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Frequency-Domain Analysis of Linear Time-Periodic Systems

Henrik Sandberg, Erik Möllerstedt, and Bo Bernhardsson

***Abstract*— In this paper we study how a system with a time-periodic impulse response may be expanded into a sum of modulated time-invariant systems. This allows us to define a linear frequency-response operator for periodic systems, called the harmonic transfer function (HTF). Similar frequency-response operators have been derived before for sampled-data systems and periodic finite-dimensional state-space systems. The HTF is an infinite-dimensional operator that captures the frequency coupling of a time-periodic system. The paper includes analysis of convergence of truncated HTFs. For this reason the concepts of input/output roll-off are developed and related to time-varying Markov parameters.**

I. Introduction

A. Notation

Signals defined in continuous time on an interval I will belong to the spaces $L_1(I)$ or $L_2(I)$. The standard norms on these spaces will be denoted by $\|\cdot\|_{L_1(I)}$ and $\|\cdot\|_{L_2(I)}$. We will denote square-summable sequences by ℓ_2, and the norm by $\|\cdot\|_{\ell_2}$. R denotes the real axis, R_+ the non-negative real axis, and Z the set of all integers. j is the imaginary unit, and jR is the imaginary axis.

B. Problem Formulation

In this paper we study linear operators G defined on signals u in L_2:

$$y = Gu.$$

We will restrict ourselves to the set of bounded operators G. The set of bounded operators will be denoted by L_∞ and has a finite L_2-induced norm:

$$\|G\|_{L_\infty} = \sup_{\|u\|_{L_2} \leq 1} \|Gu\|_{L_2}. \tag{1}$$

The norm (1) may be computed in many ways depending on how G is represented. If, for example, there exists a finite-dimensional state-space realization of G, solutions to certain Riccati equations can be used. In this paper we pursue frequency-domain methods.

We assume in the following that the given G is bounded (1) and has a representation in the time domain with a *causal* impulse response $g(t,\tau)$ ($g(t,\tau) = 0$ for $t < \tau$):

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau, \tag{2}$$

where $u(t)$ and $y(t)$ belong to $L_2(-\infty,\infty) = L_2(R)$. Conditions for when G can be represented as an integral equation (2) is given in, for example, Sandberg [1], [2].

If there is a real positive number T such that

$$g(t+T,\tau+T) = g(t,\tau), \quad \text{for all} \quad t \geq \tau, \tag{3}$$

then the operator (or the system it represents) is said to be *periodic* with period T. We will obtain a frequency-domain representation of periodic systems G, originally represented in time (2). The search for a frequency-domain representation is motivated by the fact that frequency domain methods are very successful in the study of time-invariant systems, i.e. systems whose impulse response satisfy $g(t,\tau) = g(t-\tau)$, for all $t \geq \tau$.

C. Previous Work

The study of periodic systems has a long history in applied mathematics and control. One reason for the many studies of periodic systems is that natural and man-made systems often have the periodicity property (3). Some examples are: planets and satellites in orbit, rotors of wind mills and helicopters, sampled-data systems, and AC power systems. An excellent survey of periodic systems and control is that of Bittanti and Colaneri [3].

Frequency-domain analysis of linear time-periodic systems in continuous time has been studied by several authors in the past. To the authors' best knowledge Wereley in [4] computed the first frequency-response operator for linear periodic finite-dimensional state-space systems. He called the operator the *harmonic transfer function* (HTF). It is computed by using harmonic balance on a state-space system, i.e., periodic matrices are expanded into Fourier series and the harmonics are equated. The HTF is an infinite-dimensional operator, but it was shown in [4] in several numerical examples that by truncating higher harmonics one often obtains good accuracy. In the following we will use Wereley's term for the frequency-response operator, that is, the harmonic transfer function. This is well motivated as the frequency-response operator in the following work in the same way, even though it is derived under different conditions and is computed differently.

Some existence questions were left untreated by Wereley, so Zhou et al. have written a series of papers, including [5], [6], where they have proved such results. To compute the HTF in the original definition, the inverse of an infinite-dimensional quasi-Toeplitz matrix is needed. It is nontrivial to prove convergence of simple truncations of such matrices, so Zhou et al. first applied the Floquet decomposition on the periodic system. After a Floquet decomposition has been applied the inverse is simple to compute. The drawback with

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H. Sandberg is with the Department of Automatic Control, Lund Institute of Technology, Box 118, SE-221 00 Lund, Sweden. E-mail: henriks@control.lth.se

E. Möllerstedt and B. Bernhardsson are with Ericsson Mobile Platforms AB, SE-221 83 Lund, Sweden

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this method is that the Floquet decomposition may be hard to obtain in practice.

Möllerstedt and Bernhardsson used the HTF for the modeling of power systems and converters. In [7], [8] they showed that the frequency-response operator also could be computed from the impulse response of a system. The main objectives in [7], [8] are modeling and to show that the suggested methods work well, so convergence and existence questions are not considered there. The objective of this paper is to provide this justification.

Sampled-data systems is a closely related area where a lot of work has been done. Sampled-data systems are periodic systems of a special structure. To obtain the frequency response usually two approaches are taken: the lifting or the steady-state input-output approach. The lifting approach is used in for example Bamieh et al. [9] or Yamamoto et al. [10]. The steady-state approach is used in for example Araki et al. [11] or Dullerud [12]. The frequency-response operator derived in [11] (called the FR operator) has the same form as the HTF. One can show that the two approaches mentioned above are equivalent, see Yamamoto et al. [13]. A nice property of sampled-data systems is that closed-form solutions often are obtained. This is not the case for generic periodic systems. The literature on sampled-data systems is vast and many more good references could be mentioned.

A more general study of frequency-domain representations of time-varying linear systems has also been presented in Ball et al. [14].

D. Contribution and Organization

This paper is certainly not the first paper on frequency-response operators for time-periodic systems. The main contribution of this paper is a detailed computation of the harmonic transfer function from the impulse response and an extensive analysis of the convergence of square truncation of the HTF. In section II we define input and output roll-off and relate these concepts to time-varying Markov parameters. In section III we expand the periodic impulse response into a Fourier series. In section IV we apply the Fourier transform to the Fourier series from section III and obtain the harmonic transfer function. We also see that input and output roll-off give convergence rates for square truncations.

II. Properness and Roll-off

Our goal is a frequency-domain description of G, and we will many times represent the input signal $u(t)$ and output signal $y(t)$ in $L_2(R)$, by their Fourier transforms $\hat{u}(j\omega)$ and $\hat{y}(j\omega)$. ω is the angular frequency. This presents no problems as $L_2(R)$ is isomorphic with $L_2(jR)$ under the Fourier transform, see Dym and McKean [15]. We define the norms as follows

$$\|u\|_{L_2} = \|u(\cdot)\|_{L_2(R)} = \left(\int_{-\infty}^{\infty} |u(t)|^2 dt\right)^{1/2} \tag{4}$$

$$= \|\hat{u}(\cdot)\|_{L_2(jR)} = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega\right)^{1/2}, \tag{5}$$

and the equality of (4) and (5) follows from Plancherel's theorem.

We will truncate the representations of signals and systems, and therefore it will be interesting to study how the systems treat high-frequency signals. Hence a projection operator which we call P_Ω is defined. Its representation in the frequency domain is given by

$$\hat{y}_\Omega(j\omega) = \widehat{(P_\Omega y)}(j\omega) = \begin{cases} \hat{y}(j\omega), & |\omega| \le \Omega, \\ 0, & |\omega| > \Omega. \end{cases} \tag{6}$$

Notice that P_Ω is not causal in the time domain, and $\|P_\Omega\|_{L_\infty} = 1$. It is also convenient to define $Q_\Omega = I - P_\Omega$. In order for a truncated system to be a good approximation we need some sort of roll-off, corresponding to strict properness for linear time-invariant systems, see Zhou [16]. We call a system strictly proper if

$$\|G - P_{\Omega_1} G P_{\Omega_2}\|_{L_\infty} \to 0 \quad \text{as} \quad \Omega_1, \Omega_2 \to \infty.$$

To give sufficient conditions for properness we first notice that we can decompose the problem into two separate problems:

$$\|G - P_{\Omega_1} G P_{\Omega_2}\|_{L_\infty} \le \|(I - P_{\Omega_1})G\|_{L_\infty} + \|G(I - P_{\Omega_2})\|_{L_\infty}.$$

Definition 1 (Möllerstedt [7]): If for a system $G \in L_\infty$ there are positive constants C_1 and k_1 such that

$$\|(I - P_\Omega)G\|_{L_\infty} \le C_1 \cdot \Omega^{-k_1},$$

then G is said to have *output roll-off* k_1, and if there are positive constants C_2 and k_2 such that

$$\|G(I - P_\Omega)\|_{L_\infty} \le C_2 \cdot \Omega^{-k_2},$$

then G is said to have *input roll-off* k_2.
For systems with output roll-off k_1 and input roll-off k_2 we have strict properness and the following rate of convergence for truncated operators $P_{\Omega_1} G P_{\Omega_2}$:

$$\|G - P_{\Omega_1} G P_{\Omega_2}\|_{L_\infty} \le C_1 \cdot \Omega_1^{-k_1} + C_2 \cdot \Omega_2^{-k_2}. \tag{7}$$

For time-invariant systems we have the following proposition:

Proposition 1: If G has a time-invariant impulse response, that is $g(t,\tau) = g(t-\tau)$ for all $t \ge \tau$, then if it has output roll-off k_1, it also has input-roll off k_1, and vice versa. If $|\hat{g}(j\omega)| \le C \cdot |\omega|^{-k}$ then $C_1 = C_2 = C$ and $k_1 = k_2 = k$.

Proof: Follows directly from Definition 1 and that $GQ_\Omega = Q_\Omega G$ for time-invariant G. ■

For time-varying impulse responses, input and output roll-off are more difficult to check. However, by making certain expansions of (2) we can state necessary and sufficient conditions for roll-off. For simplicity we assume that the impulse response belongs to $\mathscr{C}^m$ in the region $t > \tau$, i.e., the impulse response is m times continuously differentiable. Furthermore, we assume that all the derivatives have uniform exponential decay. Using integration by parts repeatedly on the relation (2) for inputs with support in $[\alpha, t]$ we obtain the following expansion:

$$y(t) = g(t,t)\frac{u(t)}{p} - g'_\tau(t,t)\frac{u(t)}{p^2} + \dots$$
$$(-1)^{m-1} g_\tau^{(m-1)}(t,t)\frac{u(t)}{p^m} + (-1)^m \int_\alpha^t g_\tau^{(m)}(t,\tau)\frac{u(\tau)}{p^m}d\tau. \tag{8}$$

By making a similar expansion of the adjoint operator G^* we obtain the expansion:

$$y(t) = \frac{1}{p}g(t,t)u(t) + \frac{1}{p^2}g'_t(t,t)u(t) + \dots$$
$$+ \frac{1}{p^m}g_t^{(m-1)}(t,t)u(t) + \frac{1}{p^m}\int_\alpha^t g_t^{(m)}(t,\tau)u(\tau)d\tau. \tag{9}$$

Here $u(t)/p^k$ is the k-times integration operator: $\int_\alpha^t \cdots \int_\alpha^{t_2} u(t_1)dt_1 \cdots dt_k$, $g_\tau^{(k)}(t,\tau) = \partial^k g(t,\tau)/\partial\tau^k$, and $g_t^{(k)}(t,\tau) = \partial^k g(t,\tau)/\partial t^k$. For details see [17]. The coefficients in the expansions are the time-varying Markov parameters:

Definition 2 (Output and Input Markov Parameters): For a system with impulse response $g(t,\tau)$ the *output Markov parameters* are defined as

$$\left\{g(t,t), g'_t(t,t), g''_t(t,t), \dots\right\}, \tag{10}$$

and the *input Markov parameters* are defined as

$$\left\{g(t,t), -g'_\tau(t,t), g''_\tau(t,t), \dots\right\}. \tag{11}$$

These coefficients coincide with the regular Markov parameters, $\{g(0), g'(0), g''(0), \dots\}$ for time-invariant impulse responses.

Theorem 1 (Markov Parameters and Roll-off): Assume that $g(t,\tau)$ belongs to $\mathscr{C}^m$ in $t > \tau$, that all the m derivatives of the impulse response have uniform exponential decay, and that $m > \max\{k_1, k_2\}$. Then G has

i) output roll-off k_1, if and only if the $k_1 - 1$ first output Markov parameter are zero for all t

ii) input roll-off k_2, if and only if the $k_2 - 1$ first input Markov parameter are zero for all t

Proof: We first show the sufficiency of i). If the first $k_1 - 1$ output Markov parameters are zero we have the following expansion of $Q_\Omega G$ using (9):

$$y(t) = \frac{Q_\Omega}{p^{k_1}}\left(g_t^{(k_1-1)}(t,t)u(t) + \int_{-\infty}^t g_t^{(k_1)}(t,\tau)u(\tau)d\tau\right).$$

The first factor has the Fourier transform $1/(j\omega)^{k_1}$ for $|\omega| > \Omega$ and is zero otherwise. Its induced L_2-norm is then $1/\Omega^{k_1}$. The second factors has an induced L_2-norm of $C_1 < \infty$ as all the derivatives of the impulse response have uniform exponential decay. Hence we have

$$\frac{\|y\|_{L_2}}{\|u\|_{L_2}} \le \|Q_\Omega G\|_{L_\infty} \le \frac{C_1}{\Omega^{k_1}}$$

and it follows that G has output roll-off k_1. The sufficiency of ii) follows similarly using (8) instead of (9).

To prove the necessity of i) and ii) we choose a special input signal u. One can choose a modulated Gaussian pulse of frequency Ω. See [17] for details. ■

TABLE I

INPUT AND OUTPUT MARKOV PARAMETERS OF TIME-VARYING STATE-SPACE SYSTEMS.

#	Input Markov parameter	Output Markov parameter
0.	$D(t)$	$D(t)$
1.	$C(t)B(t)$	$C(t)B(t)$
2.	$C(t)[A(t)B(t) - B'(t)]$	$[C'(t) + C(t)A(t)]B(t)$
3.	$C(t)[B''(t) - 2A(t)B'(t)$ $-A'(t)B(t) + A^2(t)B(t)]$	$[C''(t) + 2C'(t)A(t)$ $+C(t)A'(t) + C(t)A^2(t)]B(t)$

Example 1 (Finite-Dimensional State-Space Models): A time-varying system with a finite-dimensional state-space realization can be written as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned} \tag{12}$$

We assume that all matrices are bounded and as differentiable as is required. The impulse response of the system is given by $g(t,\tau) = C(t)\Phi_A(t,\tau)B(\tau) + D(t)\delta(t-\tau)$, where Φ_A is the transition matrix for $\dot{x}(t) = A(t)x(t)$. The first few Markov parameters are given in Table I. Notice how the parameters reduce to the well-known Markov-parameters of time-invariant systems, that is $D, CB, CAB, CA^2B, \dots$

III. Fourier Expansions of Periodic Systems

Until now we have only represented a system G in L_∞ with a convolution integral in the time-domain. For time-invariant systems it is well known that convolution becomes simple multiplication if we represent signals and systems in the frequency domain. We now look for an analogous frequency-domain representation of linear time-periodic systems. For reasons that will become clearer later it simplifies to consider systems in a set B. The set B is defined as

$$B = \{G : \|G\|_B < \infty \text{ , } g(t,\tau) \text{ is } T\text{-periodic and causal}\} \tag{13}$$

where

$$\|G\|_B = \sum_{k=0}^{\infty}\left(\int_{r=kT}^{(k+1)T}\int_{t=0}^{T}|g(t,t-r)|^2 dt\, dr\right)^{\frac{1}{2}}. \tag{14}$$

$\|\cdot\|_B$ is a combination of a Hilbert-Schmidt norm and an ℓ_1-norm.

If G has a causal periodic impulse response then for every given $r \in R_+$, the impulse response $g(t, t-r)$ is T-periodic in t. For $G \in B$, $g(\cdot, \cdot - r)$ belongs to $L_2[0,T]$ for almost all $r \in R_+$. This follows from Fubini's theorem (see for example Dym and McKean [15]). Hence, for almost all r we can expand the impulse response in a Fourier series with convergence in $L_2[0,T]$:

$$g(t, t-r) = \sum_{l=-\infty}^{\infty} g_l(r) e^{jl\omega_0 t}, \tag{15}$$

where

$$g_l(r) = \frac{1}{T} \int_0^T e^{-jl\omega_0 t} g(t, t-r) dt \tag{16}$$

and $\omega_0 = 2\pi/T$. We summarize the properties of the set B in the following proposition:

Proposition 2 (The set B): For a system G in B it holds that

i)
$$\|G\|_{L_\infty} = \sup_{\|u\|_{L_2} \le 1} \|Gu\|_{L_2} \le 2\|G\|_B,$$
and thus B is a subset of L_∞

ii) the time-invariant Fourier coefficients are summable and square-summable:
$$g_l(\cdot) \in L_1[0,\infty) \cap L_2[0,\infty)$$
for all $l \in Z$.

Proof: i): In [7] it shown that

$$\|G\|_{L_\infty} \le \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \int_0^T |g(t,\tau)|^2 d\tau dt \right)^{1/2}$$

by using lifting on (2). By then using (3) and the triangular inequality, i) follows.

ii): Follows by noticing that

$$|g_l(r)|^2 \le \frac{1}{T} \int_0^T |g(t, t-r)|^2 dt,$$

for all l. By then using the definition of the B-norm and the Cauchy-Schwarz inequality ii) follows. ■

Notice that $\|G\|_B$ might be a very poor upper estimate of $\|G\|_{L_\infty}$, so the norm is not used for explicit calculations. The reason for introducing the set B is that it simplifies the Fourier analysis. The set B is not empty:

Example 2 (Exponentially stable systems are in B): All periodic systems that are uniformly exponentially stable, i.e. there are positive constants K, κ such that

$$|g(t,\tau)| \le K \cdot e^{-\kappa(t-\tau)}, \quad t \ge \tau$$

are in B. Hence, the systems considered in Theorem 1 are in B.

Next we see how truncated Fourier expansions of systems in B behave. The following lemma proves convergence:

Lemma 1 (Truncated Fourier representations): A truncated Fourier expansion of G in B with N frequencies is defined as

$$G_N: \quad g_N(t,\tau) = \sum_{l=-N}^{N} g_l(t-\tau) e^{jl\omega_0 t},$$

where $\tau = t - r$. It has the following properties:

i) G_N belongs to B.

ii) $\|G - G_N\|_{L_\infty} \le 2\|G - G_N\|_B \to 0$ as $N \to \infty$.

Proof: See Appendix. ■

A. Frequency Coupling and Steady-State Response

To see the difference between a time-invariant and a time-periodic system it is instructive to study the steady-state response to a harmonic input signal, $u(t) = e^{j\omega t}$ with frequency ω. For time-invariant systems it is well known that the output also is a harmonic of the same frequency. This is, however, not the case for time-periodic systems. If we for simplicity study a finite Fourier expansion of G, we obtain

$$\begin{aligned} y_N = G_N e^{j\omega t} &= \int_{-\infty}^{t} \left(\sum_{l=-N}^{N} g_l(t-\tau) e^{jl\omega_0 t} \right) e^{j\omega\tau} d\tau \\ &= \sum_{l=-N}^{N} \hat{g}_l(j\omega) e^{j(l\omega_0 + \omega)t} \end{aligned} \tag{17}$$

This shows that the response includes a whole range of frequencies, with a difference of ω_0. This is a well-known property of linear periodic systems, see for example Wereley [4] or Zhou et al. [5]. Hence these systems have frequency coupling. It also shows that a frequency-domain approach could be successful, as there is still a fairly simple relation between frequencies in input and output.

IV. The Harmonic Transfer Function

By including a sufficient amount of frequencies in the Fourier expansion G_N of G, we can come arbitrarily close to G itself in L_∞−sense. Let $y_N = G_N u$ and $y = Gu$, then from (2) and Lemma 1 we have

$$\begin{aligned} y_N(t) &= \int_{-\infty}^{t} \left(\sum_{l=-N}^{N} g_l(t-\tau) e^{jl\omega_0 t} \right) u(\tau) d\tau \\ &= \sum_{l=-N}^{N} [g_l(\cdot) e^{jl\omega_0 \cdot} * u(\cdot) e^{jl\omega_0 \cdot}](t) \end{aligned} \tag{18}$$

where $*$ is the standard convolution product. By applying the Fourier transform on (18) we get

$$\hat{y}_N(j\omega) = \sum_{l=-N}^{N} \hat{g}_l(j\omega - jl\omega_0) \hat{u}(j\omega - jl\omega_0). \tag{19}$$

The Fourier transform of $g_l(t)$, denoted by $\hat{g}_l(j\omega)$, is well defined by Proposition 2 ii), and even bounded and continuous for all ω as $g_l \in L_1$, see [15]. By Lemma 1 ii) and Plancherel's theorem, we know that $\hat{y}_N(j\omega)$ converges to $\hat{y}(j\omega)$ in $L_2(jR)$ as $N \to \infty$. Therefore we can put $N = \infty$ in (19) if we mean convergence in L_2−sense, and not pointwise convergence.

In Araki et al. [11] the Sample-Data(SD)-Fourier transform was defined, and it is also useful here. The SD-transform is an isometric isomorphism between $L_2(jR)$ and a Hilbert space we denote by $L_2^Z(jI_0)$. It maps the Fourier transform into an infinite-dimensional column-vector-valued function. The SD-transform of $\hat{u}(j\omega)$ is denoted by $\hat{U}(j\omega)$ and is defined as

$$\hat{U}(j\omega) = [\dots \quad \hat{u}(j\omega + j\omega_0) \quad \hat{u}(j\omega) \quad \hat{u}(j\omega - j\omega_0) \quad \dots]^T.$$

As the vector contains repeated versions of $\hat{u}(j\omega)$, it is enough to define $\hat{U}(j\omega)$ for $\omega \in I_0 = (-\omega_0/2, \omega_0/2]$ to be able to take the inverse SD-transform. We define the norm in $L_2^Z(jI_0)$ as

$$\begin{aligned} \|\hat{U}(\cdot)\|_{L_2^Z(jI_0)} &= \frac{1}{\sqrt{2\pi}} \left(\int_{I_0} \|\hat{U}(j\omega)\|_{\ell_2}^2 d\omega \right)^{1/2} \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{I_0} \sum_{k=-\infty}^{\infty} |\hat{u}(j\omega + jk\omega_0)|^2 d\omega \right)^{1/2}. \end{aligned} \tag{20}$$

For signals $u \in L_2$, we now have three representations: $u(t)$, $\hat{u}(j\omega)$, and $\hat{U}(j\omega)$. In fact, the following extended Plancherel's theorem is true:

$$\|u\|_{L_2} = \|u(\cdot)\|_{L_2(R)} = \|\hat{u}(\cdot)\|_{L_2(jR)} = \|\hat{U}(\cdot)\|_{L_2^Z(jI_0)}. \tag{21}$$

If u has finite L_2-norm, then $\hat{U}(j\omega)$ is in ℓ_2 (its elements are square summable) for almost all $\omega \in I_0$, that is $\|\hat{U}(j\omega)\|_{\ell_2} < \infty$ almost everywhere.

We can write (19) when $N = \infty$ in matrix-vector form using the SD-transform:

$$\hat{Y}(j\omega) = \hat{G}(j\omega)\hat{U}(j\omega), \quad \omega \in I_0 = (-\omega_0/2, \omega_0/2]. \tag{22}$$

where $\hat{G}(j\omega)$ is equal to:

$$\begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \hat{g}_0(j\omega + j\omega_0) & \hat{g}_1(j\omega) & \hat{g}_2(j\omega - j\omega_0) & \\ \ddots & \hat{g}_{-1}(j\omega + j\omega_0) & \hat{g}_0(j\omega) & \hat{g}_1(j\omega - \omega_0) & \ddots \\ & \hat{g}_{-2}(j\omega + j\omega_0) & \hat{g}_{-1}(j\omega) & \hat{g}_0(j\omega - j\omega_0) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

We call $\hat{G}(j\omega)$ the *harmonic transfer function* (HTF) of G. This was the term used by Wereley in [4]. A similar object was called the FR operator by Araki et al. in [11] in the case of sampled-data systems. The difference between these efforts is the way the elements of $\hat{G}(j\omega)$ are computed. In the sampled-data case explicit formulas are given in [11]. In the time-periodic state-space case formulas are given in [4], [5], and in the impulse response case formulas are given here.

We may now state the counterpart of Theorem 4 and 5 in Araki et al. [11]. A similar result for time-periodic state-space systems is derived by Wereley and Zhou et al. in [4], [6]. We include it here also as it is a useful result in the following.

Theorem 2 (L_∞-norm formula): For linear periodic systems G in B, we can define the HTF $\hat{G}(j\omega)$ as above, and for any input signal $u \in L_2$ it holds that

$$\begin{aligned} \|y\|_{L_2}^2 &= \frac{1}{2\pi} \int_{I_0} \|\hat{Y}(j\omega)\|_{\ell_2}^2 d\omega \\ &= \frac{1}{2\pi} \int_{I_0} \|\hat{G}(j\omega)\hat{U}(j\omega)\|_{\ell_2}^2 d\omega. \end{aligned} \tag{23}$$

$\hat{G}(j\omega)$ is a bounded operator on ℓ_2 for almost all ω in I_0 and

$$\|G\|_{L_\infty} = \text{ess} \sup_{\omega \in I_0} \|\hat{G}(j\omega)\|_\infty, \tag{24}$$

where $\|\cdot\|_\infty$ represents the induced ℓ_2-norm.

Proof: Very similar to the proof of Theorem 5 in Araki et al. [11].

■

A. Computation and Truncation of HTFs

To compute the norm (1) of a system $G \in L_\infty$ with roll-off, the following observation, which follows directly from (7), is useful

$$0 \leq \|G\|_{L_\infty} - \|P_{\Omega_1} G P_{\Omega_2}\|_{L_\infty} \leq C_1 \cdot \Omega_1^{-k_1} + C_2 \cdot \Omega_2^{-k_2}. \tag{25}$$

Theorem 2 gives us a way to compute the induced L_2-norm, given a HTF $\hat{G}(j\omega)$. It is not essential that $\hat{G}(j\omega)$ corresponds to a causal operator in time for (24) to hold, it is true for every frequency-domain relation (22). Hence we can apply it to the approximation $P_{\Omega_1} G P_{\Omega_2}$. The central element of the HTF of $P_{\Omega_1} G P_{\Omega_2}$ becomes a finite-dimensional matrix:

Corollary 1: If $G \in B$, $\Omega_1 = (N_1 + 1/2)\omega_0$, and $\Omega_2 = (N_2 + 1/2)\omega_0$ then the HTF of $P_{\Omega_1} G P_{\Omega_2}$ is given by the matrix $\hat{G}_{(N_1,N_2)}(j\omega)$:

$$\begin{bmatrix} \hat{g}_{N_1-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{N_1+N_2}(j\omega - jN_2\omega_0) \\ \vdots & & \vdots \\ \hat{g}_{-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{N_2}(j\omega - jN_2\omega_0) \\ \vdots & & \vdots \\ \hat{g}_{-N_1-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{-N_1+N_2}(j\omega - jN_2\omega_0) \end{bmatrix},$$

of dimension $(2N_1 + 1) \times (2N_2 + 1)$.

Hence we can represent a linear periodic system in B arbitrarily well with finite-dimensional matrices and compute its norm as

$$\|P_{\Omega_1} G P_{\Omega_2}\|_{L_\infty} = \max_{\omega \in I_0} \overline{\sigma} \left(\hat{G}_{(N_1,N_2)}(j\omega) \right), \tag{26}$$

where $\overline{\sigma}(\cdot)$ is the maximum singular value of a matrix. The maximum is indeed obtained in (26) as the elements are continuous by Proposition 2 ii). By griding the frequency interval I_0 and by computing the maximum singular value we get an estimate of $\|G\|_{L_\infty}$, and the rate of convergence depends upon the roll-off of G according to (25). Square truncations of the frequency-response operator are commonly used to estimate the norm of a system, see for example [6], [11]. In [11] the rate of convergence was shown to be bounded by $K \cdot N^{-1/2}$ for $(2N+1) \times (2N+1)$ matrices and some constant K. We now see that by checking the Markov parameters we can improve this bound.

As we can approximate the infinite-dimensional HTF with finite-rank matrices arbitrarily well, it also follows that input or output roll-off implies that $\hat{G}(j\omega)$ is a compact operator on ℓ_2 for almost all ω. A large output roll-off means that the operator decays quickly in the up-down direction, and a large input roll-off means that the operator decays quickly in the left-right direction

V. Conclusion

In this paper we have studied linear time-periodic systems from a frequency-domain point of view. Previous studies in this field are often based on a state-space approach, see Wereley [4] and Zhou et al. [5], whereas we have here taken an impulse response approach. We have identified a set of periodic impulse responses, denoted by B, that allow us to expand the corresponding systems into a sum of modulated time-invariant systems, with convergence in an induced L_2-norm sense. We can construct a linear frequency-response operator for these systems, the harmonic transfer function. Similar frequency-response operators have appeared before, in for example [4], [5], [11], but they have been computed and analyzed differently.

We have put effort into the problem of how truncated harmonic transfer functions converge. This problem has been approached by introducing the concepts of input and output roll-off. For time-invariant systems the input and output roll-off are identical. Necessary and sufficient roll-off conditions have been stated in terms of time-varying Markov parameters. The HTF of a system with roll-off can be truncated into a finite-dimensional matrix, and explicit convergence rates have been given.

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Appendix

A. Proof of Lemma 1

First introduce the error function

$$\gamma_N(r) = \int_0^T |g(t,t-r) - \sum_{l=-N}^{N} g_l(r)e^{jl\omega_0 t}|^2 dt.$$

Due to the convergence of the Fourier series (15) it holds for almost all r that

$$0 \le \ldots \le \gamma_N(r) \le \ldots \le \gamma_0(r) \le \int_0^T |g(t,t-r)|^2 dt, \quad (27)$$

and $\lim_{N\to\infty} \gamma_N(r) = 0$.
i): It holds that

$$\begin{aligned}\|G-G_N\|_B &= \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \gamma_N(r)dr \right)^{1/2} \\ &\le \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \int_0^T |g(t,t-r)|^2 dtdr \right)^{1/2} = \|G\|_B,\end{aligned}$$

and statement i) follows.
ii): From (27) and the proof of i) it follows that that $\{\|G-G_N\|_B\}_{N=0}^{\infty}$ is a bounded decreasing sequence. Hence, $\lim_{N\to\infty}\|G-G_N\|_B$ do exist.

By definition and interchanging the order of limits we then have

$$\begin{aligned}\lim_{N\to\infty}\|G-G_N\|_B &= \lim_{N\to\infty} \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \gamma_N(r)dr \right)^{1/2} \\ &= \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \lim_{N\to\infty} \gamma_N(r)dr \right)^{1/2} = 0.\end{aligned}$$

To justify the interchange of the order of the limit and the summation, we notice that the sum is uniformly convergent in N because of (27). The interchange of the order of the limit and integration is a property of the Lebesgue integral of decreasing sequences of functions (a.e.), see for example Exercise 6 at page 10 of Dym and McKean [15]. The final result follows with the help of Proposition 2.