$

where σ is an operator indicating the time derivative $\frac{d}{dt}$ for continuous-time systems and a forward unit time shift for discrete-time systems. Also, $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is

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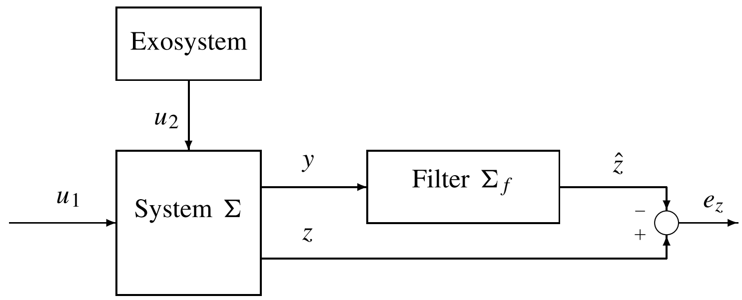

Fig. 1. General block diagram of filtering for Σ.

the measured output, and $z \in \mathbb{R}^q$ is the desired output signal to be estimated. The input $u_1 \in \mathbb{R}_1^m$ is assumed to be a zero mean wide sense stationary white noise of unit intensity. On the other hand, the input $u_2 \in \mathbb{R}_2^m$ is assumed to be a linear combination of sinusoidal signals each of which has an unknown amplitude and phase but known frequency, and thus can be modeled by

$$\Sigma_a : \ \sigma x_a = S x_a, \quad u_2 = C_a x_a, \tag{2}$$

where $x_a \in \mathbb{R}^{n_a}$ for some n_a. We consider a general proper filter of the form,

$$\Sigma_f : \begin{cases} \sigma\xi = L\xi + My \\ \hat{z} \ = N\xi + Py \end{cases} \tag{3}$$

which is assumed to be internally stable. Sometimes, we seek a family of filters parameterized in a positive parameter ε. In that case, a family of filters is described by

$$\Sigma_f^\varepsilon : \begin{cases} \sigma\xi = L_\varepsilon\xi + M_\varepsilon y \\ \hat{z} \ = N_\varepsilon\xi + P_\varepsilon y, \end{cases} \tag{4}$$

where L_ε, M_ε, N_ε, and P_ε are matrices parameterized in a positive parameter ε. Our interest here lies in estimating the desired output signal z while using only the measured output y but not the input u. As usual, let $\hat{z}$ be the estimate of z as given by a filter, and let e_z be the estimation error, defined by $e_z = z - \hat{z}$, as depicted in Figure 1. Before we formulate the specific problems of our interest, we first define what we mean by unbiased filters in this generalized setting.

Definition II.1 *Consider a continuous- or discrete-time system Σ as given in* (1) *where the input u_1 is a zero mean wide sense stationary white noise, while u_2 is generated by a linear exosystem given in* (2)*. We say a linear stable strictly proper (or proper) filter* (3) *is **(generalized) unbiased** if, in the absence of the input u_1, the estimation error e_z decays asymptotically to zero for all possible initial conditions of the given system* (1) *and the filter* (3)*, and for all input signals u_2. Equivalently, a filter is unbiased if, in the absence of the input u_1, the estimation error e_z decays asymptotically to zero for all possible initial conditions of the exosystem* (2)*, the given system* (1)*, and the filter* (3)*.*

Remark II.2 The above definition, whenever $u_2 = 0$, reduces to the familiar notion of unbiasedness of filters.

Before the next problem, let us recall the standard definition of the RMS norm of a stochastic process. In continuous time, we have:

$$\|w\|_{\text{RMS}}^2 = \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\int_0^T \|w(t)\|^2 \,\mathrm{d}t,$$

where $\mathbb{E}$ denotes the expectation under the assumption that the signal $w(t)$ is a stochastic process with bounded variance. In discrete time, we have:

$$\|w\|_{\text{RMS}}^2 = \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{k=0}^{T} \|w(k)\|^2,$$

We are now ready to define formally the following generalized optimal filtering problem under white noise input.

Problem II.3 Consider a continuous- or discrete-time system Σ as given in (1) where the input u_1 is a zero mean wide sense stationary white noise of unit intensity, and u_2 is the output of an exosystem as given in (2). Then, the **generalized optimal filtering problem under white noise input** is defined as follows: Find, whenever it exists, a linear stable strictly proper (or proper) filter such that

(i) **(Generalized unbiasedness)** the estimation error e_z, in the absence of the input u_1, decays asymptotically to zero for all possible finite initial values of the exosystem (2), the given system (1), and the filter (3), and

(ii) **(Performance)** the RMS norm of the error signal, namely $\|e_z\|_{\text{RMS}}$, is as small as possible.

We can now define the generalized optimal filtering performance under white noise input associated with the above generalized optimal filtering problem.

Definition II.4 *For the continuous- or discrete-time system Σ given in* (1) *where the input u_1 is a zero mean wide sense stationary white noise of unit intensity, and u_2 is the output of an exosystem as given in* (2)*, the infimum of the RMS norm of the error signal e_z over the set of all linear stable strictly proper (or proper) unbiased filters is called the **generalized***

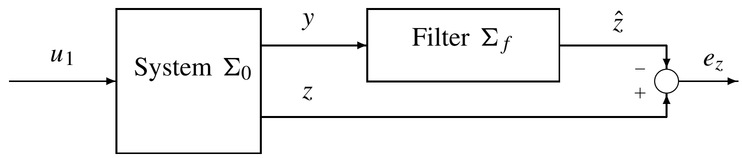

Fig. 2. General block diagram of filtering for Σ_0

optimal filtering performance under white noise input *via linear stable strictly proper (or proper) filters, and is denoted by $\gamma^*_{g,sp}$ (or $\gamma^*_{g,p}$).*

The generalized optimal filtering problem under white noise input can be given a deterministic interpretation since the RMS norm of the error signal e_z is equal to the H_2 norm of the transfer matrix from the input u_1 to the error e_z. Thus, we can interpret the generalized optimal filtering problem under white noise input as the generalized H_2 optimal filtering problem, and similarly $\gamma^*_{g,sp}$ (or $\gamma^*_{g,p}$) as the generalized H_2 optimal filtering performance via linear stable strictly proper (or proper) filters.

Clearly, if the generalized H_2 optimal filtering problem is not solvable, we can define the generalized suboptimal filtering problem as follows.

Problem II.5 Consider a continuous- or discrete-time system Σ as given in (1) where the input u_1 is a zero mean wide sense stationary white noise of unit intensity, and u_2 is the output of an exosystem as given in (2). Then, the **generalized suboptimal filtering problem under white noise input** is defined as follows: Find, whenever it exists, a family of linear stable strictly proper (or proper) filters parameterized in positive ε such that

- (i) **(Generalized unbiasedness)** for any given filter in the family, the estimation error e_z, in the absence of the input u_1, decays asymptotically to zero for all possible finite initial values of the exosystem (2), the given system (1), and the filter (3), and
- (ii) **(Performance)** the RMS norm of the error signal, namely $\|e_z\|_{\mathrm{RMS}}$, approaches $\gamma^*_{g,sp}$ (or $\gamma^*_{g,p}$) as ε tends to zero.

Whenever the input u_2 is set to zero, the generalized H_2 optimal filtering problem for the given system Σ reduces to the celebrated H_2 optimal filtering problem (Kalman filtering problem) for a system Σ_0 given by

$$\Sigma_0 : \begin{cases} \sigma x = Ax + B_1 u_1, \\ y \;\; = Cx + D_1 u_1, \\ z \;\; = Ex + F_1 u_1. \end{cases} \tag{5}$$

The block diagram of filtering for the system Σ_0 is depicted in Figure 2. Also, in this case, we denote the infimum of the RMS norm of the error signal over all the linear unbiased stable filters for the system Σ_0 by γ^*_{sp} or γ^*_p depending on whether we use strictly proper or proper filters.

III. Performance, existence and uniqueness conditions

We need to investigate several issues pertaining to generalized H_2 optimal (suboptimal) filtering. These issues include computing the generalized H_2 optimal filtering performance $\gamma^*_{g,sp}$ (or $\gamma^*_{g,p}$), and developing the existence and uniqueness conditions for the generalized H_2 optimal (suboptimal) filters. To do so, we form an expanded system $\tilde{\Sigma}$ as

$$\tilde{\Sigma} : \begin{cases} \sigma \bar{x} = A_e \bar{x} + B_e u_1, \\ y \;\; = C_e \bar{x} + D_1 u_1, \\ z \;\; = E_e \bar{x} + F_1 u_1, \end{cases} \tag{6}$$

where

$$A_e = \begin{pmatrix} A & B_2 C_a \\ 0 & S \end{pmatrix}, \quad B_e = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$
$$C_e = \begin{pmatrix} C & D_2 C_a \end{pmatrix}, \quad E_e = \begin{pmatrix} E & F_2 C_a \end{pmatrix}. \tag{7}$$

The general filtering block diagram for the expanded system $\tilde{\Sigma}$ is given in Figure 3. We also need the following assumption.

Assumption III.1 *The matrix pair (C_e, A_e) is detectable.*

We have the following results.

Theorem III.2 *Consider the filtering block diagram of Figure 1 for the system Σ given in* (1)*, the filter Σ_f given in* (3)*, and the exosystem Σ_a given in* (2)*. Let the pair (C, A) be detectable. Also, consider the filtering block diagram of Figure 3 for the expanded system $\tilde{\Sigma}$ given in* (6) *and the filter Σ_f given in* (3)*. Let Assumption III.1 be satisfied. Then, the following two statements are equivalent:*

- (i) *The filter Σ_f is a proper (or strictly proper) generalized H_2 optimal filter for Σ.*
- (ii) *The filter Σ_f is a proper (or strictly proper) H_2 optimal filter for $\tilde{\Sigma}$.*

Moreover, $\gamma^*_{g,sp}$ (or $\gamma^*_{g,p}$) equals the infimum of the RMS norm of the estimation error signal $\tilde{e}_z$ as depicted in the block diagram of Figure 3 over all unbiased strictly proper (or proper) stable filters.

Theorem III.3 *Consider the filtering block diagram of Figure 1 where Σ_f is replaced by Σ^ε_f as given in* (4)*, and where the system Σ is given in* (1)*, and the exosystem Σ_a is as given in* (2)*. Let the pair (C, A) be detectable. Also, consider*

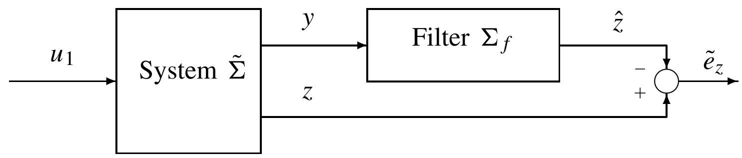

Fig. 3. General block diagram of filtering for $\tilde{\Sigma}$

the filtering block diagram of Figure 3 where Σ_f is replaced by Σ_f^{ε} as given in (4), *and where the expanded system $\tilde{\Sigma}$ is as given in* (6). *Let Assumption III.1 be satisfied by $\tilde{\Sigma}$. Then, the following two statements are equivalent:*

(i) *The family of filters Σ_f^{ε} is a family of proper (or strictly proper) generalized H_2 suboptimal filters for Σ.*

(ii) *The family of filters Σ_f^{ε} is a family of proper (or strictly proper) H_2 suboptimal filters for $\tilde{\Sigma}$.*

The above development needs the pair (C_e, A_e) be detectable. It turns out that this can be weakened by requiring that (C, A) be detectable and that Assumption III.4 is satisfied.

Assumption III.4 *For all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ (for continuous-time systems) or $|\lambda| \geq 1$ (for discrete-time systems) we have:*

$$\operatorname{rank}\begin{pmatrix} \lambda I - A & -B_2C_a \\ 0 & \lambda I - S \\ C & D_2C_a \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \lambda I - A & -B_2C_a \\ 0 & \lambda I - S \\ C & D_2C_a \\ E & F_2C_a \end{pmatrix}.$$

IV. Dependence of performance, existence and uniqueness conditions on the input u_2

As discussed in the previous section, various aspects of the generalized H_2 optimal filtering problem for the given system Σ are tantamount to the corresponding aspects of the H_2 optimal filtering for the expanded system $\tilde{\Sigma}$ that incorporates the given system Σ and the exosystem Σ_a that models the input signal u_2. Since the generalized H_2 optimal filtering has an additional requirement of rejecting the input u_2 (unbiased requirement) over and above the requirement of H_2 optimal filtering (namely, minimizing the RMS value of the estimation error signal), in this section we ask ourselves two fundamental questions:

(i) How does the performance of generalized H_2 optimal filtering for Σ differs from the performance of H_2 optimal filtering for Σ_0?

(ii) How do the solvability conditions of generalized H_2 optimal filtering problem for Σ differ from the solvability conditions of H_2 optimal filtering problem for Σ_0?

Theorems IV.1 and IV.2 answer these questions.

Theorem IV.1 *Consider the generalized H_2 optimal filtering problem as defined in Problem II.3 for the system Σ of* (1) *along with the associated exosystem Σ_a of* (2) *whose performance is indicated by $\gamma^*_{g,sp}$ or $\gamma^*_{g,p}$ depending upon whether the class of strictly proper or proper filters are used. Also, consider the H_2 optimal filtering problem for the system Σ_0 of* (5) *whose performance is indicated by γ^*_{sp} or γ^*_p depending upon whether the class of strictly proper or proper filters are used. Then, under Assumption III.1, we have*

$$\gamma^*_{g,sp} = \gamma^*_{sp} \quad \text{and} \quad \gamma^*_{g,p} = \gamma^*_p.$$

Theorem IV.2 *Consider the generalized H_2 optimal filtering problem as defined in Problem II.3 for the system Σ of* (1) *along with the associated exosystem Σ_a of* (2). *Let Assumption III.1 be satisfied. Also, consider the H_2 optimal filtering problem for the system Σ_0 of* (5). *We have the following statements:*

(i) *For the case when $F_2 = 0$, the generalized H_2 optimal filtering problem for the system Σ along with the associated exosystem Σ_a is solvable via strictly proper filters if and only if the H_2 optimal filtering problem for Σ_0 is solvable via strictly proper filters.*

(ii) *For the case when $F_2 = 0$ and additionally $D_2 = 0$, the generalized H_2 optimal filtering problem for the system Σ along with the associated exosystem Σ_a is solvable via proper filters if and only if the H_2 optimal filtering problem for Σ_0 is solvable via proper filters.*

There are examples showing that Theorem IV.2 does not hold if $F_2 \neq 0$.

V. Performance limitations of generalized H_2 optimal filtering due to the structural properties of a system

Performance limitations of generalized H_2 optimal filtering due to the structural properties of a system can be studied in two respects. One can first study the dependence of $\gamma^*_{g,sp}$ or $\gamma^*_{g,p}$ on the structural properties of a system. As revealed by Theorem IV.1 $\gamma^*_{g,sp} = \gamma^*_{sp}$ and $\gamma^*_{g,p} = \gamma^*_p$ where γ^*_{sp} and γ^*_p are the H_2 optimal performance measures for the system Σ_0 of (5). Both γ^*_{sp} or γ^*_p have been studied in a recent paper [2]. For continuous-time systems, they can be decomposed into two parts, one arising due to non-left invertible dynamics and the other due to unstable zero dynamics (i.e. the dynamics dictated by the invariant zeros in the open right half plane) of the subsystem characterized by (A, B, C, D). The part that is contributed by the unstable zero dynamics to γ^*_{sp} and γ^*_p tends to zero as the open right half plane invariant zeros move towards the imaginary axis, and in the same way it increases as the open right half plane invariant zeros move away from the imaginary axis. If the subsystem characterized by (A, B, C, D) is left-invertible and has no unstable zero dynamics, both γ^*_{sp} and γ^*_p equal zero. Similarly, for discrete-time systems, both γ^*_{sp} and γ^*_p can be decomposed into two parts, one arising due to unstable zero dynamics (i.e. the dynamics dictated by the invariant zeros outside the unit circle) and the other due to the non-left invertible dynamics as well as the dynamics dictated by the infinite zero structure of order greater than or equal to one of the subsystem characterized by (A, B, C, D). The part that is contributed by the unstable zero dynamics to γ^*_{sp} and γ^*_p tends to zero as the invariant zeros outside the unit circle move towards the unit circle, and in the same way it increases as the invariant zeros outside the unit circle move away from the unit circle.

One can also define the performance measure associated with the unbiased requirement and then examine its dependence on the structural properties of the given system as pursued next.

A. Performance limitations of unbiased filtering

In this subsection, we will define the performance measure associated with the unbiased requirement and then examine its dependence on the structural properties of the given system. The performance measure due to unbiasedness can be considered as the energy of the estimation error signal under the condition that u_1 is zero. Thus, consider the filtering block diagram of Figure 1 for the system Σ given in (1), the filter Σ_f given in (3), and the exosystem Σ_a given in (2). By combining the given system Σ and the exosystem Σ_a together, we form the expanded system $\tilde{\Sigma}$ as in (6) except we set there $u_1 = 0$, i.e., let

$$\begin{cases} \sigma\bar{x} = A_e\bar{x}, \\ y \;\;= C_e\bar{x}, \\ z \;\;= E_e\bar{x}, \end{cases} \tag{8}$$

where the matrix triple (A_e, C_e, E_e) is as in (7). Whenever the unbiased requirement is satisfied by the filter Σ_f, since the error e_z then is an energy signal, we can define the unbiasedness performance measure J^g as follows: For continuous-time systems,

$$J^g(\bar{x}_0, \xi_0, \Sigma_f) = \int_0^\infty e_z(t)'e_z(t)\,dt,$$

and for discrete-time systems,

$$J^g(\bar{x}_0, \xi_0, \Sigma_f) = \sum_{i=0}^{\infty} e_z(i)'e_z(i).$$

In the above equations,

$$\bar{x} = \begin{pmatrix} x \\ x_a \end{pmatrix}, \quad \bar{x}_0 = \bar{x}(0), \quad \xi_0 = \xi(0).$$

Clearly, the performance measure due to the unbiasedness depends on the filter used. Our aim here is to study it when Σ_f has the strictly proper CSS architecture as given by

$$\Sigma^g_{sp} : \begin{cases} \sigma\xi = (A_e - KC_e)\xi + Ky, \quad \xi(0) = \xi_0 \in \mathbb{R}^{n+n_a}, \\ \hat{z} \;\;= E_e\xi, \end{cases} \tag{9}$$

where K is the filter gain. Then, the dynamics of the error e_z, in the absence of input signal u_1 (as we assumed), is then given by

$$\begin{cases} \sigma e = (A_e - KC_e)e, \quad e(0) = e_0 = (\bar{x}_0 - \xi_0) \in \mathbb{R}^{n+n_a}, \\ e_z = E_e e. \end{cases} \tag{10}$$

We can then define the infimum of $J^g(\bar{x}_0, \xi_0, \Sigma^g_{sp})$ over all possible filter gains subject to the constraint (10), and denote such an infimum by $J^{*g}(\bar{x}_0, \xi_0, \Sigma^g_{sp})$. We can compute $J^{*g}(\bar{x}_0, \xi_0, \Sigma^g_{sp})$ as follows. Consider an auxiliary system,

$$\Sigma_{aux} : \begin{cases} \sigma x_{aux} = A_e x_{aux} + B_{aux}v, \\ y_{aux} \;\;= C_e x_{aux}, \\ z_{aux} \;\;= E_e x_{aux}, \end{cases} \tag{11}$$

where $B_{aux} = \bar{x}_0 - \xi_0$ and v is an unknown white noise input. Then, in view of Theorem III.2, we have the following result. Whenever we refer to $\gamma^*_{sp}(A, B, C, D, E, F)$, we mean by it γ^*_{sp} associated with the H_2 optimal filtering problem characterized by the sextuple (A, B, C, D, E, F)

Lemma V.1 *Consider the generalized H_2 optimal filtering problem as defined in Problem II.3 for the system Σ of* (1) *along with the associated exosystem Σ_a of* (2). *Let Assumption III.1 be satisfied. Also, consider the strictly proper filter Σ^g_{sp} given in* (9). *Let Σ_{aux} be as in* (11). *Then the infimum of the performance measure due to the unbiasedness is given by*

$$J^{*g}(\bar{x}_0, \xi_0, \Sigma^g_{sp}) = (\gamma^*_{sp}(A_e, B_{aux}, C_e, 0, E_e, 0))^2.$$

We note that the initial condition $\bar{x}_0$ of the given system is usually unknown, while one can set the initial condition ξ_0 of the filter as one likes. As before, this suggests that one can generate an average performance measure for the unbiasedness requirement. Let e_i, $i = 1, \cdots, n$, form a basis for the state space of Σ. Also, f_i, $i = 1, \cdots, n_a$, form a basis for the state space of Σ_a. Moreover, assume that one always sets the initial condition ξ_0 of the filter to zero. Then, we can define a new average performance measure for the unbiasedness requirement while utilizing a strictly proper filter of CSS architecture as

$$\tilde{J}^g(\Sigma, \Sigma_a, \Sigma^g_{sp}) = \tilde{J}^g_1(\Sigma, \Sigma_a, \Sigma^g_{sp}) + \tilde{J}^g_2(\Sigma, \Sigma_a, \Sigma^g_{sp}) \tag{12}$$

where

$$\tilde{J}^g_1(\Sigma, \Sigma_a, \Sigma^g_{sp}) = \sum_{i=1}^{n} J^g(\begin{pmatrix} e_i \\ 0 \end{pmatrix}, 0, \Sigma^g_{sp}),$$

and

$$\tilde{J}^g_2(\Sigma, \Sigma_a, \Sigma^g_{sp}) = \sum_{i=1}^{n_a} J^g(\begin{pmatrix} 0 \\ f_i \end{pmatrix}, 0, \Sigma^g_{sp}).$$

To study the limitations imposed by the given system on $\tilde{J}^g(\Sigma, \Sigma_a, \Sigma_{sp}^g)$, we can define

$$\tilde{J}^{*g}(\Sigma, \Sigma_a, \Sigma_{sp}^g)$$

as the infimum over all possible filter gains K subject to the constraint (10). Once again it is straightforward to show that $\tilde{J}^{*g}(\Sigma, \Sigma_a, \Sigma_{sp}^g)$ is also related to the H_2 optimal performance of an appropriately defined auxiliary system. Let

$$\tilde{\Sigma}_{aux} : \begin{cases} \sigma \tilde{x}_{aux} = A_e \tilde{x}_{aux} + I v, \\ \tilde{y}_{aux} \;\; = C_e \tilde{x}_{aux}, \\ \tilde{z}_{aux} \;\; = E_e \tilde{x}_{aux}, \end{cases} \tag{13}$$

where v is an unknown white noise input. We have the following result.

Lemma V.2 *Consider the generalized H_2 optimal filtering problem as defined in Problem II.3 for the system Σ of* (1) *along with the associated exosystem Σ_a of* (2). *Let Assumption III.1 be satisfied. Also, consider the strictly proper filter Σ_{sp}^g given in* (9). *Let Σ_{aux} be as in* (13). *Then the infimum of the average performance measure due to the unbiasedness is given by*

$$\tilde{J}^{*g}(\Sigma, \Sigma_a, \Sigma_{sp}^g) = (\gamma_{sp}^*(A_e, I, C_e, 0, E_e, 0))^2.$$

In an expanded version of the paper, a lower bound for $\tilde{J}^{*g}(\Sigma, \Sigma_a, \Sigma_{sp}^g)$ has been computed, i.e. it is shown that

$$\tilde{J}^{*g}(\Sigma, \Sigma_a, \Sigma_{sp}^g) \geq \frac{2 \operatorname{Re} z_i}{|z_i|^2 \Delta}$$

where z_i is any unstable invariant zero (non-minimum phase invariant zero) of the system characterized by (A, B, C, D). Also, Δ depends on (A, B, C, D), however it is always greater than a number α for some $\alpha > 0$. This implies that *the infimum of the average performance measure due to the unbiased requirement, namely $\tilde{J}^{*g}(\Sigma, \Sigma_a, \Sigma_{sp}^g)$ (as defined in* (12)*), tends to infinity as any of the unstable invariant zeros (non-minimum phase invariant zeros) of the subsystem characterized by (A, B, C, D) tends towards the origin.*

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