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Robust Output Feedback Controller Design via Genetic Algorithms and LMIs: The Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

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***Abstract*— This paper deals with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for uncertain continuous-time linear systems. The polytopic parametric uncertainties are studied. Based on Genetic Algorithms (GAs) and linear matrix inequalities (LMIs), a hybrid algorithm is presented for numerical computation of an optimal fixed-order static or dynamic output feedback robust controller. The genetic algorithm is used to obtain the population of controllers and the LMI based routines are used to minimize some performance criterion. The problem is formulated in terms of different Lyapunov functions. This approach can be used for synthesis of reduced or full-order controllers. Some examples borrowed from the literature are discussed to illustrate and validate this approach.**

I. Introduction

Designing robust controllers with guaranteed performance in the face of plant uncertainty has been the aim of robust multivariable control theory over the last two decades. If there are uncertainties in the system model, some quantity combining the $\mathcal{H}_2$ norm and the $\mathcal{H}_\infty$ norm can be a desirable measure of a system's robust performance [29]. Thus the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance criterion provides an interesting measure for evaluating controllers. The theoretic motivation for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has been extensively discussed in [4] [10] [17] [24]. Some important results about output feedback control can be found in [7] [8] [14] [26] and the references therein. However, the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem does not have yet a closed-form solution except for special cases presented in the literature.

In this work, a hybrid approach based on Genetic Algorithms (GAs) and LMIs in order to find an internally stabilizing output feedback controller which solves the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem is proposed.

The first motivation for this work arises from the fact the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust controller, in general, is not easy to be designed. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem was reduced to a convex optimization problem by considering a formulation with a common Lyapunov function in [3] [5] [15] [20] [25], nevertheless this assumption results in some degree of conservatism.

The second motivation comes from the change of variable presented in [20] [25], developed to turn output feedback specifications into LMIs. In this change of variable the system matrices A and B_2 are involved, thus the dynamic output feedback control design problem for systems subject to polytopic uncertainties cannot be reduced to a convex optimization problem.

The last motivation, but not the least, is the interesting properties of the GAs [19] [21]. The underlying principles of GAs were first published by Holland in 1962. GA can be applied to a number of control methodologies for the improvement of the overall system performance. Previous researches into this area include [6] [11] [16] [18]. In [6] [16] [18], GAs are used to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for SISO systems using transfer functions. Some related approaches to ours, developed in this work, can be found in [11] [23].

This paper is organized as follows. Section II presents the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem. In Section III, the hybrid algorithm based on GAs and LMIs for obtaining a solution to this problem is proposed. Some examples from the literature are discussed in Section IV.

II. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Robust Control Problem

Consider an uncertain continuous-time linear system described by the following state-space equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z_\infty(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ z_2(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \\ y(t) = C_y x(t) + D_{y1} w(t) + D_{y2} u(t) \end{cases}, \quad (1)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$, $z_\infty(t) \in \Re^{p_1}$, $z_2(t) \in \Re^{p_2}$, $w(t) \in \Re^l$. $x(t)$ is referred to as the state, $u(t)$ is the control input, $y(t)$ is the sensor output, $z_\infty(t)$ and $z_2(t)$ are the controlled outputs and $w(t)$ is the exogenous input. All matrices are assumed to be real of appropriate and known dimensions. Assume also that the matrices A and B_2 belong to the convex-bounded domains defined as

$$\mathcal{D}_A = \left\{ A; A = \sum_{i=1}^{N} \alpha_i A_i, \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0 \right\} \quad (2)$$

$$\mathcal{D}_B = \left\{ B_2; B_2 = \sum_{j=1}^{M} \beta_j B_{2j}, \sum_{j=1}^{M} \beta_j = 1, \beta_j \geq 0 \right\}. \quad (3)$$

Furthermore, suppose that all pairs (A, B_2) and (C_y, A) are stabilizable and detectable, respectively, and without any loss of generality, $D_{y2} = 0$. No assumptions are necessary about singular plants with jw-axis zeros or rank deficiencies in matrices D.

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The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem under consideration can be formulated as follows. Consider a standard system block diagram on the figure 1.

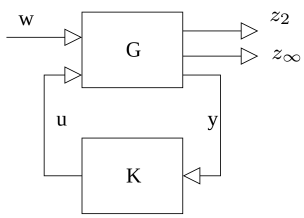

Fig. 1. Block Diagram.

where the uncertain plant G is given by (1). The linear time invariant (LTI) output feedback controller $K(s)$ can be represented in the state-space form by:

$$\begin{cases} \dot{\eta}(t) &= A_K\eta(t) + B_K y(t) \\ u(t) &= C_K\eta(t) + D_K y(t) \end{cases}, \tag{4}$$

where $A_K \in \Re^{n_c\times n_c}$. In this approach, the order n_c must be fixed but the controller can be of reduced ($n_c < n$) or full-order ($n_c \geq n$). Both G and K are real-rational and proper.

The resulting closed-loop system by dynamic output feedback is described by

$$\begin{cases} \dot{x}_f(t) &= A_f x_f(t) + B_{1f}w(t) \\ z_\infty(t) &= C_{1f}x_f(t) + D_{11f}w(t) \\ z_2(t) &= C_{2f}x_f(t) + D_{21f}w(t) \\ y(t) &= C_{yf}x_f(t) + D_{y1}w(t) \end{cases}, \tag{5}$$

where

$$\begin{aligned} A_f &= \begin{bmatrix} A + B_2D_KC_y & B_2C_K \\ B_KC_y & A_K \end{bmatrix}, \\ B_{1f} &= \begin{bmatrix} B_1 + B_2D_KD_{y1} \\ B_KD_{y1} \end{bmatrix}, \\ C_{if} &= \begin{bmatrix} C_i + D_{i2}D_KC_y & D_{i2}C_K \end{bmatrix}, \ i = 1,2, \\ D_{i1f} &= \begin{bmatrix} D_{i1} + D_{i2}D_KD_{y1} \end{bmatrix}, \ i = 1,2, \\ C_{yf} &= \begin{bmatrix} C_y & 0 \end{bmatrix}, \end{aligned} \tag{6}$$

with $A \in \mathcal{D}_A$ and $B \in \mathcal{D}_B$. Rewriting the equations (5) as

$$\begin{cases} \dot{x}_f(t) &= \bar{A}x_f(t) + \bar{B}_1w(t) + \bar{B}_2u_s(t) \\ z_\infty(t) &= \bar{C}_1x_f(t) + D_{11}w(t) + \bar{D}_{12}u_s(t) \\ z_2(t) &= \bar{C}_2x_f(t) + D_{21}w(t) + \bar{D}_{22}u_s(t) \\ y(t) &= C_yx_f(t) + D_{y1}w(t) \\ y_s(t) &= \bar{C}_yx_f(t) + \bar{D}_{y1}w(t) \end{cases}, \tag{7}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0_{n_c\times n_c} \end{bmatrix}, \ \bar{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$\bar{B}_2 = \begin{bmatrix} B_2 & 0 \\ 0 & I_{n_c\times n_c} \end{bmatrix}, \ \bar{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}, \ i = 1,2, \tag{8}$$

$$\bar{D}_{i2} = \begin{bmatrix} D_{i2} & 0 \end{bmatrix}, \ i = 1,2,$$

$$\bar{C}_y = \begin{bmatrix} C_y & 0 \\ 0 & I_{n_c\times n_c} \end{bmatrix}, \ \bar{D}_{y1} = \begin{bmatrix} D_{y1} \\ 0 \end{bmatrix},$$

the investigated control law is given by $u_s(t) = L_Ky_s(t)$, with

$$L_K = \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix} \in \Re^{(m+n_c)\times(p+n)}. \tag{9}$$

Thus, the dynamic output feedback problem can be treat as a static output feedback one.

Defining $T_{z_\infty w}(s)$ as the closed-loop transfer matrix from w to z_∞ and $T_{z_2w}(s)$ the one from w to z_2, that is

$$\begin{aligned} T_{z_\infty w}(s) &= (\bar{C}_1\bar{D}_{12}L_K\bar{C}_y)[sI - (\bar{A} + \bar{B}_2L_K\bar{C}_y)]^{-1} \\ &\quad (\bar{B}_1 + \bar{B}_2L_K\bar{D}_{y1}) + (D_{11} + \bar{D}_{12}L_K\bar{D}_{y1}) \\ T_{z_2w}(s) &= (\bar{C}_2\bar{D}_{22}L_K\bar{C}_y)[sI - (\bar{A} + \bar{B}_2L_K\bar{C}_y)]^{-1} \\ &\quad (\bar{B}_1 + \bar{B}_2L_K\bar{D}_{y1}) + (D_{21} + \bar{D}_{22}L_K\bar{D}_{y1}) \end{aligned} \tag{10}$$

the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem under consideration can be formulated as follows.

Find a proper, real-rational admissible robust controller L_K which minimizes the $\mathcal{H}_2$ norm $\|T_{z_2w}\|_2$ subject to the $\mathcal{H}_\infty$ norm constraint $\|T_{z_\infty w}\|_\infty < \gamma$, for a given achievable $\mathcal{H}_\infty$-norm bound γ, for $\forall A \in \mathcal{D}_A$ and $\forall B_2 \in \mathcal{D}_B$.

Note that $(D_{21} + \bar{D}_{22}L_K\bar{D}_{y1})$ must be zero in order to guarantee that the $\mathcal{H}_2$ problem be properly posed. Although this problem consider the minimization of $\|T_{z_2w}\|_2$, our approach allows to deal with a more general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance cost the trade-off criterion:

$$\underset{L_K}{Min} \quad \alpha\|T_{z_\infty w}\|_\infty^2 + \beta\|T_{z_2w}\|_2^2 \tag{11}$$

with $\alpha > 0$ and $\beta > 0$ as defined in [12]. It is important to observe that the $\mathcal{H}_\infty$ norm constraint keeps the closed-loop system internally stable under the action of uncertainties ($\forall A \in \mathcal{D}_A$ and $\forall B \in \mathcal{D}_B$).

In order to solve this problem we recall some standard results in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theory. We first present the $\mathcal{H}_\infty$ problem in terms of LMIs.

Using the bounded real lemma [7] [28] and the concept of quadratic stability [5], the $\mathcal{H}_\infty$ constraint is equivalent to the existence of a unique solution $X_\infty = X_\infty^T > 0$ that satisfies the matrix inequality

$$\begin{pmatrix} A_fX_\infty + X_\infty A_f^T & B_{1f} & X_\infty C_{1f}^T \\ B_{1f}^T & -I & D_{11f}^T \\ C_{1f\infty}X_\infty & D_{11f} & -\gamma^2 I \end{pmatrix} < 0, \tag{12}$$

for all vertices $(A_i, B_{2j}), i = 1,2\ldots N$ and $j = 1,2\ldots M$, as defined in (2) and (3).

Now consider the $\mathcal{H}_2$ performance measure [5] [14] without taking into account the $\mathcal{H}_\infty$ norm constraint. Supposing that the output feedback controller L_K is computed in such a way that A_f is asymptotically stable, the $\mathcal{H}_2$ norm of $T_{z_2w}(s)$ can be calculated by [9]:

$$\|T_{z_2w}(s)\|_2^2 = Trace(C_{2f}L_cC_{2f}^T), \quad (13)$$

where $L_c = L_c^T > 0$ is the solution of the Lyapunov equation

$$A_fL_c + L_cA_f^T + B_{1f}B_{1f}^T = 0. \quad (14)$$

Therefore, for the uncertain plant G described in (5), $\|T_{z_2w}\|_2^2 \leq Trace(C_{2f}X_2C_{2f}^T)$ for any $X_2 = X_2^T > 0$ such that

$$A_fX_2 + X_2A_f^T + B_{1f}B_{1f}^T < 0 \quad (15)$$

for all matrices (A_i, B_{2j}), $i = 1, 2 \ldots N$ and $j = 1, 2 \ldots M$. As earlier, this problem is not tractable unless requiring the same Lyapunov matrix $X_2 > 0$ (quadratic stability) in (15). It is worth noticing that the inequalities (12) and (15) are LMIs for fixed controller L_K and γ.

This leads to the following suboptimal matrix inequality formulation for our mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem:

$$\begin{array}{cc} \underset{X_2>0, X_\infty>0, L_K}{Min} & Trace(C_{2f}X_2C_{2f}^T) \\ s.t. & \end{array} \quad (16)$$

$$\begin{bmatrix} M_{ak}X_\infty + X_\infty(M_{ak})^T & M_{bj} & X_\infty M_c^T \\ * & -I & M_d^T \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (17)$$

$$M_{ak}X_2 + X_2M_{ak}^T + M_{bj}M_{bj}^T < 0, \quad (18)$$

$$D_{21} + \bar{D}_{22}L_K\bar{D}_{y1} = 0. \quad (19)$$

where

$$\begin{array}{c} M_{ak} = \bar{A}_i + \bar{B}_{2j}L_K\bar{C}_y \\ M_{bj} = \bar{B}_1 + \bar{B}_{2j}L_K\bar{D}_{y1} \\ M_c = \bar{C}_1 + \bar{D}_{12}L_K\bar{C}_y \\ M_d = D_1 + \bar{D}_{12}L_K\bar{D}_{y1} \end{array}, \quad (20)$$

with $i = 1, 2 \ldots N$ and $j = 1, 2 \ldots M$.

This problem is not jointly convex in the variables (X_2, X_∞, L_K), but it is still convex for a fixed controller L_K. This performance criterion gives an upper bound of the optimal $\mathcal{H}_2$ performance subject to the $\mathcal{H}_\infty$ norm constraint.

It is important to point out that our approach does not require the hypothesis of common Lyapunov matrices $X_2 = X_\infty$. This reduces the conservatism and provides better results. Furthermore, the usual change of variable necessary to recast the mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem as a LMI problem [20] [25] can also be eliminated. It allows to readily solve the dynamic or static output feedback control case for plants subject to uncertainties. In the change of variable introduced in [25] using matrices R and S, the system matrices A and B_2 are involved, hence certain limitations are imposed on extending this result to deal with control synthesis problem for polytopic uncertain systems. Relaxing this assumptions, the design of both reduced and full-order controller can also be considered in an unified state-space framework by our approach.

III. GAs and LMIs Based Optimization Approach

In this section we introduce a hybrid design procedure of robust output feedback controller for solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem presented in the precedent section. The hybrid procedure we proposed combines the reliability properties of the Genetic Algorithms [19] [21] and their typical search heuristics with the accuracy and efficiency of LMI solving methods [5] [22].

Based on GAs and LMIs this algorithm searches an optimal robust controller L_K (9) and consequently determines $X_2 > 0$ and $X_\infty > 0$ that solve the optimization problem (16) satisfying (17)-(19). Thus, our algorithm works with a population of candidate solutions L_K (individuals).

This choice has been made in order to decrease the algorithm required computation effort. Hence, the optimization can be achieved via the genetic operators, selection, crossover and mutation, applying only to the L_K population, instead of Lyapunov matrices. Many works based on Lyapunov's method found in the literature [11] [23] [27] adopt jointly the Lyapunov matrices and the controller parameters as individual to compose the GA population.

At each generation the optimization problem (16) is solved for all candidate individuals L_K of the population using the Matlab package LMI-Lab [12]. Remember that for fixed $L_K = [l_{ij}]_{(m+n_c)\times(p+n_c)}$ the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem (17)-(19) is convex. The $\mathcal{H}_\infty$ norm constraint γ is given but nevertheless the more general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem defined with the performance cost (11) and constraints (17)-(19) can be solved with this algorithm.

A simple genetic algorithm for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem is described as follows.

A. Initialization

Our genetic algorithm works with a float point representation for the population, i.e., the parameters themselves. The individuals L_K are created randomly. Unlike other GAs, some solutions may be infeasible in this case. An L_K is said to be infeasible if it does not fulfill the LMI constraints (17)-(19). Therefore, every time an individual L_K (controller) is generated or changed, it's necessary to guarantee its feasibility, otherwise L_K is discarded. This proceeding goes on until the number of individual of the population is reached. The search domain for the parameters $l_{ij}, \forall i, \forall j$, cover by this algorithm must be limited. The GA employed in this work adopt an elitist strategy, in which at each generation the individual with worst fitness value is replaced by the one with best fitness value from the last generation. It stops when a fixed number of generations N_I is achieved or when the value returned by the fitness function remains constant for a number of generations.

B. Objective and Fitness Function

The objective (cost) function provides the mechanism for evaluating each individual L_K. We consider two differents objective functions. The first one uses the $\mathcal{H}_2$ norm:

$$\begin{array}{cc} J(L_K) = Min & Trace(C_{2f} X_2 C_{2f}^T) \\ X_2 > 0, X_\infty > 0 & \\ subject\ to & (17) - (19) \end{array} . \quad (21)$$

The second one minimizes a trade-off criterion (11):

$$\begin{array}{cc} J(L_K) = Min & \alpha\gamma^2 + \beta Trace(C_{2f} X_2 C_{2f}^T) \\ X_2 > 0, X_\infty > 0, \gamma > 0 & \\ subject\ to & (17) - (19) \end{array} . \quad (22)$$

To maintain uniformity over different problems, we use the fitness function to rescale the objective function. In this work, the fitness function is defined as

$$fitness(L_K) = \frac{1}{1 + \sqrt{J(L_K)}} \quad (23)$$

Minimizing the objective function $J(L_K)$ is equivalent to getting a maximum fitness value in the genetic search. No penalty functions are used in this algorithm.

C. Selection

Based on the principle of survival of the fitness, some individuals are selected to populate the next generation. The selection is executed based on ”roulette wheel”method [21].

D. Crossover

Selected individuals are then recombined, through a crossover operation, with a crossover probability, by mixing and matching genetic information between pairs of individuals contained in the current population. Two types of crossover are used: heuristic and modified arithmetic.

The heuristic operator generates a single offspring $\bar{L}_K$ from two current parents L_{K1} and L_{K2} according to the following rule:

$$\bar{L}_K = r(L_{K2} - L_{K1}) + L_{K2}, \quad (24)$$

where r is a random number chosen in $[0, 1]$, and the parent L_{K2} is not worst than L_{K1}, that is, $J(L_{K1}) \geq J(L_{K2})$. The heuristic crossover is a special operator which uses values of the objective function in determining the direction of the search. It is possible for this operator to generate an offspring which is not feasible. In this case another random value r is chosen and another offspring is created. If after 3 attempts no new candidate solution is feasible, the parents are crossed by using the modified arithmetic crossover.

The modified arithmetic crossover operator is not only a simple convex combination between two current individuals, but an exchanging of genetic information of heuristic nature:

$$\begin{aligned} \bar{L}_{k1} &= (\alpha L_{k1} C_f X_{21} + (1-\alpha) L_{k2} C_f X_{22}) C_f^T \\ &\quad (C_f(\alpha X_{12} + (1-\alpha) X_{22}) C_f^T)^{-1} \\ \bar{L}_{k2} &= ((1-\alpha) L_{k1} C_f X_{21} + \alpha L_{k2} C_f X_{22}) C_f^T \\ &\quad (C_f((1-\alpha) X_{21} + \alpha X_{22}) C_f^T)^{-1}, \end{aligned} \quad (25)$$

where $C_f = \bar{C}_2 + \bar{D}_{22} L_K \bar{C}_y$, X_{21} and X_{22} are the Lyapunov matrices solutions to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ convex problem (17)-(19) for the fixed controllers L_{K1} and L_{K2}, respectively, and α is a randomly generated number in the range 0 to 1. $\bar{L}_{K1}$ and $\bar{L}_{K2}$ are the two new offspring. To guarantee feasibility of the two new offspring, this procedure can be repeated up to 5 times.

E. Mutation

After crossover, each individual is subjected to mutation with a given probability. The mutation operator used in this book is similar to that one of classic genetic algorithms using real numbers. The mutation operator performed on a single individual L_K is defined as

$$\bar{L}_K = L_K + \rho L_{K0} \quad (26)$$

where L_{K0} is a random matrix with all entries uniformly chosen in $[-1, 1]$, ρ is a positive real constant and $\bar{L}_K$ is the mutated individual.

F. GA Parameters

The following parameters were used:

- $l_{ij} \in [-\bar{l}_{ij}, \bar{l}_{ij}]$;
- Heuristic crossover probability p_{ch} = 0.3;
- Arithmetic crossover probability p_{ca} = 0.8;
- Mutation probability p_m = 0.1;
- $\rho = 1$;

These parameters are empiric, and were obtained after many simulations. The others parameters (population size, number of generation N_I and $\bar{l}_{ij}$) are different for each problem.

IV. Examples

Example 1. Borrowed from [13], it represents the model of the linearized dynamic equation of the VTOL helicopter. In [13], only the optimal $\mathcal{H}_2$ control by output feedback is considered. The matrices are

$$A = \begin{pmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & a_{32} & -0.7070 & a_{34} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $a_{32} \in [-0.6319, 1.3681]$; $a_{34} \in [1.22, 1.62]$

$$B_2 = \begin{pmatrix} -0.4422 & -0.1761 \\ b_{21} & 7.5922 \\ 5.5200 & -4.4900 \\ 0 & 0 \end{pmatrix},\ C_y^T = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where $b_{21} \in [2.7446, 4.3446]$ and

$$B_1 = I_4,\ C_2 = \begin{pmatrix} I_4 \\ 0 \end{pmatrix},\ D_{22} = \begin{pmatrix} 0 \\ I_2 \end{pmatrix},\ D_{21} = 0,$$
$$D_{y1} = 0,\ D_{y2} = 0,\ C_1 = I_4,\ D_{11} = 0,\ D_{12} = 0.$$

Setting $N_I = 10$ and with a population of 10 individuals and $n_c = 2$, the algorithm finds the dynamic output feedback controller

$$A_K = \begin{pmatrix} -7.1164 & -4.1802 \\ 4.5548 & -41.5504 \end{pmatrix}, B_K = \begin{pmatrix} 32.4228 \\ 23.3667 \end{pmatrix}$$

$$C_K = \begin{pmatrix} 15.7138 & 24.5692 \\ -37.8962 & -26.8289 \end{pmatrix}, D_K = \begin{pmatrix} 4.9199 \\ -7.0081 \end{pmatrix}$$

and the associated guaranteed cost $\|H\|_2^2 = 18.9888$ and $\|H\|_\infty = 18.9714$. Designing a quadratic stabilizing output feedback in order to compare our results to those obtained in [13], we found $L_K = [1.0032, -10.0781]^T$ and the associated guaranteed cost $\|H\|_2^2 = 18.8209$, with $N_I = 10$ and a population size 20. In [13], the optimal solution provided is $L_K = [-1.0086, 6.1051]^T$ and associated guaranteed cost $\|H\|_2^2 = 20.2509$.

Example 2. Consider the following system discussed in [2]:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ C_y = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

$$B_{12} = I_2,\ B_{1\infty} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & 1 \end{pmatrix},\ D_{22} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Instead of having just one perturbation w, there are two different perturbations, represented by w_2 (H_2 Norm) and w_∞ (H_∞ Norm). The system becomes:

$$\begin{cases} \dot{x} = Ax + B_{12}w_2 + B_{1\infty}w_\infty + B_2u \\ z_2 = C_2x + D_{22}u \\ z_\infty = C_1x \\ y = C_yx \end{cases} \tag{27}$$

The GA begins by randomly generating population of 20 individuals. After 10 generations, the output feedback control was $L_K = [-3.4176]$ with associated norms $\|H\|_2 = 1.5651$ and $\|H\|_\infty = 1.0000$. In [2], the results were $L_K = [-0.8165]$, $\|H\|_2 = 1.5651$ and $\|H\|_\infty = 1.3416$.

V. Conclusion

In this work, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem has been addressed. An algorithm based on GAs and LMIs has been proposed to find a fixed structure output feedback robust controller which minimizes an $\mathcal{H}_2$ performance cost subject to $\mathcal{H}_\infty$ norm constraint. The near mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust optimal design is achieved by genetic operations and LMI based routines. Simulations results indicate that this approach can offer an effective and simple method to solve the open mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem.

The proposed algorithm is suitable for full-order or reduced-order dynamic output controller design. In this approach the assumption $X_2 = X_\infty$ is not required reducing the conservatism and allowing better solutions. It is important to note that the genetic operators are applied only to the gain matrices L_K. Hence this implementation can present less computation burden.

Due to the properties of this approach, additional performance constraints could be incorporated to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem (multiobjective synthesis), for example, the performance requirements which can be reached by pole placement. This method can be applied to the design of control laws for discrete systems subject to state and control constraints [1]. Future work will be dedicated in this direction. The results of this work can also be readily extended to the state feedback case.

References

[1] H.X. de Araújo, I. Queinnec and S. Tarbouriech, "$\mathcal{H}_2$ guaranteed cost control for uncertain discrete-time systems under state and control constraints", *in 36th Conference on Decision And Control*, 1997, pp. 345-350.

[2] D. Arzelier and D. Peaucelle, "An Iterative Method for Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Synthesis via Static Output Feedback", *in 41th Conference on Decision And Control*, 2002, pp. 3464-3469.

[3] R. Bambang, E. Shimemura and K. Uchida, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control with pole placement, state-feedback case", *in Proceedings of American Control Conference*, 1993, pp. 2777-2779.

[4] D.S. Bernstein and W.M. Haddad, "LQG Control with an $\mathcal{H}_\infty$ Performance Bound: A Riccati Equation Approach", *in IEEE Transactions on Automatic Control*, vol. 34, no. 3, 1989, pp. 293-305.

[5] S. Boyd, L.E. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Society for Industrial and Applied Mathematics, Philadelphia, PA; 1994.

[6] B.-S. Chen, Y.-M. Cheng and C.-H. Lee, "A Genetic Approach to Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Optimal PID Control", *in IEEE Control Systems*, 1995, pp. 51-60.

[7] M. Chilali and P. Gahinet, "$\mathcal{H}_\infty$ Design with Pole Placement Constraints: An LMI Approach", *in IEEE Transactions on Automatic Control*, vol. 41, no. 3, 1996, pp. 358-367.

[8] M. Chilali, P. Gahinet and P. Apkarian, "Robust Pole Placement in LMI Regions", *in IEEE Transactions on Automatic Control*, vol. 44, no. 12, 1999, pp. 2257-2270.

[9] J.C. Doyle, K. Glover, P.K. Khargonekar and B.A. Francis, "State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems", *in IEEE Transactions on Automatic Control*, vol. 34, no. 8, 1989, pp. 831-847.

[10] J. Doyle, K. Zhou, K. Glover and B. Bodenheimer, "Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance Objectives II: Optimal Control", *in IEEE Transactions on Automatic Control*, vol. 39, no. 8, 1994, pp. 1575-1586.

[11] H. Du and X. Shi, "Low-Order $\mathcal{H}_\infty$ Controller Design using LMI and Genetic Algorithm", *in 15th IFAC World Congress*, 2002.

[12] P. Gahinet, A. Nemirovski, A.J. Laub and M. Chilali, *LMI Control Toolbox*, The Mathworks, Inc., Natick, MA; 1995.

[13] J.C. Geromel, P.L.D. Peres and S.R. Souza, "Convex Analysis of Output Feedback Control Problems: Robust Stability and Performance", *in IEEE Transactions on Automatic Control*, 1996, vol. 41, no. 7, pp. 997-1003.

[14] J.C. Geromel, C.C. de Souza and R.E. Skelton, "Static Output Feedback Controllers: Stability and Convexity", *in IEEE Transactions on Automatic Control*, 1998, vol. 43, no. 1m pp. 120-125.

[15] B. Halder and T. Kailath, "LMI Based Design of Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controllers: The State Feedback Case", *in Proceedings of American Control Conference*, 1999, pp. 1866-1870.

[16] J.-C. Hung and B.-S. Chen, "Genetic Algorithm Approach to Fixed-Order Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Optimal Deconvolution Filter Designs", *in IEEE Transactions on Signal Processing*, 2000, vol. 48, no. 12, pp. 3451-3461.

[17] P.K. Khargonekar and M.A. Rotea, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control: A Convex Optimization Approach", *in IEEE Transactions on Automatic Control*, vol. 36, no. 7, 1991, pp. 824-837.

[18] R.A. Krohling, L.S. Coelho and A.A.R. Coelho, "Evolution Strategies for Synthesis of Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Fixed-Structure Controllers", *in 14th IFAC World Congress*, 1999, vol. F, pp. 471-476.

[19] K.F. Man, K.S. Tang and S. Kwong, "Genetic Algorithms: Concepts and Applications", *in IEEE Transactions on Industrial Electronics*, 1996, vol. 43, no. 5, pp. 519-534.

[20] I. Masubuchi, N. Suda and A. Ohara, "LMI-based controller synthesis: A unified formulation and solution", *in Proceedings of American Control Conference*, 1995, pp. 3473-3477.

[21] Z. Michalewicz, *Genetic Algorithms + data structures = evolution programs* , Spring-Verlag Berlin Heidelberg, New York, NY; 1996.

[22] A. Nemirovskii and P. Gahinet, "The Projective Method for Solving Linear Matrix Inequalities", *in Proceedings of American Control Conference*, 1994.

[23] G.K.M. Pedersen, A.S. Langballe and R. Wísniewski, "Synthesizing Multi-Objective $\mathcal{H}_2/\mathcal{H}_\infty$ Dynamic Controller Using Evolutionary Algorithms", *in 15th IFAC World Congress*, 2002.

[24] C. Scherer, "Mutiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ Control", *in IEEE Transactions on Automatic Control*, vol. 40, no. 6, 1995, pp. 1054-1062.

[25] C. Scherer, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control", *in Trends in Control*, *A European Perspective*, A. Isidori, Ed. Berlin: Springer-Verlag, 1995, pp. 173-216.

[26] C. Scherer, P. Gahinet and M. Chilali, "Multiobjective Output-Feedback Control via LMI Optimization", *in IEEE Transactions on Automatic Control*, vol. 42, no. 7, 1997, pp. 896-911.

[27] I. Sekaj and V. Veselý, "Robust Output Feedback Controller Design via Genetic Algorithm", *in 15th IFAC World Congress*, 2002.

[28] K. Zhou, J.C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ; 1996.

[29] K. Zhou, K. Glover and B. Bodenheimer and J. Doyle, "Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance Objectives I: Robust Analysis", *in IEEE Transactions on Automatic Control*, vol. 39, no. 8, 1994, pp. 1564-1574.