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Mapping Integral Quadratic Constraints into Parameter Space

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***Abstract*— This paper presents the mapping equations for Integral Quadratic Constraints (IQCs). In particular it is shown that IQCs with bounded rational multipliers can be mapped into parameter space.**

Using IQCs not only provides a uniform framework to map specifications into parameter space, but provides a unified approach to parameter space based robustness analysis and synthesis with respect to nonlinearities, time variations, and uncertain parameters.

The exploitation of additional degrees of freedom contained in the multipliers is also addressed. To this end convex optimization problems in terms of Linear Matrix Inequalities (LMIs) can be formulated and solved during the mapping process.

I. INTRODUCTION

The parameter space approach is a well established method for robustness analysis of systems with uncertain parameters [1]. Initially the parameter space approach considered eigenvalue specifications for linear systems.

Recently, this approach was extended to frequency domain specifications [3], [4]. Static nonlinearities were considered in the parameter space approach in [2]. Finally [10] derived mapping equations for multi-input multi-output systems, including $\mathcal{H}_2, \mathcal{H}_\infty$ norms and passivity specifications.

Our goal is to use the unifying framework of IQCs to find mapping equations which allow to incorporate a large set of specifications. Having found the mapping equations for general IQC specifications enables us to consider specifications from the input-output theory, absolute stability theory and the robust control field. Specifications from all these research fields can be used in conjunction with the parameter space approach. Using the same mathematical formulation the same computational methods can be used for these different specifications.

Outline: In Section II, a brief treatment of IQCs will be given. The following section then states the main result, the mapping equations for IQCs. We will first consider fixed, frequency independent multipliers. Section IV will then show how to map specifications based on IQCs with frequency dependent multipliers into parameter space.

For many uncertainties not just a single multiplier but sets of multipliers exist to characterize the uncertainty structure. Often these sets can be described by parameterized multipliers. We will show how to utilize these additional degrees of freedom in order to minimize the conservativeness inherent in mapping only a single multiplier in Section V. As an example a nonlinear system is analyzed in Section VI using a frequency-dependent multiplier. Finally we state some open problems for future work.

TABLE I
NOTATION

Symbol	Meaning
$\Re$	real part
$\mathbb{R}$	set of all real numbers
$\mathcal{L}_2^m[0,\infty)$	space of square summable functions
$\mathcal{H}_\infty$	space of bounded, analytic functions for $s > 0$
I_q	$q \times q$ identity matrix
$A(s)^*$	conjugate transpose $A(-s)^T$
$\succ$	positive definite

II. INTEGRAL QUADRATIC CONSTRAINTS

In general, IQCs provide a characterization of the structure of a given operator and the relations between signals of a system component.

In a system theoretical context the following IQC is widely used. Two signals $w \in \mathcal{L}_2^m[0,\infty)$ and $v \in \mathcal{L}_2^l[0,\infty)$ satisfy the IQC defined by the multiplier $\Pi(\mathrm{j}\omega) = \Pi(\mathrm{j}\omega)^*$, if

$$\int_\infty^\infty \begin{bmatrix}\hat{v}(\mathrm{j}\omega)\\ \hat{w}(\mathrm{j}\omega)\end{bmatrix}^* \Pi(\mathrm{j}\omega) \begin{bmatrix}\hat{v}(\mathrm{j}\omega)\\ \hat{w}(\mathrm{j}\omega)\end{bmatrix} d\omega \geq 0, \tag{1}$$

holds for the Fourier transforms of the signals. Consider the bounded and causal operator Δ defined on the extended space of square integrable functions on finite intervals. If the signal w is the output of Δ, i.e. $w = \Delta(v)$, then the operator Δ is said to *satisfy the IQC defined by* Π, if (1) holds for all signals $v \in \mathcal{L}_2^l[0,\infty)$. Thus the multiplier Π gives a characterization of the operator Δ. The operator Δ represents the nonlinear, time-varying, uncertain or delayed components of a system. For example, let Δ be a saturation $w = sat(v)$ then the multiplier

$$\Pi = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

defines an IQC which holds for this nonlinear operator. Note, that this multiplier is not necessarily unique. Actually there might be an infinite set of valid multipliers. See [9] for a summarizing list of important IQCs, and [7] for a detailed treatment.

We consider the general configuration of a causal and bounded linear time-invariant (LTI) transfer function $G(s)$, and a bounded and causal operator Δ which are interconnected in a feedback manner

$$\begin{aligned} v &= Gw + e_2, \\ w &= \Delta(v) + e_1, \end{aligned}$$

where e_1 and e_2 are exogeneous inputs. See Fig. 1.

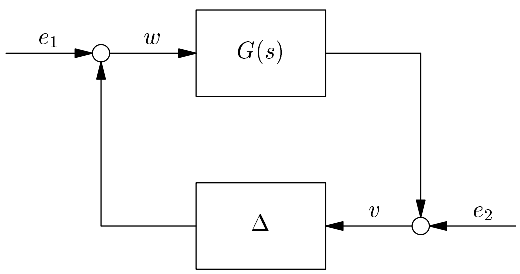

Fig. 1. General IQC feedback structure

The stability of this system can be verified using the following theorem.

Theorem II.1 ([9]): *Let $G(s) \in \mathcal{RH}_\infty^{l\times m}$, and let Δ be a bounded causal operator. Assume that*

(i) *for $\tau \in [0;1]$, the interconnection $(G, \tau\Delta)$ is well-posed,*

(ii) *for $\tau \in [0;1]$, the IQC defined by Π is satisfied by $\tau\Delta$,*

(iii) *there exists $\varepsilon > 0$ such that*

$$\begin{bmatrix} G(\mathrm{j}\omega) \\ I \end{bmatrix}^* \Pi(\mathrm{j}\omega) \begin{bmatrix} G(\mathrm{j}\omega) \\ I \end{bmatrix} \leq -\varepsilon I, \quad \forall \omega \in \mathbb{R}. \quad (2)$$

Then, the feedback interconnection of $G(s)$ and Δ is stable. □

Note that the considered feedback interconnection uses positive feedback only.

III. Parameter Space Mapping

In general, the parameter space approach maps a given specification, e.g. a permissible eigenvalue region, into a space of uncertain parameters $q \in \mathbb{R}^p$. Usually the specification is mapped into a parameter plane because this leads to understandable and powerful graphical results. Moreover, since we can map several specifications consecutively, this approach actually allows multi-objective analysis and synthesis of control systems.

In other words we are interested in the set of all parameters $\mathcal{P}_{good}$ which fulfill a given specification. The boundary of this "good" set is characterized by the equality case of the specification. Mathematically this good set is given by a mapping equation. We will show, how to get mapping equations for IQCs in the sequel.

A. Uncertain parameter systems

Consider an uncertain LTI system $G(s, q) \in \mathcal{RH}_\infty^{l\times m}$ which is interconnected to a bounded causal operator Δ. The parameters $q \in \mathbb{R}^p$ are uncertain but constant parameters with possibly known range. The operator Δ might represent various types of uncertainties, including constant uncertain parameters not included in q. Let Π be a constant multiplier which characterizes the uncertainty Δ with partition

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix}. \quad (3)$$

Conditions (i) and (ii) in Theorem II.1 are parameter-independent. Hence, the parameter-dependent stability condition (iii) which can be written as

$$G(\mathrm{j}\omega)^*\Pi_{11}G(\mathrm{j}\omega) + \Pi_{22} + \Pi_{12}^T G(\mathrm{j}\omega) + G(\mathrm{j}\omega)^*\Pi_{12} \leq -\varepsilon I, \quad (4)$$

has to be fulfilled by parameters in $\mathcal{P}_{good}$.

The avenue to determine mapping equations will be the application of the Kalman-Yakubovich-Popov (KYP) lemma. Previously known results on how to map specifications expressible as algebraic Riccati equations (ARE) are then easily applied to IQC specifications.

B. Kalman-Yakubovich-Popov Lemma

The KYP lemma relates very different mathematical descriptions of control theoretical properties to each other. In particular, it shows close connections between frequency-dependent inequalities, AREs and Linear Matrix Inequalities.

The KYP lemma is also known as the "positive real lemma". Nowadays a very popular application of the KYP lemma is to derive LMIs for frequency domain inequalities, since efficient numerical algorithms for the solution of LMI problems exist.

Theorem III.1 (*Kalman, Yakubovich, Popov*): *Let (A, B) be a given pair of matrices that is stabilizable and A has no eigenvalues on the imaginary axis. Then the following statements are equivalent.*

(i) *$R \succ 0$ and the ARE*

$$Q + XA + A^TX - (XB + S)R^{-1}(XB + S)^T = \mathbf{0} \quad (5)$$

has a stabilizing solution $X = X^T$,

(ii) *the LMI with unknown X*

$$\begin{bmatrix} XA + A^TX & XB \\ B^TX & \mathbf{0} \end{bmatrix} + \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ 0 \quad (6)$$

has a solution $X = X^T$,

(*iii*) *for a spectral factorization the condition*

$$\begin{bmatrix}(\mathrm{j}\omega I - A)^{-1}B\\ I\end{bmatrix}^* \begin{bmatrix}Q & S\\ S^T & R\end{bmatrix}\begin{bmatrix}(\mathrm{j}\omega I - A)^{-1}B\\ I\end{bmatrix} > 0 \tag{7}$$

holds $\forall\omega \in [0;\infty]$,

(*iv*) $R \succ 0$ *and the Hamiltonian matrix*

$$H = \begin{bmatrix}A - BR^{-1}S^T & BR^{-1}B^T\\ Q - SR^{-1}S^T & -A^T + SR^{-1}B^T\end{bmatrix} \tag{8}$$

has no eigenvalues on the imaginary axis.

Proof. See, for example, [11] or [12]. □

In order to apply the KYP lemma we will need the following remark.

Remark III.1 The KYP lemma can be extended such that the spectral factorization condition applies to the case were a transfer function $G(s) = C(sI - A)^{-1}B + D$ appears in the outer factors similar to the IQC condition (2):

$$\begin{bmatrix}G(\mathrm{j}\omega)\\ I\end{bmatrix}^* \begin{bmatrix}Q & S\\ S^T & R\end{bmatrix}\begin{bmatrix}G(\mathrm{j}\omega)\\ I\end{bmatrix} = \begin{bmatrix}(\mathrm{j}\omega I - A)^{-1}B\\ I\end{bmatrix}^* M \begin{bmatrix}(\mathrm{j}\omega I - A)^{-1}B\\ I\end{bmatrix}, \tag{9}$$

where

$$M = \begin{bmatrix}C & D\\ \boldsymbol{0} & I\end{bmatrix}^T \begin{bmatrix}Q & S\\ S^T & R\end{bmatrix}\begin{bmatrix}C & D\\ \boldsymbol{0} & I\end{bmatrix}.$$

□

C. Mapping Equations

We are now ready to derive the main result, the mapping equations for IQC conditions. Let $G(s,q)$ have a state-space realization $A(q), B(q), C(q), D(q)$, i.e.

$$G(s,q) = C(q)(sI - A(q))^{-1}B(q) + D(q).$$

We will not express the parametric dependence of matrices in the remainder for notational convenience.

Using Remark III.1 the basic IQC condition (2) in Theorem II.1 for a constant multiplier Π with partition (3) can be transformed into the condition

$$\begin{bmatrix}(\mathrm{j}\omega I - A)^{-1}B\\ I\end{bmatrix}^* M \begin{bmatrix}(\mathrm{j}\omega I - A)^{-1}B\\ I\end{bmatrix} \leq -\varepsilon I, \tag{10}$$

where the multiplier is transformed into

$$M = \begin{bmatrix}C^T\Pi_{11}C & C^T(\Pi_{12}+\Pi_{11}D)\\ (D^T\Pi_{11}+\Pi_{12}^T)C & D^T\Pi_{11}D+\Pi_{12}^TD+D^T\Pi_{12}+\Pi_{22}\end{bmatrix}.$$

Since we are interested in mapping equations describing the boundaries of a parameter set $\mathcal{P}_{good}$, we consider marginal satisfaction of (10), i.e. $\varepsilon = 0$.

Now, use statements (*iii*) and (*iv*) of the KYP lemma to get the equivalent condition that the Hamiltonian matrix

$$H = \begin{bmatrix}A & \boldsymbol{0}\\ C^T\Pi_{11}C & -A^T\end{bmatrix} - \begin{bmatrix}B\\ C^T(\Pi_{12}+\Pi_{11}D)\end{bmatrix}\tilde{\Pi}_{22}^{-1}\begin{bmatrix}C^T(\Pi_{12}+\Pi_{11}D)\\ -B\end{bmatrix}^T \tag{11}$$

has no eigenvalues on the imaginary axis, where we let $\tilde{\Pi}_{22} = \Pi_{22} + D^T\Pi_{12} + \Pi_{12}^TD + D^T\Pi_{11}D$.

We have now formulated the adherence of a given IQC specification as the non-existence of pure imaginary eigenvalues of an associated Hamiltonian matrix. Using results from [8] we can extend this equivalence to systems with analytic dependence on uncertain parameters.

Actually, given a specific parameter $q^* \in \mathbb{R}^p$ for which a maximal, hermitian solution $X_+(q^*)$ of (5) exists, the Hamiltonian matrix (11) has no pure imaginary eigenvalues. We can extend this property as long as the number of eigenvalues on the imaginary axis is constant. In other words, having found a parameter for which a specification described by an IQC holds, the same specification holds as long as the number of imaginary eigenvalues of the associated Hamiltonian (11) is zero and does not change. Hence the boundary of the subspace for which the desired specification holds is given by all parameters for which the number of pure imaginary eigenvalues of (11) changes. A new pair of imaginary eigenvalues of (11) only arises if either two complex eigenvalue pairs become a double eigenvalue pair on the imaginary axis or if a double real pair becomes a pure imaginary pair. Note: Another possibility is a drop in the rank of H, which corresponds to eigenvalues which go through infinity.

Let us first discuss the appearance of pure imaginary eigenvalues through a double pair on the imaginary axis. The matrix $H(q)$ has a double eigenvalue at $\lambda = \mathrm{j}\omega$ if and only if

$$\begin{aligned}|\mathrm{j}\omega I - H(q)| &= 0,\\ \tfrac{\partial}{\partial\omega}|\mathrm{j}\omega I - H(q)| &= 0.\end{aligned} \tag{12}$$

A necessary condition for a real eigenvalue pair which becomes a pure imaginary pair through parameter changes is

$$|\mathrm{j}\omega I - H(q)|_{\omega=0} = |H(q)| = 0. \tag{13}$$

Additionally the opposite end of the imaginary axis has to be considered

$$|\mathrm{j}\omega I - H(q)|_{\omega=\infty}. \tag{14}$$

Equation (14) is just the coefficient of the term with the highest degree in ω of $|\mathrm{j}\omega I - H|$, observing that this determinant is an even function in ω.

Equation (13) is not sufficient, since it determines all parameters for which (11) has a pair of eigenvalues at the origin. This includes real pairs which are just interchanging on the real axis. To get sufficiency we have to check for all parameters satisfying (13), if there are only real eigenvalues.

The mapping equations (12), (13), and (14) have a similar structure like the familiar equations for eigenvalue loci specifications. Actually (12), (13), and (14) can be interpreted as the complex, real, and infinite root boundary, respectively.

IV. Frequency-dependent Multipliers

Consider the case when the multiplier Π is frequency-dependent, i.e. $\Pi = \Pi(\omega)$. A particular example is the strong result by Zames and Falb [13] for nonlinearities. Put into the IQC framework, an odd nonlinear operator, e.g. saturation, satisfies the IQC defined by

$$\Pi(\omega) = \begin{bmatrix} 0 & 1 + H(\mathrm{j}\omega) \\ 1 + H(\mathrm{j}\omega)^* & -2 - H(\mathrm{j}\omega) - H(\mathrm{j}\omega)^* \end{bmatrix},$$

where $H(s)$ has an impulse response with $\mathcal{L}_1$ norm less than one.

Following [9], any bounded rational multiplier $\Pi(\mathrm{j}\omega)$ can be factorized as

$$\Pi(\mathrm{j}\omega) = \Psi(\mathrm{j}\omega)^* \Pi_s \Psi(\mathrm{j}\omega), \tag{15}$$

where $\Psi(\mathrm{j}\omega)$ absorbs all dynamics of $\Pi(\mathrm{j}\omega)$ and Π_s is a static matrix. Hence the general IQC condition can be written as a spectral factorization condition

$$\begin{bmatrix} (\mathrm{j}\omega I - \tilde{A})^{-1}\tilde{B} \\ I \end{bmatrix}^* U^T \Pi_s U \begin{bmatrix} (\mathrm{j}\omega I - \tilde{A})^{-1}\tilde{B} \\ I \end{bmatrix} > 0, \tag{16}$$

where $(\tilde{A}, \tilde{B})$ represent an augmented system composed of both the LTI system dynamics and the multiplier dynamics, and U is a static matrix which takes the C, D matrices of $G(s) = C(sI - A)^{-1}B + D$ into account. See [7] for the actual transformation equations.

Hence frequency-dependent bounded rational multipliers $\Pi(\mathrm{j}\omega)$ can be mapped into parameter space using basic matrix transformations and the results from the previous section.

V. LMI Optimization

For a system with fixed parameters, all multipliers considered so far led to a simple stability test which could be evaluated by computing the eigenvalues of a Hamiltonian matrix (8). For systems with uncertain parameters $q \in \mathbb{R}^p$, we showed how to map an IQC condition into a parameter plane. But the uniqueness of a multiplier is in general not given.

While the main idea behind the IQC framework is to find a suitable multiplier for an uncertainty, for many uncertainties a set of possible multipliers exists. Especially for nonlinearities and time-delay systems there is an enormous list of publications involving different multipliers. See [9] for some references. Depending on the considered LTI system one or the other multiplier might prove advantageous and yield less conservative results.

For example consider the following multiplier from [6] for a system

$$\dot{x}(t) = (A + U\Delta V)x(t)$$

with slowly time-varying uncertainty Δ and known rate bounds

$$\Pi(\mathrm{j}\omega) = \begin{bmatrix} Z & Y^T - \mathrm{j}\omega\Lambda^T \\ Y + \mathrm{j}\omega\Lambda & -X \end{bmatrix}.$$

Jönsson [6] derives a set of LMI conditions to check stability involving $X = X^T, Y$ and $Z = Z^T$. These matrices can be easily obtained solving a convex optimization problem. The result of the optimization is not only a binary stability check, but also an optimal multiplier $\Pi(\mathrm{j}\omega)$.

There are two different possibilities to exploit the degrees of freedom in the multiplier formulation during the mapping process.

One approach would be to use a limited set of parameter points (q_1, q_2) for which we obtain optimal multipliers and subsequently determine the set of good parameters $\mathcal{P}_{good}$ for each individual multiplier. The actual overall set of uncertain parameters which fulfill the specification is than given as the union of all individual good sets.

The second approach could be denoted as *adaptive multiplier mapping*. Hereby we obtain successive multipliers as we actually generate and move along the boundary of the set $\mathcal{P}_{good}$. Thus we adaptively correct the optimal multiplier on the way as we generate the boundary by solving an underlying optimization problem.

While the first approach needs to solve a limited and predefined number of optimization problems, the *adaptive multiplier mapping* requires a possibly large number of optimization which is not known a priori. Nevertheless the second approach gives the actual set $\mathcal{P}_{good}$ directly and there is no need to determine the union of individual sets. Furthermore if the actual mapping is expensive, it might be favourable to use a single adaptive mapping run.

Note that conditions (i) and (ii) of Theorem II.1 have to be evaluated separately if the multiplier Π depends on a parameter of the current parameter plane, e.g. q_1 or q_2.

VI. Example

Consider the following nonlinear control example depicted in Fig. 2 with a PI controller, a deadzone which models the actuator and a linear plant $G(s)$. The transfer function of the controller is given by $G_{PI}(s) = k_1 + \frac{k_2}{s}$. The plant is given by

$$G(s) = \frac{qs + 1}{s^2 + s + 1},$$

where $q \in \mathbb{R}$ is an uncertain parameter.

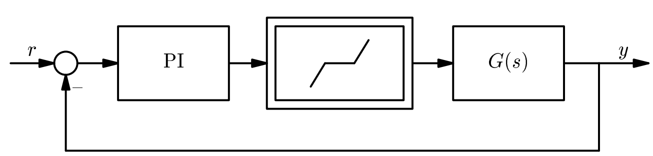

Fig. 2. Deadzone PI example

We aim at analyzing the robustness of the system with respect to variations in q. Furthermore we want to tune the controller such that robustness to parameter variations is achieved.

The given feedback interconnection is called critical since the worst case linearization is at best neutrally stable. Note that the transfer function $G_{PI}(s)G(s)$ is unbounded which prevents the application of standard stability criteria for nonlinear systems which require bounded operators.

We use the Zames-Falb IQC derived in [5], where it was shown that an integrator and a sector bounded nonlinearity can be encapsulated in a bounded operator that satisfies the following IQC

$$\Pi(\mathrm{j}\omega) = \begin{bmatrix} 0 & 1 - H(\mathrm{j}\omega)^* \\ 1 - H(\mathrm{j}\omega) & -\frac{2}{k}\Re(1 - H(\mathrm{j}\omega) - kF(\mathrm{j}\omega)) \end{bmatrix}, \tag{17}$$

where

$$F(s) = \frac{H(s) - H(0)}{s},$$

where $H(s)$ is a stable transfer function with l_1 norm less than one, and the parameter k equals the static gain of the open loop linear part $k = k_2 G(0)$. This IQC corresponds to Zames and Falbs IQC for slope restricted nonlinearities [13].

Let the integral gain $k_2 = 2/5$ and $H(s) = 1/(s+1)$. For our particular example the parameter k equals the proportional gain $k = k_2$. We map the stability condition into the (k_1, q) parameter plane. This allows to evaluate robustness with respect to q, while we can select the controller gain k_1 to maximize the robustness.

Since the multiplier $\Pi(\mathrm{j}\omega)$ in (17) is frequency-dependent, we use the method described in Section IV to reformulate the IQC stability problem with a constant multiplier. For this particular example the augmented system $(\tilde{A}, \tilde{B})$ in (16) is of forth order, the corresponding mapping equations are of eighth order.

The resulting stability boundaries are shown in Fig. 3. The set of stable parameters $\mathcal{P}_{good}$ contains the origin.

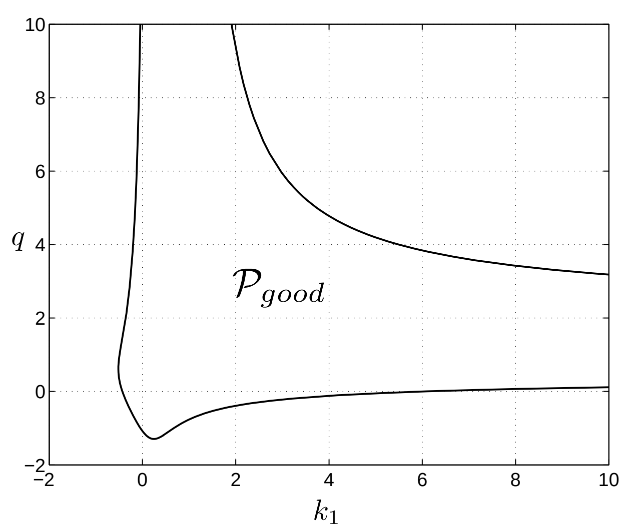

Fig. 3. Stability boundaries

To evaluate the conservativeness of the results numerical simulations were performed using the nonlinear system. The simulations showed that the upper line shown in Fig. 3 is far from the real boundary, while the lower boundary is very close to the actual boundary.

Fig. 4 not only shows the nonlinear boundaries (solid) but also the stability boundaries for a linear system (dashed) which lacks the nonlinear deadzone actuator. The results show that the nonlinear stability region is only slightly smaller than the linear counterpart. Although the mathematical description of the curves is different.

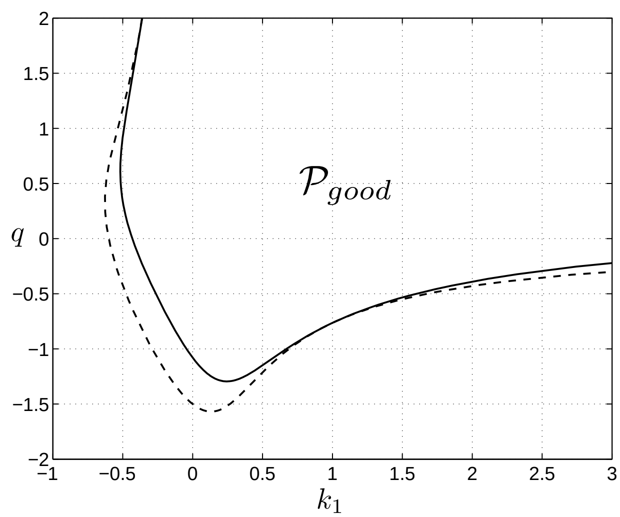

Fig. 4. Comparison of linear and nonlinear system

VII. CONCLUSION AND FUTURE WORK

A. Conclusion

The objective of this paper was to determine previously unknown mapping equations for IQC-based stability tests. Using the results in this paper we can draw from the vast number of available IQCs and incorporate them into the parameter space approach. Using standard parameter space methods allows then to include an even larger list of specifications into control system analysis and design.

B. Future Work

In order to exploit the various degrees of freedom inherent in various IQCs, the implementation of the described underlying LMI optimization should be investigated.

REFERENCES

[1] J. Ackermann, P. Blue, T. Bünte, L. Güvenç, D. Kaesbauer, M. Kordt, M. Muhler, and D. Odenthal, *Robust Control: The Parameter Space Approach*, Springer–Verlag, London, 2002.

[2] J. Ackermann and T. Bünte, *Robust Prevention of Limit Cycles for Robustly Decoupled Car Steering Dynamics*, Kybernetika, 35 (1999), pp. 105–116.

[3] V. Besson and A. T. Shenton, *Interactive Control System Design by a mixed H_∞-Parameter Space Method*, IEEE Trans. on Automatic Control, 42 (1997), pp. 946–955.

[4] S. HARA, T. KIMURA, AND R. KONDO, *H_∞ control system design by a parameter space approach*, in Proc. 10th Int. Symposium on Math. Theory of Networks and Systems, Kobe, Japan, 1991, pp. 287–292.

[5] U. JÖNSSON AND A. MEGRETSKI, *The Zames-Falb IQC for Critically Stable Systems*, Technical Report, LIDS-P 2405, MIT, Cambridge, MA, 1997.

[6] U. JÖNSSON, *A Popov Criterion for Systems with Slowly Time-Varying Parameters*, IEEE Trans. on Automatic Control, 44 (1999), pp. 844–846.

[7] U. JÖNSSON, *Lectures on Input-Output Stability and Integral Quadratic Constraints*. Lecture Notes, Royal Institute of Technology, Stockholm, Sweden, 2001.

[8] P. LANCASTER AND L. RODMAN, *Algebraic Riccati equations*, Oxford Science Publications, 1995.

[9] A. MEGRETSKI AND A. RANTZER, *System analysis via integral quadratic constraints*, IEEE Trans. on Automatic Control, 42 (1997), pp. 819–830.

[10] M. MUHLER, *Mapping MIMO Control Specifications into Parameter Space*, in Proc. IEEE Conf. on Decision and Control, Las Vegas, NV, 2002, pp. 4527–4532.

[11] A. RANTZER, *On the Kalman-Yakubovich-Popov lemma*, Systems and Control Letters, 28 (1996), pp. 7–10.

[12] J. C. WILLEMS, *Least squares stationary optimal control and the algebraic Riccati equation*, IEEE Trans. on Automatic Control, 16 (1971), pp. 621–634.

[13] G. ZAMES AND P. L. FALB, *Stability conditions for systems with monotone and slope-restricted nonlinearities*, SIAM Journal of Control, 6 (1968), pp. 89–108.