Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Asymptotic Properties of an Output-Feedback Suboptimal Control Scheme for Stochastic Bilinear Systems

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***Abstract*— The asymptotic properties of the filtering section in a feedback-control scheme for the stochastic regulation problem of noisy-observed linear systems with state-dependent noise, are studied in the present work. The feedback-control scheme consists in the suboptimal quadratic controller for which we proved a separation property and gave the complete set of equation in a previous paper. In this paper we focus our attention on the filtering part of the control scheme and prove that, under some (reasonable) conditions involving the system to be controlled, the set of matrix differential equations describing the evolution of the covariances of the system state, state-estimate, and error-estimate, have a limiting solution that can be used to implement the overall control scheme.**

I. INTRODUCTION

The problem of "extending", in some sense, the LQG control-scheme ([1], [2]) to wider classes of nonlinear systems and cost functionals has been widely studied in the literature. In the pioneeristic work [3], Mortensen proved that the original incomplete-information nonlinear stochastic optimal control problem is equivalent to a complete-information but infinite-dimensional one, where the stochastic partial differential equation giving the state conditional probability density replaces the original (finite-dimensional) system equations. The reader is referred to [4], [5] and references therein for details about this control methodology, and to [6], [7], [8] for the problem of finding the evolution equation of the conditional density function of a general diffusion process. Although in these papers the setting of the optimal control problem is very general, when an application of the resulting control scheme is required, one is led to a typically huge computational effort in order to solve the partial differential equations giving the conditional density. At this purpose a research effort has been recently devoted in searching general conditions under which the conditional statistics of a nonlinear system are given by a *finite-dimensional* filter (see for instance [9]). Moreover we point out the results in [4], [5] where it is shown that, under suitable assumptions, finite-dimensional sufficient statistics are available and allow to reduce the original (incomplete information) optimal control problem to a *finite*-dimensional and complete-information one.

Since, *in general*, finite-dimensional filters giving the *optimal estimate* of the state process of a nonlinear system do not exist (or, they are up to now unknown), it make sense to attack the control problem from a different point of view which is appealing in that it offers the opportunity of saving most of the computational effort required to implement a complete calculation of the conditional statistics. This consists essentially in relaxing the requirement for the controller to be optimal among *all* the observation functions. We are thus led to another class of methods we will call *suboptimal* control methods. The papers [10], [11] represent a first attempt of exploiting this suboptimal methodology in order to give a meaningful solution for regulation problems of discrete-time non Gaussian linear systems. In particular, in [11], using the results of [12] concerning the *polynomial filtering* for linear discrete-time systems, a general *polynomial* controller was found. The meaningful class of continuous-time linear systems with state and control dependent noise is considered in [13], [14]. This is a class of systems occurring in engineering applications: control-dependent noise occurs, for instance, in modeling thrust malalignment in a gas-jet thrusting system for the attitude control of a satellite, whereas state-dependent noise occurs in aerospace systems; one example is the momentum exchange method for regulating the angular procession of a rotating spacecraft (see [14] and reference therein for more details). For this class of systems Wohnam considered a quadratic index and a general class of nonlinear state-feedback controllers (complete information case). Then "optimality lemma" was proven giving sufficient conditions that allowed Wonham to find the optimal state-feedback controller, resulting in a linear map of the current state which is computable by solving a backward Riccati-like equation. In a later work [15] Wonham proved the existence and uniquenes of the solution for this kind of matrix differential equation. However, the "optimality lemma" and the subsequent result hold only for state-feedbacks, that is in the complete-information case. In [14] a different technique than the more mathematically sophisticated one used by Wonham in [13] is used. However, since McLane searches directly the *linear*-state-feedback controller, he is able to derive in a much more direct way the same result of Wonham concerning the state-feedback controller. In the same paper, the output-feedback case is treated for a noise-free observation equation.

The article [16] is a first attempt to attack the problem of building up a general output-feedback polynomial controller for the class of bilinear systems described by Ito stochastic differential equations, and in the case of a general *noisy* observation equation (incomplete information case). In this paper the *quadratic* feedback-controller was found. The resulting overall control-scheme results indeed to be similar to the LQG one, in that it results by the composition of a control part, that is a linear map of the optimal-quadratic state-estimate, and of a *quadratic filter* (producing the quadratic state-estimate, see [17] for the general polynomial filter in the open-loop case). The resulting control-scheme is given as a set of differential equations in closed form that can be implemented provided that the probability distribution of the initial state is known. As a matter of fact, a Riccati-like differential equation is derived, giving the evolution of the error-covariance and state-covariance matrices (both indeed are necessary in order to compute the filter-gain, differently than in the LQG case). However, a question remained unsolved over what the solution of this set of differential matrix-valued equations converges to, as time goes to infinity. Indeed, this is a central question from an application point of view, in that it allows, possibly, to find a steady-state gain for the filter, and then to implement a steady-state version of the filter. This indeed makes the overall control scheme really implementable.

In the present paper we treat this problem and give sufficient conditions for the stability of the covariance equation. The set of Riccati-like equations involved in the filter are derived in the more easy case of a linear filter and linear feedback control. Indeed, as we will better explain later, since the quadratic filter defined in [16] has the structure of a *linear* filter for a suitable *linear* augmented system there is no loss of generality in considering the linear case.

The paper is organized as follows. In §II notations and symbols used thorough the paper are defined, and the set of Riccati-like equations for the linear filtering are derived. In §III we present a preliminary result, that nevertheless is fundamental in order to get an overall steady-state control system: we show that, under suitable conditions, the control matrix in the controlled system has a limiting value as the time horizon goes to infinity. In §IV the main result is presented, concerning the filtering part of the control system. Finally, in §V some numerical results are presented for a simulated system.

II. SETTING OF THE PROBLEM

First of all we introduce the basic notations and symbols that will be used throughout the paper. $(\Omega,\mathscr{F},P)$ will denote the basic probability triple. $\mathbf{E}\{\cdot\}$ represents the expectation operator. $L^2(\mathscr{E})$, with $\mathscr{E}$ a linear space, denotes the space of all the $\mathscr{E}$-valued square-integrable random variables defined on $(\Omega,\mathscr{F},P)$. Let I be a real interval and $\xi,\eta : I\to L^2(\mathbb{R}^n)$. We shall symbolize with $\mathscr{F}_t^\xi$ the σ-algebra generated by $\{\xi_s;\ s\in I, s\le t\}$. $\mathbf{E}\{\xi_t\}$ will be often denoted by $\mu_\xi(t)$, and $\mathbf{E}\{(\xi_t-\mathbf{E}\{\xi_t\})(\eta_t-\mathbf{E}\{\eta_t\})^T\}$ with $\Psi_{\xi,\eta}(t)$. $\Psi_{\xi,\xi}(t)$ will be shortened as $\Psi_\xi(t)$. Let $\mathscr{S}\subset L^2(\mathscr{E})$ be a closed linear space and $X\in L^2(\mathscr{E})$, then $\mathrm{Proj}\left\{X/\mathscr{S}\right\}$ will symbolize the orthogonal projection of X onto $\mathscr{S}$. Anytime the underlying space is understood we will use the notation $\widehat{X}$. Let us consider the following stochastic system:

$$dX_t = AX_tdt+Hu_tdt+\sum_{k=1}^{q}B^kX_tdW_t^k+GdN_t, \qquad (1)$$

$$dY_t = CX_tdt+dW_t', \qquad (2)$$

with $X_{t_0}=\overline{X}$, where $\overline{X}$ is a second order random variable with known covariance $\Psi_{\overline{X}}=\overline{\Psi}_X$, $Y_{t_0}=0$. W, W' and N_t are, respectively, the standard, mutually independent, q, m, and d-dimensional Brownian motions. Moreover, $t\in I$, $I=[t_0,t_f]\subset\mathbb{R}$, $A\in\mathbb{R}^{n\times n}$, $H\in\mathbb{R}^{n\times p}$, $C\in\mathbb{R}^{m\times n}$, $B^k\in\mathbb{R}^{n\times n}$, $k=1,\dots,q$. The control function $u:\Omega\times I\to\mathbb{R}^p$ is assumed to be adapted to the non-decreasing family $\{\mathscr{F}_t^Y\}_{t\in I}$. We will denote with $\mathscr{L}_t^i(Y)$ the set of $\mathbb{R}^i$-valued linear transformations of $\{Y_s;\ s\in I, s\le t\}$. One has that $\mathscr{L}_t^i(Y)$ is a closed linear subspace of $L^2(\mathbb{R}^i)$ and hence it is well defined $\widehat{X}_t=\mathrm{Proj}\left\{X_t/\mathscr{L}_t^n(Y)\right\}$, that will be referred as the *linear-optimal estimate* of the state X. We shall represent with $\mathscr{L}^i(Y)$ the set of functions $\xi : I\times\Omega\to\mathbb{R}^i$ such that $\xi_t\in\mathscr{L}_t^i(Y)$, for all $t\in I$. The statement of the (finite-horizon) suboptimal linear-feedback control problem is the following:

$$\min_{u\in\mathscr{L}^p(Y)} J(u), \qquad (3)$$

$$J(u)=\frac{1}{2}\mathbf{E}\left\{X_{t_f}^TFX_{t_f}+\int_{t_0}^{t_f}\left(X_t^TQX_t+u_t^TRu_t\right)dt\right\} \qquad (4)$$

where $\forall t$, $Q=Q^T\ge 0$, $R=R^T>0$, and $F=F^T\ge 0$, under the differential constraints represented by system (1), (2). In the present paper we are interested to the asymptotic properties of the Riccati-like equation arising from the filtering problem associated to the feedback control problem (3), (4). Since a quadratic filter (and we guess this is true in general for a polynomial one) is obtained by reducing the original problem to a linear-filtering one for a linear system with wide-sense diffusion terms, it follows that – in view of a steady-state analysis of the Riccati equations involved – the polynomial degree of the estimate is unessential, so in the following we will carry out the analysis for the less cumbersome case of a linear-optimal estimate. Let $\widehat{X}_t$ be the linear-optimal estimate of the state X_t of (1). Then, as shown in [16] for the more general quadratic case, the filter equation have the following form:

$$d\widehat{X}_t = A\widehat{X}_tdt+D(t)\widehat{X}_tdt+P(t)C^T\left(dY_t-C\widehat{X}_tdt\right), \qquad (5)$$

where the matrix $P(t)$ is the error-estimate covariance, $P(t)=\mathbf{E}\{(X_t-\widehat{X}_t)(X_t-\widehat{X}_t)^T\}$. An ordinary matrix differential equation satisfied by $P(t)$ can be readily derived as follows. Substitute eq. (2) in (5), and then subtract from (1). In this way a stochastic differential equation is obtained for the error-estimate $X_t-\widehat{X}_t$. From the definition of $P(t)=\mathbf{E}\{(X_t-\widehat{X}_t)(X_t-\widehat{X}_t)^T\}$, and by the orthogonality principle $\mathbf{E}\{(X_t-\widehat{X}_t)\widehat{X}_t^T\}=0$ (that is $\Psi_{X\widehat{X}}(t)=\Psi_{\widehat{X}}(t)=\Psi_{\widehat{X}X}(t)$), one has

$$P(t)=\Psi_X(t)-\Psi_{\widehat{X}}(t). \qquad (6)$$

Moreover, with standard calculations, the following ordinary differential equation is derived:

$$\begin{aligned}\dot{P}(t) = {} & AP(t)+P(t)A^T+\sum_{k=1}^{q}B^k\left(\Psi_X(t)+\mu\mu^T(t)\right)B^{k^T}\\ & +GG^T-P(t)C^TC(t)P(t),\end{aligned} \qquad (7)$$

with initial condition: $P(t_0) = \overline{\Psi}_X$. Equation (7), endowed with (6), is a Riccati-like equation that does not belongs to the class considered by Wonham in [15] (indeed an additional term, namely $\sum B^k \Psi_{\widehat{X}} B^{k^T}$, is present in (7)). Equation (7) should be endowed with evolution equations for $\Psi_X(t)$ and $\Psi_{\widehat{X}}(t)$, that can be found by considering the aggregate process $Z_t = \begin{bmatrix} X_t \\ \widehat{X}_t \end{bmatrix}$, and by computing $\Psi_Z(t) = \begin{bmatrix} \Psi_X(t) & \Psi_{\widehat{X}}(t) \\ \Psi_{\widehat{X}}(t) & \Psi_{\widehat{X}}(t) \end{bmatrix}$. Thus one directly derives the following equations for $\Psi_X, \Psi_{\widehat{X}}$ and $\mu = \mu_X = \mu_{\widehat{X}}$:

$$\begin{aligned}
\dot{\Psi}_X(t) =& A\Psi_X(t) + \Psi_X(t)A^T \\
&+ D(t)\Psi_{\widehat{X}}(t) + \Psi_{\widehat{X}}(t)D^T(t) \\
&+ \sum_{k=1}^{q} B^k\big(\Psi_X(t) + \mu\mu^T(t)\big)B^{k^T} + GG^T, \qquad (8) \\
\dot{\Psi}_{\widehat{X}}(t) =& (A + D(t))\Psi_{\widehat{X}}(t) + \Psi_{\widehat{X}}(t)(A + D(t))^T \\
&+ P(t)C^T C(t)P(t), \qquad (9) \\
\dot{\mu}(t) =& (A + D(t))\,\mu(t), \qquad (10)
\end{aligned}$$

with the initial conditions $\Psi_X(t_0) = \Psi_{\widehat{X}}(t_0) = \overline{\Psi}_X$, $\mu(t_0) = \mathbf{E}\{\overline{X}\}$, respectively. Now, we are in a position to state the problem. The question is under what conditions the solutions of (8)–(10) converges to a constant matrix as time goes to infinity (from the above and from the general theory, solutions indeed exist unique and are non-negative symmetric matrices for any time). Before doing this we need to answer to a preliminary question concerning the control matrix $D(t)$ appearing in the filter equation (5). Such a matrix is defined as $D(t) = HL^o(t)$, where $L^o(t)$ is the optimal *universal* controller given by (see [16] Theorem 1):

$$L^o(t) = -R^{-1}H^T V(t), \qquad (11)$$

where R is the weight of the control function in the performance index, and $V(t)$ is given by the following backward Riccati-like equation:

$$\begin{aligned}
\dot{V}(t) =& -A^T V(t) - V(t)A - Q - \sum_{k=1}^{q} B^{k^T} V(t)B^k \\
&+ V(t)HR^{-1}H^T V(t) \qquad V(t_f) = F. \qquad (12)
\end{aligned}$$

Now, since we are concerned with a filtering problem over an infinite time-interval, the question we should answer in advance is over what $D(t)$ converges to as the terminal time t_f goes to infinity. As this purpose in the following section we precisely state and give a solution to the infinite-horizon control problem.

III. THE INFINITE HORIZON CONTROL PROBLEM

Let us consider system (1), (2). The following theorem holds.

Theorem 1. *Let us suppose the couple (A,H) is stabilizable, and the matrices B^k, $k = 1,\dots,q$, satisfy the following condition:*

$$\inf_{M \in \mathbb{R}^{p\times n}} \left| \int_0^\infty e^{t(A-HM)^T} \left(\sum_{k=1}^{q} B^{k^T} B^k \right) e^{t(A-HM)} dt \right| < 1. \qquad (13)$$

Moreover, let $Q \in \mathbb{R}^{n\times n}$, $Q \geq 0$, $Q = Q^T$, be such that $\left(Q^{(\frac{1}{2})}, A\right)$ is observable. Then the solution of the infinite-horizon control problem:

$$\min_{u \in \mathscr{L}^p(Y)} J(u), \qquad (14)$$

$$J(u) = \lim_{t_f \to +\infty} \frac{1}{2t_f} \mathbf{E}\left\{ \int_0^{t_f} \left(X_t^T Q X_t + u_t^T R u_t \right) dt \right\}, \qquad (15)$$

is given by $u_t^o = L^o \widehat{X}_t = -R^{-1}H^T V \widehat{X}_t$, where V is the unique symmetric nonnegative matrix solution of the algebraic Riccati equation:

$$A^T V + VA^T + Q + \sum_{k=1}^{q} B^{k^T} V B^k - VHR^{-1}H^T V = 0, \qquad (16)$$

and $\widehat{X}_t$ is given by (5), with $D(t) = D = HL^o$. Moreover, $(A + D)$ is stable. If the filter covariance $P(t)$ satisfies the condition:

$$\lim_{t_f \to +\infty} \frac{1}{2t_f} \int_0^{t_f} P(t)dt = \Phi < +\infty, \qquad (17)$$

then the value of the optimal cost functional is finite and given by

$$J^o = J(u^o) = \frac{1}{2} \sum_{i,j=1}^{n} V_{ij}(GG^T)_{ij} + tr\{L^{o^T} R L^o \Phi\}. \qquad (18)$$

Proof. Let $F(t) > 0$ $\forall t \in [0, +\infty)$ be a family of matrices such that $\lim_{t\to+\infty} F(t) = 0$. Then we can rewrite the cost functional in (15) as:

$$\begin{aligned}
J(u) \;=\; & \lim_{t_f \to +\infty} \frac{1}{2t_f} \mathbf{E}\Bigg\{ X_{t_f}^T F(t_f) X_{t_f} \\
&+ \int_0^{t_f} \left(X_t^T Q X_t + u_t^T R u_t \right) dt \Bigg\}. \qquad (19)
\end{aligned}$$

Now, let us consider the solution $V(t)$ of (12). From [13] we have that, under the hypotheses of stabilizability of (A,H), observability of $\left(Q^{(\frac{1}{2})}, A\right)$, and the condition (13), there exists a matrix $V \geq 0$, $V = V^T$, such that, for any $t \in [0, +\infty)$, $V(t) \to V$ as $t_f \to +\infty$, and moreover V is the unique nonnegative solution of the algebraic Riccati equation (16). Then, defining $L^o(t) = -R^{-1}H^T V(t)$, one has $L^o(t) \to L^o = -R^{-1}H^T V$ as $t_f \to \infty$. Thus, with similar calculations as in [16], the index $J(u)$ in (19) can be rewritten as:

$$\begin{aligned}
J(u) \;=\; & \lim_{t_f \to +\infty} \frac{1}{2t_f} \mathbf{E}\Bigg\{ \int_0^{t_f} \big(u_t - L^o X_t\big)^T R \big(u_t - L^o X_t\big) dt \\
&+ \int_0^{t_f} \sum_{i,j=1}^{n} V_{ij}(GG^T)_{ij} dt + X_0^T V X_0 \Bigg\}, \qquad (20)
\end{aligned}$$

from which, in a similar way as in the proof of Theorem 3.2, we infer that $u_t^o = L^o\widehat{X}_t = -R^{-1}H^TV\widehat{X}_t$. The value of the optimal cost functional is obtained by substituting $u_t^o = L^o\widehat{X}_t$ in (20), and using the property $x^TMx = tr\{Mxx^T\}$ for any $M \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$. One has:

$$
\begin{aligned}
J(u^o) = &\frac{1}{2}\sum_{i,j=1}^{n} V_{ij}(GG^T)_{ij} \\
&+ \lim_{t_f\to+\infty} \frac{1}{2t_f}\mathrm{tr}\Big\{V\mathbf{E}\{X_0X_0^T\} \\
&+ L^{oT}RL^o\int_0^{t_f}\mathbf{E}\{(X_t-\widehat{X}_t)^T(X_t-\widehat{X}_t)\}\Big\},
\end{aligned}
$$

and since $\mathbf{E}\{(X_t-\widehat{X}_t)^T(X_t-\widehat{X}_t)\} = P(t)$, with $P(t)$ satisfying (17), taking the limit results in (18). •

Note that condition (17) simply represents the requirement for the linear optimal filter to be *stable*, that is, it has a bounded error covariance over the filtering interval $[0,+\infty)$. Thus, from theorem 1, we have that, as one can reasonably expect, the cost index has a finite value at the optimum only if the linear optimal filter has bounded error covariance.

IV. ASYMPTOTIC PROPERTIES OF THE COVARIANCE EQUATIONS

Let $\lambda_i(M)$ denote the *i*-th eigenvalue of a matrix M. For the filter dynamic matrix $(A+D) = A-HR^{-1}H^TV$ (which is stable by Theorem 4.1), we define $\lambda_{max} < 0$ as follows:

$$\lambda_{max} \doteq \max_i \{\mathrm{Re}[\lambda_i(A+D)]\}. \tag{21}$$

Moreover, define $\overline{\beta} \in \mathbb{R}^+$ as follows:

$$\overline{\beta} = \inf\Big\{\beta > 0 \;\big/\; \|e^{(A+D)^Tt}\| \le \beta^{\frac{1}{2}}e^{-|\lambda_{max}|t}, \quad \forall t \ge 0\Big\}.$$

Finally, let us denote for short $\mathbf{B}(M) = \sum_{i=1}^q B^kMB^{k^T}$.

Theorem 2. *Let us suppose that the pair* (A,H) *of* (1) *is stabilizable, and condition* (13) *is satisfied. Then, consider the controlled system:*

$$
\begin{aligned}
dX_t &= AX_tdt + D\widehat{X}_tdt + \sum_{k=1}^{q} B^kX_tdW_t^k + GdN_t, \\
dY_t &= CX_tdt + dW_t',
\end{aligned}
$$

$X_{t_0} = \overline{X}$, $Y_{t_0} = 0$, *where* $D = -HR^{-1}H^TV$. *Suppose that the couple* (A^T,C^T) *is stabilizable, and moreover the following conditions hold:*

$$\int_0^\infty \overline{\beta}\; e^{-2|\lambda_{max}|t}\|\mathbf{B}(I)\|dt = \theta < 1, \tag{22}$$

$$\inf_K \left\| \int_0^\infty e^{(A^T-C^TK)^Tt}\;\mathbf{B}(I)\;e^{(A^T-C^TK)t}dt \right\| < 1, \tag{23}$$

than the matrix functions $P(t), \Psi_X(t), \Psi_{\widehat{X}}(t)$, *solution of* (6), (8), (9) *(with* $D(t) \equiv D$*), are bounded over* $[0,+\infty)$.

Proof. Denote by $I_n \in \mathbb{R}^{n\times n}$ and $O_{m\times n} \in \mathbb{R}^{m\times n}$ the identity and zero matrices respectively. Let us cast together (8), (9) thus obtaining the following equation for $\Psi_Z(t) \doteq \begin{bmatrix}\Psi_X(t) & \Psi_{\widehat{X}}(t)\\ \Psi_{\widehat{X}}(t) & \Psi_{\widehat{X}}(t)\end{bmatrix}$:

$$\dot{\Psi}_Z(t) = \Phi(\Psi_Z(t),K(t)), \qquad \Psi_Z(t_0) = \overline{\Psi}_Z \tag{24}$$

where, denoting $\widetilde{\Theta} \doteq \Theta - \chi^T\widetilde{K}$:

$$
\begin{aligned}
\Phi(\Psi_Z(t),K(t)) &\doteq \widetilde{\Theta}^T\Psi_Z(t) + \Psi_Z(t)\widetilde{\Theta} \\
&\quad + \mathbf{B}(\Psi_Z(t)) + \overline{GG}^T + 2K^TK(t), \\
\mathbf{B}(\Psi_Z(t)) &= \sum_{k=1}^{q}\overline{B}^k\big(\Psi_Z(t) + \mu_Z\mu_Z^T(t)\big)\overline{B}^{kT},
\end{aligned}
$$

$$
\Theta = \begin{bmatrix} A^T & O_n \\ D^T & (A+D)^T \end{bmatrix}, \quad \chi = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix},
$$

$$
K(t) = \tfrac{1}{2}\begin{bmatrix} O_n & -CP(t) \\ O_n & CP(t) \end{bmatrix}, \overline{B}^k = \begin{bmatrix} B^k & O_n \\ O_n & O_n \end{bmatrix}, \overline{G} = \begin{bmatrix} G \\ O_n \end{bmatrix}
$$

with $O_n \doteq O_{n\times n}$. Let $L_1 = \begin{bmatrix} O_n & O_n \\ -I_n & I_n \end{bmatrix}$. By (6) one has

$$K(t) = \frac{1}{2}\chi\Psi_Z(t)L_1^T. \tag{25}$$

Multiply (24) for L_1 and L_1^T (left and right respectively):

$$L_1\dot{\Psi}_Z(t)L_1^T = \Phi'(\Psi_Z(t),K(t)), \tag{26}$$

where $\Phi'(\Psi_Z(t),K(t)) \doteq L_1\Phi(\Psi_Z(t),K(t))L_1^T$. For any matrix $\widetilde{K}(t) \in \mathbb{R}^{2m\times 2n}$ one has:

$$
\begin{aligned}
\Phi'(\Psi_Z(t),K(t)) &= L_1\widetilde{\Theta}^T\Psi_Z(t)L_1^T + L_1\Psi_Z(t)\widetilde{\Theta}L_1^T \\
&+ L_1\big(\mathbf{B}(\Psi_Z(t)) + \overline{GG}^T\big)L_1^T + 2L_1\widetilde{K}^T\widetilde{K}(t)L_1^T \\
&- 2L_1\big(\widetilde{K}(t)-K(t)\big)^T\big(\widetilde{K}(t)-K(t)\big)L_1^T,
\end{aligned}
$$

where the following identity has been exploited:

$$L_1K^TK(t)L_1^T = \frac{1}{2}L_1K(t)^T\chi\Psi_Z(t)L_1^T,$$

which is easily derived from (25) and taking into account that L_1 is nihilpotent ($L_1L_1^T = L_1$). Hence, $\forall\widetilde{K}(t) \in \mathbb{R}^{2m\times 2n}$,

$$
\begin{aligned}
\Phi'(\Psi_Z(t),K(t)) &= \Phi'(\Psi_Z(t),\widetilde{K}(t)) \\
&\quad - 2L_1\big(\widetilde{K}(t)-K(t)\big)^T\big(\widetilde{K}(t)-K(t)\big)L_1^T.
\end{aligned} \tag{27}
$$

Let us choose $\widetilde{K}(t) \equiv \widetilde{K} = \begin{bmatrix} \widetilde{K}_1 & O_{m\times n} \\ O_{m\times n} & O_{m\times n} \end{bmatrix}$, with $\widetilde{K}_1 \in \mathbb{R}^{m\times n}, \widetilde{K}_1 \neq 0$, such that $(A^T - C^T\widetilde{K}_1)$ is stable (which is possible, since (A^T,C^T) is stabilizable by hypothesis). With this choice, and taking into account that $2K(t) = \begin{bmatrix} O_n & -CP(t) \\ O_n & CP(t) \end{bmatrix}$, it results:

$$
\begin{aligned}
&2L_1\big(\widetilde{K}(t)-K(t)\big)^T\big(\widetilde{K}(t)-K(t)\big)L_1^T \\
&= \begin{bmatrix} O_n & O_n \\ O_n & \widetilde{K}_1^T\widetilde{K}_1 + \big(\widetilde{K}_1 + CP(t)\big)^T\big(\widetilde{K}_1 + CP(t)\big) \end{bmatrix}.
\end{aligned}
$$

Hence, by(27), denoting $\Delta \doteq \begin{bmatrix} O_n & O_n \\ O_n & \widetilde{K}_1^T\widetilde{K}_1 \end{bmatrix}$, one has

$$\Phi'(\Psi_Z(t),K(t)) \le \Phi'(\Psi_Z(t),\widetilde{K}(t)) - \Delta. \tag{28}$$

Let us define the subset $U \subset \mathbb{R}^{2n}$:

$$U = \left\{ \zeta \in \mathbb{R}^{2n} \;\Big/\; \zeta^T = [0 \quad \eta^T\,],\ \eta \in \mathbb{R}^n \backslash \ker(\widetilde{K}_1) \right\}.$$

Then $\zeta^T \Delta \zeta > 0,\ \forall \zeta \in U$ and hence (by identifying a symmetric matrix with a quadratic-form) one has:

$$\Phi'(\Psi_Z(t), K(t)) < \Phi'(\Psi_Z(t), \widetilde{K}(t)) \quad \text{on } U. \qquad (29)$$

Relation (29) implies that, recalling that $\Phi' = L_1 \Phi L_1^T$:

$$\Phi(\Psi_Z(t), K(t)) \leq \Phi(\Psi_Z(t), \widetilde{K}(t)) \quad \text{on } U. \qquad (30)$$

Now, let $\widetilde{\Psi}_Z(t)$ the solution of the differential equation: $\dot{\widetilde{\Psi}}_Z(t) = \Phi(\widetilde{\Psi}_Z(t), \widetilde{K}(t))$, by the hypothesis of stabilizability of (A^T, C^T), one can choose $\widetilde{K}_1$ such that, denoting $\lambda'_{max} = \max_i \left\{ \text{Re}[\lambda_i(A^T - C^T \widetilde{K}_1)] \right\}$, it results $|\lambda'_{max}| \geq |\lambda_{max}|$ with λ_{max} defined as in (21). By definition of Φ, $\widetilde{\Psi}_Z(t)$ can be written as:

$$\widetilde{\Psi}_Z(t) = \mathrm{e}^{\widetilde{\Theta}^T t} \widetilde{\Psi}_Z(0) \mathrm{e}^{\widetilde{\Theta} t} + \int_0^t \mathrm{e}^{\widetilde{\Theta}^T (t-s)} \left[\mathbf{B}\big(\widetilde{\Psi}_Z(s)\big) + \overline{GG}^T + 2\widetilde{K}^T \widetilde{K} \right] \mathrm{e}^{\widetilde{\Theta}(t-s)} ds.$$

The above equation can be solved by successive approximation setting $\widetilde{\Psi}_Z^{(0)}(t) \equiv 0$ and defining

$$\widetilde{\Psi}_Z^{(i+1)}(t) = \mathrm{e}^{\widetilde{\Theta}^T t} \widetilde{\Psi}_Z(0) \mathrm{e}^{\widetilde{\Theta} t} + \int_0^t \mathrm{e}^{\widetilde{\Theta}^T (t-s)} \left[\mathbf{B}\big(\widetilde{\Psi}_Z^{(i)}(s)\big) + \overline{GG}^T + 2\widetilde{K}^T \widetilde{K} \right] \mathrm{e}^{\widetilde{\Theta}(t-s)} ds.$$

Thus

$$\widetilde{\Psi}_Z(t) = \lim_{i \to +\infty} \widetilde{\Psi}_Z^{(i)}(t), \qquad (31)$$

moreover

$$\widetilde{\Psi}_Z^{(i+1)}(t) \leq \gamma I + \int_0^t \mathrm{e}^{\widetilde{\Theta}^T (t-s)} \mathbf{B}\big(\widetilde{\Psi}_Z^{(i)}(s)\big) \mathrm{e}^{\widetilde{\Theta}(t-s)} ds, \qquad (32)$$

with

$$\gamma = \sup_{t \geq 0} \left\| \mathrm{e}^{\widetilde{\Theta}^T t} \widetilde{\Psi}_Z(0) \mathrm{e}^{\widetilde{\Theta} t} + \int_0^\infty \mathrm{e}^{\widetilde{\Theta}^T t} \left[\overline{GG}^T + 2\widetilde{K}^T \widetilde{K} \right] \mathrm{e}^{\widetilde{\Theta} t} dt \right\|.$$

By hypothesis (22), since $\theta < 1$, from (32), for $i = 0, 1, 2, \ldots$ one has:

$$\left\| \widetilde{\Psi}_Z^{(i+1)}(t) \right\| \leq \gamma (1 + \theta + \ \ldots\ + \theta^i) \leq \frac{\gamma}{1 - \theta}, \quad \forall t > 0,$$

and hence, by (31), $\|\widetilde{\Psi}_Z(t)\| \leq \gamma/(1-\theta)$, $\forall t > 0$. Using (30), it results (on the subset U):

$$0 = \dot{\widetilde{\Psi}}_Z(t) - \Phi(\widetilde{\Psi}_Z(t), \widetilde{K}(t)) = \dot{\Psi}_Z(t) - \Phi(\Psi_Z(t), K(t)) \geq \dot{\Psi}_Z(t) - \Phi(\Psi_Z(t), \widetilde{K}(t))$$

and then, denoting $Q(t) = \widetilde{\Psi}_Z(t) - \Psi_Z(t)$, one has:

$$\dot{Q}(t) - \Phi(Q(t), \widetilde{K}(t)) \geq 0 \quad \text{on } U,$$

that is, for some symmetric $R \geq 0$:

$$\dot{Q}(t) - \Phi(Q(t), \widetilde{K}(t)) - R = 0 \quad \text{on } U.$$

The above matrix equation has an unique, symmetric solution $Q(t) \geq 0,\ \forall t \in [0, +\infty)$. Hence

$$\widetilde{\Psi}_Z(t) - \Psi_Z(t) \geq 0 \quad \text{on } U,\ \forall t \in [0, +\infty). \qquad (33)$$

By partitioning $\widetilde{\Psi}_Z(t) = \begin{bmatrix} \widetilde{\Psi}_{1,1}(t) & \widetilde{\Psi}_{1,2}(t) \\ \widetilde{\Psi}_{2,1}(t) & \widetilde{\Psi}_{2,2}(t) \end{bmatrix}$, and recalling that $\Psi_Z(t) = \begin{bmatrix} \Psi_X(t) & \Psi_{\widehat{X}}(t) \\ \Psi_{\widehat{X}}(t) & \Psi_{\widehat{X}}(t) \end{bmatrix}$, relation (33) implies that, for any $\zeta \in U$:

$$0 \leq \zeta^T \left(\widetilde{\Psi}_Z(t) - \Psi_Z(t) \right) \zeta = \eta^T \left(\widetilde{\Psi}_{2,2}(t) - \Psi_{\widehat{X}}(t) \right) \eta,$$

$\forall \eta \in \mathbb{R}^n \backslash \ker(\widetilde{K}_1)$ and hence, since $\mathbb{R}^n \backslash \ker(\widetilde{K}_1) \neq \emptyset$ (because $\widetilde{K}_1$ has been chosen non-zero) it results $\widetilde{\Psi}_{2,2}(t) \geq \Psi_{\widehat{X}}(t)$ that is, recalling the boundedness of $\widetilde{\Psi}_Z(t)$, $\Psi_{\widehat{X}}(t)$ results to be bounded over the interval $[0, +\infty)$.

Now, consider the differential equation of $P(t) \doteq \Psi_{\widehat{X}-X}$ given by eq. (7). This is a Riccati equation in the form of Wohnam [15], but it is forward and it has the additional term: $\sum_{k=1}^q B^k \left(\Psi_{\widehat{X}}(t) + \mu \mu^T(t) \right) B^{k^T}$. Nevertheless, since the matrix $(A + D)$ is stable, the matrix $\mu \mu^T(t)$ is bounded on $[0, +\infty)$. Moreover, we have shown before that $\Psi_{\widehat{X}}(t)$ is bounded on $[0, +\infty)$, hence the additional term is bounded. Using this, since the couple (A^T, C^T) is stabilizable by hypothesis, and exploiting condition (23), by successive approximations, in a similar way as before, one can shows that $P(t)$ is bounded over $[0, +\infty)$. Thus, taking into account of (6), the proof is completed. •

Conditions (22), (23), and moreover condition (13) for the existence of a steady-state controller, simply states that **B** is not too large; in other words: the state-dependent noise have a not too large "power". From theorem 2, since $P(t)$ is a function of nonnegative matrices and it is bounded over $[0 \ +\infty)$, it follows that there exists a nonnegative limiting solution, namely $P = \lim_{t \to +\infty} P(t)$ that is independent of the initial condition. This allows to implement the filter using a stationary gain.

V. NUMERICAL RESULTS

To test the effectiveness of the proposed algorithm, let us consider the optimal control problem for a bilinear continuous-time, second-order, unstable, scalar output, stationary stochastic system as (1), (2) with $H = I_2$, $C = [1 \quad 2]$,

$$A = \begin{bmatrix} 0.7 & 0 \\ 2 & 1.8 \end{bmatrix}, B^1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, G = \begin{bmatrix} 70 & 7 \\ 170 & 77 \end{bmatrix};$$

with the following choice for the weight matrices of the quadratic cost criterion:

$$Q = Q^T = \begin{bmatrix} 144 & 0 \\ 0 & 144 \end{bmatrix} \geq 0, R = R^T = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} > 0.$$

The hypotheses of Theorems 1 and 2 are satisfied. The optimal linear regulator has been implemented and the results are displayed in figs. 1, 2. Fig. (1), shows the traces of the matrix functions $P(t)$, $\Psi_X(t)$, $\Psi_{\widehat{X}}(t)$. We see that, as expected, these variables have indeed a limiting value. Using the value of $P = \lim_{t \to +\infty} P(t)$, it is possible to implement the suboptimal linear regulator. In Fig. 2 we can see the effectiveness of it, in that the stabilizing performance is apparent. As an example we have plotted the second component of the state, for the open-loop system ($u_t \equiv 0$) and for the closed-loop system ($u_t \equiv u_t^o$).

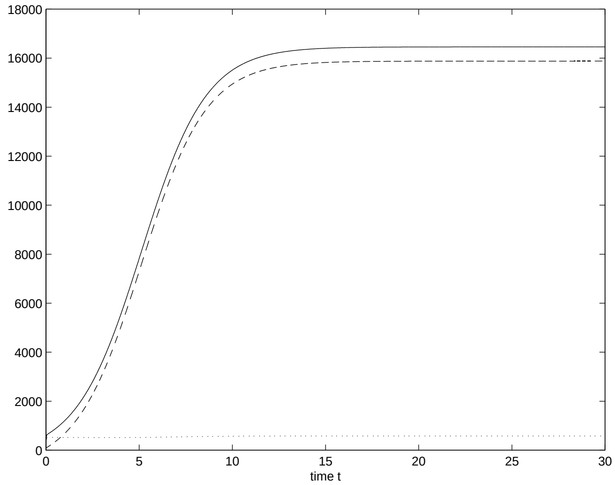

Fig. 1. trace$\{\Psi_X(t)\}$ (solid line), trace$\{P(t)\}$ (dashed line), trace$\{\Psi_{\hat{X}}(t)\}$ (dotted line).

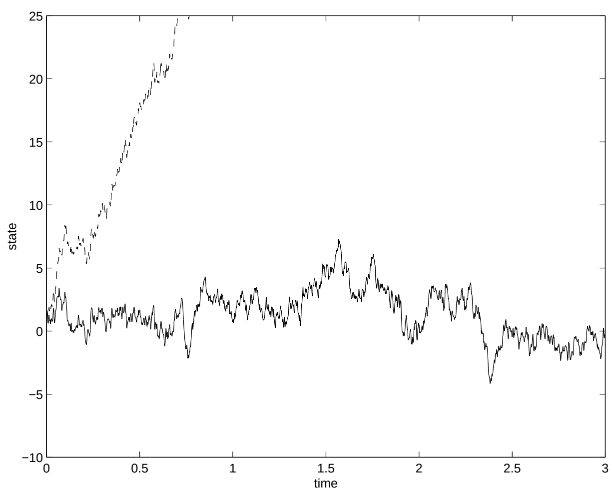

Fig. 2. Behavior of the state second component: open-loop system ($u_t \equiv 0$) (dashed line) and closed-loop system ($u_t \equiv u_t^o$) (solid line).

VI. CONCLUSIONS

The asymptotic properties of the set of Riccati-like equations, namely (8), (9), giving the filter-gain in a suboptimal output-feedback control-scheme has been studied. The main result is theorem 2 that states the existence of a limiting solution for the set of differential equations giving the error-estimate covariance, under the reasonable hypothesis of stabilizability of the pair (A^T, C^T) and under the further hypotheses (22), (23), that simply assure that the state-dependent noise of the system has a not too large power. Moreover, theorem 2 guarantees that the steady-state control problem is well posed and the filtering section of the overall control scheme can be really implemented by using the steady-state value of the error-covariance $P(t)$ in the filter equation.

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