Nonlinear stability via sum of squares programming

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Extended abstract

In this talk we present a tutorial introduction to sum of squares (SOS) techniques, emphasizing their applications to nonlinear stability questions via Lyapunov functions [1], [2].

Sum of squares and multivariate polynomials: A multivariate polynomial is a *sum of squares* (SOS) if it can be written as a sum of squares of other polynomials, i.e.,

$$p(x) = \sum_i q_i^2(x), \quad q_i(x) \in \mathbb{R}[x].$$

If $p(x)$ is SOS then clearly $p(x) \geq 0$ for all x.

SOS programs as SDPs: Sum of squares conditions can be written as semidefinite programs (SDPs). The reason is that a polynomial $p(x)$ is SOS if and only if $p(x) = z^T Q z$, where z is a vector of monomials in the x_i variables and $Q \succeq 0$. In other words, every SOS polynomial can be written as a quadratic form in a set of monomials, with the corresponding matrix being positive semidefinite. The vector of monomials z depends in general on the degree and sparsity pattern of $p(x)$.

In the last few years there have been some very interesting new developments surrounding sums of squares, where several independent approaches have produced a wide array of results linking foundational questions in algebra with computational possibilities arising from convex optimization.

SOS and Lyapunov functions: The possibility of reformulating conditions for a polynomial to be a sum of squares as an SDP is very useful, since we can use the SOS property in a control context as a convenient sufficient condition for polynomial nonnegativity. This approach allows one to search over affinely parametrized polynomial or rational Lyapunov functions for systems with dynamics of the form

$$\dot{x}_i(t) = f_i(x(t)) \qquad \text{for all } i = 1, \ldots, n$$

where the functions f_i are polynomials or rational functions. Then the condition that the Lyapunov function be positive, and that its Lie derivative be negative, are both directly imposed as sum of squares constraints in terms of the coefficients of the Lyapunov function.

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Example 1: Consider the following system, suggested by M. Krstić.

$$\begin{aligned}\dot{x} &= -x + (1+x)y \\ \dot{y} &= -(1+x)x.\end{aligned}$$

Using SOSTOOLS [3] we easily find a quartic polynomial Lyapunov function, which after rounding is given by

$$V(x,y) = 6x^2 - 2xy + 8y^2 - 2y^3 + 3x^4 + 6x^2y^2 + 3y^4.$$

It can be readily verified that both $V(x,y)$ and $(-\dot{V}(x,y))$ are SOS, since

$$V = \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 & 0 & 0 \\ -1 & 8 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & -1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix},$$

$$-\dot{V} = \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}^T \begin{bmatrix} 10 & 1 & -1 & 1 \\ 1 & 2 & 1 & -2 \\ -1 & 1 & 12 & 0 \\ 1 & -2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix},$$

and the matrices in the expression above are positive definite.

The techniques introduced in [2], [1] have been significantly extended by several researchers. Among them, we mention the results in [4], [5], that have enabled the application of the methods to time delay and hybrid systems, as well as the related work in [6], [7].

References

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