Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Control of a planar system with quantized input/output

A. Cepeda and A. Astolfi

***Abstract*— In this paper the stabilization problem for a simple (unstable) planar system in the presence of input and output quantization is addressed. It is shown that global stability to a terminal set is achieved by means of a hybrid output feedback control law, which reads out the plant only three values and yields a control action composed of three values. Simulations results complete the work.**

I. Introduction and problem formulation

The problem of stabilization of linear and nonlinear systems by means of hybrid control laws, *i.e.* control laws which can be described by automata or by a finite set of rules, has become increasingly popular in the last decade. This is mainly because such control laws can be easily coded and implemented using computers. However, while the implementation issue is fairly simple, the use of hybrid control laws raises several difficult theoretical issues. This is due to the fact that hybrid control laws, even in a linear framework, yields a nonlinear control problem, as for example inputs and outputs have to be considered quantized (and bounded) [1], [2], [3]. As a result, even standard properties, such as controllability and observability may be lost or difficult to assess.

The aim of this work is to show that, for a planar unstable linear system with quantized input and output, the global practical stabilization problem is solvable by means of a simple hybrid control law. This work is partly motivated by the results developed in [2], which in turn relies on the general theory of Wonham and Ramadge [4], and by the results on stabilization of cascaded nonlinear systems developed in [5].

In what follows, we consider a system described by equations of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u \tag{1}$$

$$y = \phi_y([c_1\ c_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) \tag{2}$$

where a, b, c_1 and c_2 are known constants such that

$$abc_1 \neq 0, \tag{3}$$

This work has been (partially) supported through a European Community Marie Curie Fellowship in the framework of the Control Training Site programme (http://www.supelec.fr/lss/CTS)

A. Cepeda is with Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n. 41092 Sevilla, Spain cepeda@cartuja.us.es

A. Astolfi is with Department of Electrical and Electronic Engineering, Imperial College, Exhibition Road, London SW7 2AZ, UK a.astolfi@imperial.ac.uk

and $\phi_y(\cdot)$ is a quantization function defined as

$$\phi_y(s) = \begin{cases} 1, & \text{if } s > \sigma_y \\ 0, & \text{if } |s| \leq 1 \\ -1, & \text{if } s < -\sigma_y \end{cases} .$$

for some $\sigma_y > 0$.

Note that without loss of generality we can assume that $a > 0$, $b > 0$ and $c_1 > 0$. If this were not the case, then it is possible to apply a preliminary state and input transformation yielding a system with this properties.

By condition (3), the underlying linear system is controllable and observable. By construction, the output y of the system takes value in the finite set $\mathcal{Y} = \{1, 0, -1\}$, and it is assumed that the input u is also constrained to take value in the finite set $\mathcal{U} = \{-\varepsilon, 0, \varepsilon\}$[1], for some $\varepsilon > 0$.

The input u (the output y) is modified (read) at some given sampling time[2]. At each sampling time the *controller* computes the value of the control signal on the basis of the available information and of the actual time[3].

The goal of the control law is to make sure that, for any initial condition $(x_1(0), x_2(0))$, the trajectories of the closed loop system remain bounded and converge toward a residual set Ω, containing the origin of the state space, and the size of which is a function of σ_y, ε and the sampling time T_s.

System (1)-(2) is the simplest system for which the considered problem makes sense, and yet it is nontrivial. In fact, if the system were exponentially unstable then global boundedness cannot be achieved via bounded control. On the contrary, if the system were stable then global boundedness would be trivial to achieve, and practical stabilization can be (in principle) achieved using *damping* injection. Note moreover that the system is not required to be minimum phase, hence the semiglobal stabilization tools developed in [6] are not applicable.

The present paper is organized as follows. In Section II we describe a state feedback control law achieving global boundedness and practical stabilization. In Section III we show how it is possible to asymptotically reconstruct the state of the system by means of a time-varying observer. The results above are exploited in Section IV where an output feedback control law yielding practical asymptotic stability is designed, and the properties of the resulting closed loop system are studied in details. Finally, Sections V and VI contain some illustrative simulations and some concluding remarks.

[1] Similar conclusions can be drawn if $\mathcal{U} = \{-\varepsilon, \varepsilon\}$.
[2] For simplicity we assume that sampling times are evenly spaced, however this is not necessary.
[3] See later for detail.

II. The state feedback controller

Consider system (1) and assume that the measurable variable is

$$\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} \phi_{x_1}(x_1) \\ \phi_{x_2}(x_2) \end{bmatrix}, \tag{4}$$

where, for $i \in \{1,2\}$, ϕ_{x_i} is a four-valued function defined as

$$\phi_{x_i}(s) = \begin{cases} 1, & \text{if } s > \sigma_{x_i} \\ 1/2, & \text{if } 0 \le s \le \sigma_{x_i} \\ -1/2, & \text{if } -\sigma_{x_i} \le s < 0 \\ -1, & \text{if } s < -\sigma_{x_i} \end{cases}, \tag{5}$$

for some $\sigma_{x_1} > 0$ and $\sigma_{x_2} > 0$.

Lemma 1: Consider system (1) with output (4) and the control law

$$u = \begin{cases} -\varepsilon \operatorname{sign}(\tilde{y}_2), & \text{if } |\tilde{y}_2| = 1 \\ -\varepsilon \operatorname{sign}(\tilde{y}_1), & \text{otherwise,} \end{cases} \tag{6}$$

for some $\varepsilon > 0$.

Let σ_{x_1}[4], σ_{x_2}, ε and the sampling time T_s be such that

$$\sigma_{x_2} > \varepsilon T_s b \tag{7}$$

$$\sigma_{x_1} > a(T_s\sigma_{x_2} + \frac{T_s^2 b\varepsilon}{2} + \frac{(\sigma_{x_2} + T_s b\varepsilon)^2}{2b\varepsilon}). \tag{8}$$

Then the closed loop system is globally practically asymptotically stable, *i.e.* the closed loop system is globally stable and for any initial condition the trajectories of the closed loop system are such that

$$\limsup_{t\to\infty} |x_1| < \sigma_{x_1} \tag{9}$$

$$\limsup_{t\to\infty} |x_2| < 2\sigma_{x_2}. \tag{10}$$

Remark 1: The output (4) is a four valued function, whereas the control law (6) is a two valued function. Hence, Lemma 1 shows that system (1) with a 1-bit decoder at the input and two 2-bits decoders at the output[5] can be globally practically stabilized. Note that, similar results can be proved in the presence of decoders with higher resolution. However, it would be interesting to show if the information pattern considered is the *minimal* one. This issue is under investigation.

Instead of providing a formal proof of the above Lemma we briefly illustrate its stabilization mechanism. Figure 1 shows the behaviour of the system (1)-(4) with the control law 6. For $t = 0$, $x_2 > \sigma_{x_2}$ (hence $\tilde{y}_2 = 1$) and the controller *responds* with $u = -\varepsilon$, and this reduces $|x_2|$. The control signal remains constant till $t = (\bar{k}+1)T_s$. In that moment, the second rule of the controller becomes active, because $|x_2| < \sigma_{x_2}$, and the controller output depends on the value of x_1.

In this example $x_1 > 0$, and this implies that the output remains $u = -\varepsilon$. The state x_1 is increasing because $x_2 > 0$ and it reachs its maximum value at $t = \bar{\bar{t}}$, when $x_2 = 0$. In this period, the control input is $u = -\varepsilon$, therefore x_2 decreases until $x_2 < -\sigma_{x_2}$. At this time, the first rule of the controller becomes active and the control signal is $u = \varepsilon$. This value of u is applied for only a sampling period, because x_2 is increasing and it reachs the boundary $-\sigma_{x_2}$. Therefore the controller applies the first rule, yielding $u = -\varepsilon$, provided x_1 remains positive.

We have therefore a period of time in which x_2 is *around* $-\sigma_{x_2}$, and this makes x_1 decrease. At $t = (k+1)T_s$ the controller notices that $x_1 < 0$, so the output of the second rule is $u = \varepsilon$. That finishes the curl of x_2, and x_2 begins to increase. The state x_1 is decreasing till $t = \hat{t}$ due to the sign of x_2. Then, x_2 reachs σ_{x_2} and the above series of switches is repeated.

Note that for $t > (\bar{k}+1)T_s$ the state x_2 remains between $[-2\sigma_{x_2}, 2\sigma_{x_2}]$ and for $t > (k+1)T_s$ the state x_1 remains between $[-\sigma_{x_1}, \sigma_{x_1}]$.

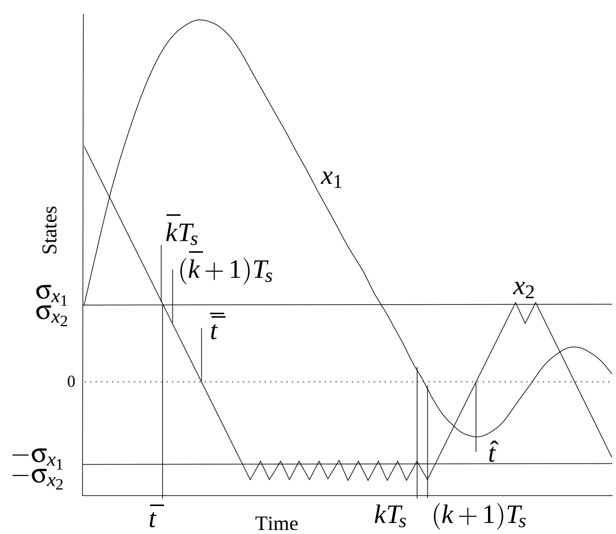

Fig. 1. Definitions.

Remark 2: The control law (6) is such that all trajectories of the closed loop system converge to a residual set Ω which is defined by the values of σ_{x_1} and σ_{x_2}. These values should be such that conditions (7) and (8) hold. Note however that, for any positive σ_{x_1} and σ_{x_2} it is always possible to select $\varepsilon > 0$ and $T_s > 0$ such that conditions (7) and (8) hold.

III. The observability issue

In this section we show that it is possible to approximately asymptotically reconstruct the state of the system (1)

[4]By equation (7) $a(T_s\sigma_{x_2} + \frac{T_s^2 b\varepsilon}{2} + \frac{(\sigma_{x_2}+T_s b\varepsilon)^2}{2b\varepsilon}) < \frac{7a\sigma_{x_2}^2}{2\varepsilon b}$, therefore σ_{x_1} can be simply selected such that $\sigma_{x_1} > \frac{7a\sigma_{x_2}^2}{2\varepsilon b}$.

[5]Note however that, as the control law is simply using the signum of the output variable $\tilde{y}$, it would be possible to use one 1-bit and one 2-bits decoder for the output channel.

from the output (2). Note that this is a nontrivial problem, as system (1) with output (2) is not uniformly observable nor there exists a universal input.

Remark 3: One further difficulty arises because of the sampled nature of the output. In fact, if the output were measured for all t then it would be possible to reconstruct exactly the state. This reconstruction can be achieved as follows.

Consider system (1) with output (2) and assume that $y(t)$ is known for all t. Suppose also that $y(0) = 1$. Then, setting $u = -\varepsilon$ yields an output trajectory $y(t)$ which is such that

$$c_1 x_1(t_1) + c_2 x_2(t_1) = \sigma_y, \tag{11}$$

for some $t_1 > 0$, and

$$c_1 x_1(t_2) + c_2 x_2(t_2) = -\sigma_y, \tag{12}$$

for some $t_2 > t_1$. Moreover, by the dynamics of the system,

$$\begin{aligned} x_2(t_2) &= x_2(t_1) - b\varepsilon(t_2 - t_1) \\ x_1(t_2) &= x_1(t_1) + a(x_2(t_1)(t_2 - t_1) - b\varepsilon\tfrac{(t_2-t_1)^2}{2}). \end{aligned} \tag{13}$$

Hence, replacing equation (13) into equation (12), one may solve for $x_1(t_1)$ and $x_2(t_1)$ from such equation and equation (11). Note that this procedure is feasible by observability of the underlying linear system. (If $y(0) = -1$ or $y(0) = 0$ a similar argument may be used.)

To solve the considered state estimation problem, consider a discrete time realization of the system given by

$$\begin{aligned} x^{k+1} = \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} &= A_d x^k + B_d u^k = \\ &= \begin{bmatrix} 1 & aT_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix} + \begin{bmatrix} ab\frac{T_s^2}{2} \\ bT_s \end{bmatrix} u^k \\ y^k = \phi_y(cx^k) &= \phi_y([c_1\, c_2] \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix}). \end{aligned} \tag{14}$$

Note that the underlying linear system is controllable and observable as $abc_1 \neq 0$.

As already remarked, observability of system (14) depends upon the selection of the input signal. To illustrate this property, assume, for example, $c_1 = 1$, $c_2 = 0$, $u^k = \varepsilon$ for all k, $x_1^0 > \sigma_y$ and $x_2^0 > 0$. Then a simple computation shows that $y^k = 1$ for all k, which implies that $x_1^k > \sigma_y$ for all k, but does not yield any information on the size of x_1^k or on x_2^k.

To provide an intuitive explanation of the observer design proposed in Lemma 2 below, note that by its quantized nature, at each sampling time it is only possible to decide if the state of the system is in one of the three regions defined by

$$cx^k > \sigma_y$$

$$cx^k < -\sigma_y$$

or

$$-\sigma_y \le cx^k \le \sigma_y.$$

However, it is possible to obtain further information on the state x^k considering how the states in the above regions *move* with k, provided the output switches. Hence, the observer has to *remember* not only the output at the sampling time, but also its previous values. In this way, it is possible to construct a sequence of regions R_k, of decreasing area, in the state space, with the property that $x_k \in R_k$, for all k.

Lemma 2: Consider the system (14).
Define

$$\begin{aligned} c(k) &= \begin{cases} [-c], & \text{if } y^k = 1 \\ \begin{bmatrix} c \\ -c \end{bmatrix}, & \text{if } y^k = 0 \\ [c], & \text{if } y^k = -1 \end{cases} \\ d(k) &= \begin{cases} [-\sigma_y], & \text{if } y^k = 1 \\ \begin{bmatrix} \sigma_y \\ \sigma_y \end{bmatrix}, & \text{if } y^k = 0 \\ [-\sigma_y], & \text{if } y^k = -1 \end{cases} \end{aligned} \tag{15}$$

and consider also the sequences of matrices

$$C(k) = \begin{cases} c(0), & \text{if } k = 0 \\ \begin{bmatrix} C(k-1)A_d^{-1} \\ c(k) \end{bmatrix}, & \text{if } k > 0 \end{cases}$$

$$D(k) = \begin{cases} d(0), & \text{if } k = 0 \\ \begin{bmatrix} D(k-1) + C(k-1)A_d^{-1}B_d u^{k-1} \\ d(k) \end{bmatrix}, & \text{if } k > 0 \end{cases} \tag{16}$$

Then the state x^k is such that[6]

$$x^k \in R_k = \{(x_1, x_2) \in I\!R^2 \mid C(k)x \preceq D(k)\}. \tag{17}$$

Lemma 2 cannot be used directly to construct an (approximate) estimate for the state x^k of system (14), as the regions R_k may be unbounded. However, it may be proved that if $R_{\bar{k}}^-$ is bounded for some $\bar{k}$ then R_k is bounded and it is not *larger* than $R_{\bar{k}}^-$, for all $k > \bar{k}$. We do not provide details of these facts, as in the next section we will show that using a proper selection of the control sequence u^k it is possible to prove a convergence result for the sequence $\{R_k\}$.

IV. Output feedback controller

In this section it is shown how the results in Sections II and III can be combined to design an output feedback control law globally, practically, stabilizing system (1) with output (2). To this end, we first prove a few preliminary lemmas.

[6]The notation $a \preceq b$, with a and b vectors has to be understood componentwise.

A. Preliminary lemmas

Lemma 3: Consider system (1) with output (2). Let $\psi(x_1^k, x_2^k)$ be defined as

$$\psi(x_1^k, x_2^k) = \begin{cases} -\varepsilon\ \mathrm{sign}(\phi_{x_2}(x_2^k)), & \text{if } |\phi_{x_2}(x_2^k)| = 1 \\ -\varepsilon\ \mathrm{sign}(\phi_{x_1}(x_1^k)), & \text{otherwise,} \end{cases}$$

where $\phi_{x_1}(\cdot)$ and $\phi_{x_2}(\cdot)$ are as in equation (5), and assume that σ_{x_1}, σ_{x_2}, ε and the sampling time T_s are such that conditions (8) and (7) hold.

Consider the regions R_k defined in Lemma 2. Define, for $i = \{1, 2\}$,

$$\begin{aligned} x_{i,min}^k &= \min\nolimits_{x \in R_k} x_i \\ x_{i,max}^k &= \max\nolimits_{x \in R_k} x_i \end{aligned} \tag{18}$$

and

$$\begin{aligned} \bar{x}_1^k &= \begin{cases} x_{1,min}^k, & \text{if } -y^k > 0 \\ x_{1,max}^k, & \text{if } -y^k < 0 \end{cases} \\ \bar{x}_2^k &= \begin{cases} x_{2,min}^k, & \text{if } -y^k > 0 \\ x_{2,max}^k, & \text{if } -y^k < 0 \end{cases} \end{aligned} \tag{19}$$

Let

$$u = \begin{cases} -\varepsilon y^k, & \text{if } R_k \text{ is not compact} \\ \psi(\bar{x}_1^k, \bar{x}_2^k), & \text{if } R_k \text{ is compact.} \end{cases} \tag{20}$$

Then there exists $\bar{k} > 0$ such that for all $k \geq \bar{k}$, R_k is compact.

Remark 4: Note that (19) can be computed online. The complexity of that operations is just an LP. In order to decrease the complexity of the computation, redundant inequalities can be removed.

Lemma 4: Let the assumptions and definitions of Lemmas 2 and 3 hold. Consider the system (1) with the output (2), the controller (20) and the sequence $\{R_k\}$.

Assume that, for some $\bar{k}$, $R_{\bar{k}}$ is compact. Then for all $\tilde{k} \geq \bar{k}$ there exists $\hat{k} \geq \tilde{k}$ such that, for all $k > \hat{k}$

$$\begin{aligned} x_{2,max}^k &\leq 2\sigma_{x_2} + x_{2,max}^{\tilde{k}} - x_{2,min}^{\tilde{k}} \\ x_{2,min}^k &\geq -2\sigma_{x_2} - x_{2,max}^{\tilde{k}} + x_{2,min}^{\tilde{k}}. \end{aligned} \tag{21}$$

Lemma 5: Consider the system (1) with the output (2), the controller (20) and the sequence $\{R_k\}$ defined on Lemma 2.

Then for all $\tilde{k}$ such that $y^{\tilde{k}} \neq 0$ there exists a $k > \tilde{k}$ such that $y^k = \{-y^{\tilde{k}}, 0\}$.

Lemma 6: Let the assumptions and definitions of Lemmas 2 and 3 hold. Consider the system (1) with the output (2), the controller (20) and the sequence $\{R_k\}$.

Then for all $\tilde{k}$ such that $R_{\tilde{k}}$ is compact there exists $\hat{k} \geq \tilde{k}$ such that, for all $k > \hat{k}$

$$x_{1,max}^k \leq \tfrac{|c_2|}{c_1}(2\sigma_{x_2} + x_{2,max}^{\tilde{k}} - x_{2,min}^{\tilde{k}}) + \tfrac{\sigma_y}{c_1} + \frac{a(2\sigma_{x_2} + x_{2,max}^{\tilde{k}} - x_{2,min}^{\tilde{k}})^2}{2b\varepsilon} + aT_s(2\sigma_{x_2} + x_{2,max}^{\tilde{k}} - x_{2,min}^{\tilde{k}}) \tag{22}$$

$$x_{1,min}^k \geq -\tfrac{|c_2|}{c_1}(2\sigma_{x_2} + x_{2,max}^{\tilde{k}} - x_{2,min}^{\tilde{k}}) - \tfrac{\sigma_y}{c_1} - \frac{a(2\sigma_{x_2} + x_{2,max}^{\tilde{k}} - x_{2,min}^{\tilde{k}})^2}{2b\varepsilon} - aT_s(2\sigma_{x_2} + x_{2,max}^{\tilde{k}} - x_{2,min}^{\tilde{k}}). \tag{23}$$

Lemma 7: If Lemmas 3 and 6 hold then for all $\gamma > 0$ there exists a $\bar{k}$ such that for all $k > \bar{k}$, $x_{2,max}^k - x_{2,min}^k < \gamma$.

B. Output feedback control

The preliminary results proved so far allow to state the following fact, the proof of which is a trivial consequence of the facts established so far.

Proposition 1: Consider system (1) with output (2). Consider the output feedback control law defined in Lemma 3.

Then, for any initial condition $x(0)$ the trajectories of the closed loop system are bounded and converge to a residual set Ω, the size of which can be arbitrarily reduced with a proper selection of the design parameters σ_{x_2}, σ_y, ε and T_s.

V. Simulations

In this section we provide some simulations to illustrate the theoretical results. We consider two systems of the form (1) with output $y = \phi_y(c_1 x_1 + c_2 x_2)$ with a, b, c_1 and c_2 such that the underlying linear systems have transfer functions

$$\text{S1:} \quad G_1(s) = \frac{2(s-1)}{s^2}$$

$$\text{S2:} \quad G_2(s) = \frac{2(s+1)}{s^2}.$$

In all cases the sampling time is $T_s = 0.1$, the initial conditions are $x_1(0) = 3$ and $x_2(0) = -2$, $\varepsilon = 2$, $\sigma_{x_1} = 1$ and $\sigma_{x_2} = 1$.

Figures 2, 3 and 4 display the phase portrait, the history of the state variables and of the control signal, respectively, for the case S1. On Figure 2 we have also plotted the boundary of the region R_k for some values of k. Note that the region is shrinking and, as proven, $x_{2,max} - x_{2,min}$ is decreasing.

Figures 5, 6 and 7 display the phase portrait, the history of the state variables and of the control signal, respectively, for the case S2. On Figure 5 we have also plotted the boundary of the region R_k for some values of k.

In both simulations, and as predicted by the theory, the state approaches a residual set and remains therein. It is interesting to observe that the phase portraits are completely different, and this is mainly due to the fact that one of the underlying linear system is minimum phase, whereas the other is not.

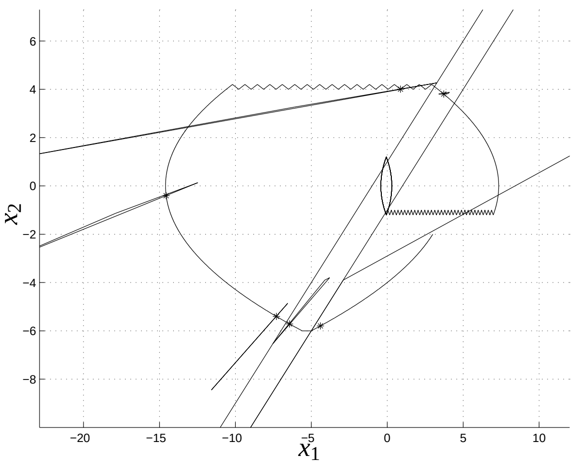

Fig. 2. Phase portrait for case S1.

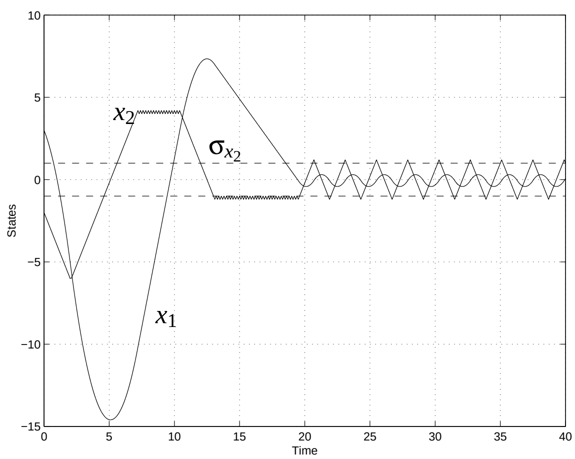

Fig. 3. State histories for case S1.

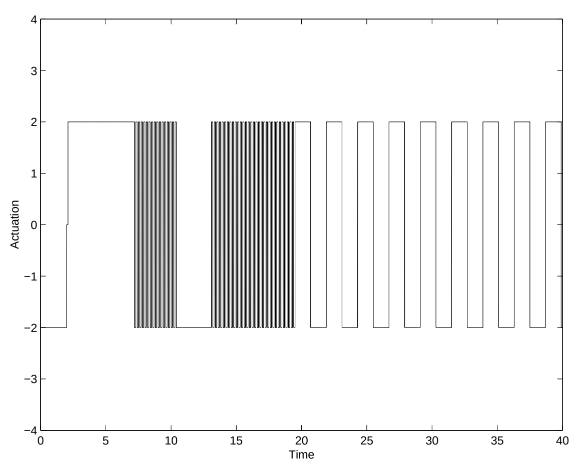

Fig. 4. Control signal for case S1.

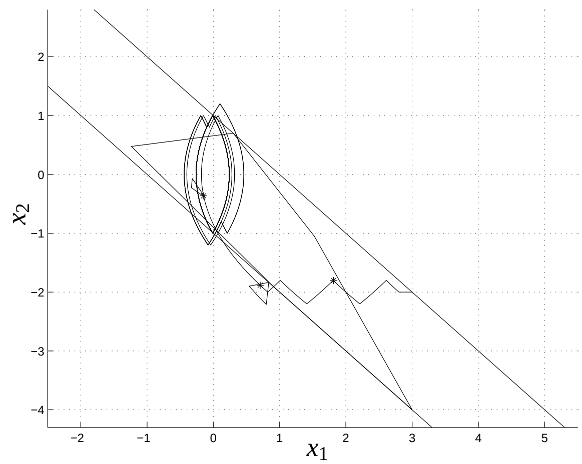

Fig. 5. Phase portrait for case S2.

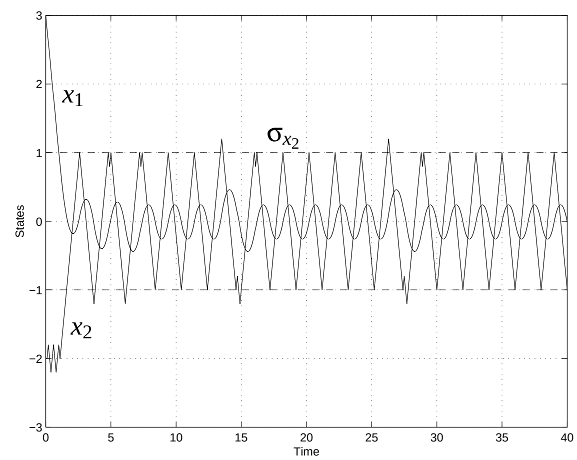

Fig. 6. State histories for case S2.

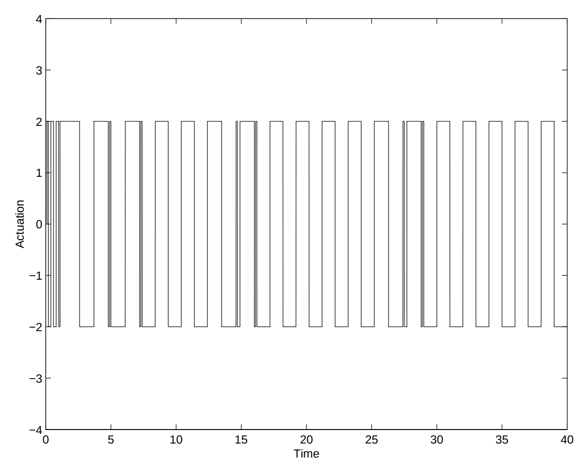

Fig. 7. Control signal for case S2.

VI. Conclusions

In this paper the stabilization problem for a simple planar system in the presence of input and output quantization has been considered. A globally practically stabilizing output feedback control law has been designed. This relies on the construction of a practically stabilizing state feedback control law and of a *practical* observer. The theoretical findings have been illustrated via some simple simulations.

The results presented in this paper can be extended to n dimensional systems described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= \phi_y(Cx) \end{aligned}$$

with A with all zero eigenvalues, (A,B) controllable, and (C,A) observable.

References

[1] J.Raish, "Controlability and observability of simple hybrid control systems-fdlti plants with symbolic measurements and quantized control inputs," *International Conference on Control*, vol. 1, pp. 595–600, 1994.

[2] J. R. et al., "Approximating automata and discrete control for continuous systems - two examples from process control," *Hybrid Systems V*, pp. 279–333, 1999.

[3] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, pp. 1543–1554, 2003.

[4] P. J. Ramadge and W. M. Wonham, "The control of discrete event systems," *Proceedings of the IEEE*, vol. 77, pp. 81–98, 1989.

[5] G. Kaliora and A. Astolfi , "Stabilization with positive and quantized control," *Proceeding of the IEEE Conference on Decision and Control*, vol. 1, pp. 1892–1897, 2002.

[6] Z. Lin and A. Saberi, "Robust semiglobal stabilization of minimum-phase input-output linearizable systems via partial stable and output feedback," *Transaction on Automatic control*, vol. 40, pp. 1029–1041, 1995.

[7] R. W. Brokett and D. Liberzon, "Quantized feedback stabilization of linear systems," *Transaction on Automatic control*, vol. 45, pp. 1279–1289, 2000.

[8] T. S. E. Konaka and S. Okuma, "Indirect adaptive control of two wheeled vehicle by quantized input and output," *Proceeding on the 2002 International Conference on Control Applications*, vol. 1, pp. 600–605, 2002.

[9] D. Liberzon, "On stabilization of linear systems with limited information," *Transaction on Automatic control*, vol. 48, pp. 304–307, 2003.

[10] J. Sur and B. Padden, "Observers for linear systems with quantized outputs," *Proceedings of the 1997 American Control Conference*, vol. 5, pp. 3012–3016, 1997.