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Global Output Feedback Stabilization for Uncertain Nonlinear Systems with Output Dependent Incremental Rate

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***Abstract*— The solvability of global robust stabilization problem of nonlinear systems by output feedback has been given under various growth conditions on the vector fields of the systems. One of the most common cases is that the vector field is bounded by a linear growth with constant incremental rate. Recently, the problem was solved in [10] for some systems with output dependent incremental rate, without considering the uncertainties. This paper further considers the uncertain systems and gives the robust result.**

I. Introduction

Over the years, there have been constant progresses on the problem of global stabilization of nonlinear systems by output feedback control [3], [5], [8], [9], [10], [12], [13], etc. In particular, in [10], the problem is solved for systems in lower triangular form under the global Lipschitz-like condition on the unmeasurable states with output dependent incremental rate. The results reply to systems without model uncertainties. In [12], under an alternative linear growth assumption with constant incremental rate, the global stabilization of a class of uncertain nonlinear systems is considered by output feedback control.

These papers employ high gain observers derived from the standard form (e.g. [7]) to reproduce the unmeasured states, and then synthesize an observer-controller based control laws. While [12] uses a static sufficiently high gain under the assumption that the incremental rate is constant, [10] uses a dynamically generated high gain in the observer to account for the incremental rate of the vector field. In this paper, we consider the global output feedback stabilization problem for uncertain systems in lower triangular form with output dependent incremental rate. Due to the presence of the model uncertainty, the method in [10] cannot be directly applied to our case because [10] has used the vector filed that defines the system in the construction of the observer. In particular, the error system is not globally asymptotically stable, but is driven by some other states. In this paper, we have managed to overcome the difficulty incurred by the model uncertainty by devising a novel high gain generator.

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The remaining sections are organized as follows. Section II formulates the main problem. In Section III, we design an observer and analyze the error system. And in Section IV we design the controller and analyze the stability of overall system. An example is given in Section V to illustrate the effectiveness of our method. Finally, we close this paper by some concluding remarks.

II. Problem Formulation and Preliminaries

Consider the following nonlinear system

$$\begin{aligned}
\dot{z} &= f_0(z, x_1, \mu(t)) \\
\dot{x}_1 &= x_2 + f_1(z, x_1, \mu(t)) \\
&\vdots \\
\dot{x}_{n-1} &= x_n + f_{n-1}(z, x_1, \cdots, x_{n-1}, \mu(t)) \\
\dot{x}_n &= u + f_n(z, x_1, \cdots, x_n, \mu(t)) \\
y &= x_1
\end{aligned} \quad (2.1)$$

where $z \in \Re^{n_z}$ and $x = \text{col}(x_1, \ldots, x_n) \in \Re^n$ are the states, $u, y \in \Re$ the input and output respectively, $\mu : [0, \infty) \mapsto \Re^{n_\mu}$ a piecewise continuous vector valued function representing disturbance and/or unknown parameters, and the functions $f_0, \cdots, f_n$ are continuously differentiable in their arguments.

Roughly, the objective of this paper is to find an output feedback controller such that, for all $z(0) \in \Re^{n_z}$ and $x(0) \in \Re^n$, the trajectories of the closed-loop system composed of (2.1) and this controller are bounded for all $t \geq 0$, and the states of (2.1) converge 0 asymptotically, i.e., $\lim_{t\to\infty} \text{col}(z(t), x(t)) = 0$. A more precise description of our objective will be given in the statement of Theorem 4.1.

As pointed out in [9], the global stabilization on nonlinear systems via output feedback is usually impossible, without extra growth conditions on the unmeasurable states of the system. Here, we impose the following growth conditions on system (2.1).

A2.1: For $i = 1, \cdots, n$, there exist non-negative continuous functions $c_i^j(\cdot)$, $j = 0, \cdots, i$, such that, for all $\mu \in \Re^{n_\mu}$,

$$|f_i(z, x_1, \cdots, x_i, \mu(t))| \leq c_i^0(x_1)\|z\| + \sum_{j=1}^{i} c_i^j(x_1)|x_j|. \quad (2.2)$$

And a minimum phase assumption is imposed on the inverse dynamics of (2.1) as follows.

A2.2: There exists a C^1 positive definite proper function $V_z(z)$ satisfying

$$\underline{a}_z\|z\|^2 \leq V_z(z) \leq \bar{a}_z\|z\|^2$$

for positive numbers $\underline{a}_z$ and $\bar{a}_z$, such that, its derivative, along the trajectories of

$$\dot{z} = f_0(z, x_1, \mu(t)), \tag{2.3}$$

satisfies

$$\frac{dV_z(z)}{dt} \leq -\|z\|^2 + c_z x_1^2 \tag{2.4}$$

for some non-negative number c_z.

Remark 2.1: The global stabilization problem of the lower triangular systems of the form (2.1) by state (or partial state) feedback control has been extensively studied in [1], [4], [6], among others. Recently, this problem was solved by output feedback control in [10] when no uncertainty $\mu(t)$ appears in system (2.1). Also [12] solved this problem under assumption A2.1 with c_i^j being constant. ∎

Remark 2.2: It should be noted the systems considered in [12] are more general that (2.1). Nevertheless, the results in this paper can also be straightforwardly extended to the same class of systems in [12]. Specifically, when the functions f_i, $i = 1, \ldots, n$, in system (2.1) explicitly depend on all states and the input, that is, f_i can be rewritten as $f_i(z, x_1, \ldots, x_n, u, \mu)$, the results still hold as long as the absolute values of f_i's are still bounded by (2.2). ∎

III. Observer and Error System

The output feedback stabilization problem of plant (2.1) will be solved by the explicit construction of an observer-controller type control law. In this section, we will design a dynamic high gain observer and analyze the error system between the original system and the observer. Then, a controller will be given in the next section.

A. Observer

First we design a high gain observer (see [7]) as follows,

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + a_1 r(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + a_2 r^2(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_n &= u + a_n r^n (x_1 - \hat{x}_1) \end{aligned} \tag{3.1}$$

where $\hat{x} = \text{col}(\hat{x}_1, \cdots, \hat{x}_n) \in \Re^n$ is the state, and $a_1, \cdots, a_n$ are the coefficients of arbitrary Hurwitz polynomial $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, that is, the matrix

$$A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}$$

is Hurwitz. Further, let P be a symmetric positive definite matrix satisfying the Lyapunov equation

$$PA + A^T P = -I. \tag{3.2}$$

Denote $B = \text{diag}(0, 1, \cdots, n-1)$, from [11], it is possible to choose $a_1, \cdots, a_n$ satisfying (3.2) and

$$P(B + I) + (B + I)P > 0. \tag{3.3}$$

Hence, there exists $p > 0$ such that

$$x^T[P(B + I) + (B + I)P]x \geq p\|x\|^2, \ \forall x \in \Re^n.$$

In the observer, r is the dynamically generated high gain. To design the generator, we will introduce a so-called *ramp* function ramp$(\cdot)$, which is continuous non-negative function defined as follows

$$\text{ramp}(r) = \begin{cases} 0, \ r < 0 \\ r, \ r \geq 0 \end{cases}. \tag{3.4}$$

Now, it is ready to define the high gain generator as follows,

$$\dot{r} = \frac{r}{p}\text{ramp}\left[\varpi\left(\frac{x_1}{r}\right) - \frac{r}{2}\right], \ r(0) = r_0 \geq 1 \tag{3.5}$$

where $\varpi(\cdot)$ is a continuously differentiable positive function to be specified later.

Remark 3.1: When the incremental rate of the system is constant, the high gain r in (3.1) can be chosen as a sufficiently high constant [12]. However, when the incremental rate depends on the output of the system, a dynamically generated high gain has to be considered. The generator (3.5) is motivated from the one in [10] but is defined differently so that it has the following two properties,

$$\text{i:} \qquad \dot{r}(t) \geq 0, \ t \geq 0 \tag{3.6}$$

$$\text{ii:} \qquad \frac{r}{2} + p\frac{\dot{r}}{r} \geq \varpi\left(\frac{x_1}{r}\right). \tag{3.7}$$

These properties are important in stability analysis of the closed-loop system. ∎

B. Error System

For $i = 1, \ldots, n$, denote $e_i = x_i - \hat{x}_i$, then the error system between the original system and the observer becomes

$$\begin{aligned} \dot{e}_1 &= e_2 + f_1(z, x_1, \mu) - a_1 r e_1 \\ \dot{e}_2 &= e_3 + f_2(z, x_1, x_2, \mu) - a_2 r^2 e_1 \\ &\vdots \\ \dot{e}_n &= f_n(z, x_1, \cdots, x_n, \mu) - a_n r^n e_1. \end{aligned} \tag{3.8}$$

Now, the system composed of plant, observer and high gain generator, i.e., (2.1)+(3.1)+(3.5), is diffeomorphic to (2.3)+(3.1)+(3.8)+(3.5).

Furthermore, scale the coordinate by defining $\epsilon_0 = \frac{z}{r}$, $\epsilon_i = \frac{e_i}{r^i}$, $i = 1, \cdots, n$. Thus, system (2.3)+(3.8) is changed into the following form

$$\begin{aligned}
\dot\epsilon_0 &= \frac{f_0(z,x_1,\mu)}{r} - \frac{\dot r}{r}\epsilon_0 \\
\dot\epsilon_1 &= \frac{f_1(z,x_1,\mu)}{r} + r\epsilon_2 - ra_1\epsilon_1 - \frac{\dot r}{r}\epsilon_1 \\
\dot\epsilon_2 &= \frac{f_2(z,x_1,x_2,\mu)}{r^2} + r\epsilon_3 - ra_2\epsilon_1 - 2\frac{\dot r}{r}\epsilon_2 \\
&\vdots \\
\dot\epsilon_n &= \frac{f_n(z,x_1,\cdots,x_n,\mu)}{r^n} - ra_n\epsilon_1 - n\frac{\dot r}{r}\epsilon_n. \quad (3.9)
\end{aligned}$$

It is seen that system (2.1)+(3.1)+(3.5) is diffeomorphic to (3.9)+(3.1)+(3.5).

For simplicity of the presentation, define the following notations,

$$\begin{aligned}
\epsilon &= \mathrm{col}(\epsilon_1,\cdots,\epsilon_n) \\
F(z,x,r,\mu) &= \mathrm{col}\bigg(\frac{f_1(z,x_1,\mu)}{r}, \frac{f_2(z,x_1,x_2,\mu)}{r^2},\cdots, \\
&\qquad \frac{f_n(z,x_1,\cdots,x_n,\mu)}{r^n}\bigg).
\end{aligned}$$

We now show that, by choosing some appropriate function $\varpi(\cdot)$ to generate a sufficiently large gain r, there exists a Lyapunov-like function for the error system (3.9).

Lemma 3.1: Suppose system (2.1) satisfies assumptions A2.1 and A2.2, and for $i=1,\cdots,n$,

$$\begin{aligned}
c_i^0(x_1) &\le a_i^0|x_1|^{i-1} + b_i^0, \quad (3.10)\\
c_i^j(x_1) &\le a_i^j|x_1|^{i-j} + b_i^j,\ j=1,\ldots,i \quad (3.11)
\end{aligned}$$

for some nonnegative numbers a_i^j, b_i^j, $j=0,\ldots,i$. Then, under some appropriate choice of the function $\varpi(\cdot)$ in (3.5), there exists a continuously differentiable positive semi-definite function $V_0(\epsilon_0,\epsilon,r)$, such that, its derivative along (3.9) and (3.5) satisfies

$$\frac{dV_0(\epsilon_0,\epsilon,r)}{dt} \le -\|\epsilon_0\|^2 - \frac{r}{2}\|\epsilon\|^2 + C\sum_{j=1}^{n}\frac{\hat x_j^2}{r^{2j}}$$

for some non-negative real number C.

Moreover, if r is bounded, i.e., $\lim_{t\to\infty} r(t) = r_\infty < \infty$, then

$$\underline{\alpha}_0(\|\mathrm{col}(\epsilon_0,\epsilon)\|) \le V_0(\epsilon_0,\epsilon,r) \le \bar\alpha_0(\|\mathrm{col}(\epsilon_0,\epsilon)\|)$$

for some class K_∞ functions $\underline{\alpha}_0(\cdot), \bar\alpha_0(\cdot)$, which depend on r_∞.

Proof: It can verified that

$$V_0(\epsilon_0,\epsilon,r) = 2\frac{V_z(r\epsilon_0)}{r^2} + \epsilon^T P\epsilon$$

is the derived Lyapunov function, and the details are omitted due to the space limit. ∎

Remark 3.2: It is obvious that if $c_i^j(x_1)$ is constant for $j=0,\cdots,i,\ i=1,\ldots,n$, then the conditions (3.10) and (3.11) are automatically satisfied with $a_i^j = 0$ and $b_i^j = c_i^j$. ∎

IV. Controller Design

The objective of this section is to design an appropriate u in (3.1), dependent on the observer states $\hat x_1,\cdots,\hat x_n$, such that there exists a Lyapunov function for the overall system (3.9)+(3.1)+(3.5). The construction of the controller is based on a recursive procedure as follows.

1) Let $\tilde x_1 = \hat x_1$, $\tilde x_2 = \hat x_2 + \kappa_1 r\tilde x_1$, for some positive number κ_1 satisfying

$$\kappa_1 \ \ge\ \frac{n+1}{2}a_1^2 + \frac{1}{p} + \frac{1}{4}. \quad (4.1)$$

And let

$$V_1(\epsilon_0,\epsilon,r,\tilde x_1) = V_0(\epsilon_0,\epsilon,r) + \frac{1}{2r^2}\tilde x_1^2,$$

then, under the condition

$$\varpi(s) \ \ge\ 2C(1+2\kappa_1^2)p, \quad (4.2)$$

its derivative satisfies

$$\begin{aligned}
&\frac{dV_1(\epsilon_0,\epsilon,r,\tilde x_1)}{dt} \\
\le\ & -\|\epsilon_0\|^2 - \frac{r}{2}\|\epsilon\|^2 + C\sum_{j=3}^{n}\frac{\hat x_j^2}{r^{2j}} + C\frac{\tilde x_1^2}{r^2} + 2C\frac{\tilde x_2^2}{r^4} \\
& +2C\frac{\kappa_1^2\tilde x_1^2}{r^2} + \frac{1}{r^2}\tilde x_1\left(\tilde x_2 - \kappa_1 r\tilde x_1 + a_1 r^2\epsilon_1\right) - \frac{\dot r}{r^3}\tilde x_1^2 \\
\le\ & -\|\epsilon_0\|^2 - \frac{nr}{2(n+1)}\|\epsilon\|^2 + C\sum_{j=3}^{n}\frac{\hat x_j^2}{r^{2j}} + 2C\frac{\tilde x_2^2}{r^4} \\
& +\frac{1}{r^3}\tilde x_2^2 + \bigg[C(1+2\kappa_1^2) + \frac{r}{4} - \kappa_1 r + \frac{n+1}{2}a_1^2 r \\
& +\frac{r}{2p} - \frac{\varpi(x_1/r)}{p}\bigg]\frac{\tilde x_1^2}{r^2} \\
\le\ & -\|\epsilon_0\|^2 - \frac{nr}{2(n+1)}\|\epsilon\|^2 - \frac{1}{2p}\left[r + \varpi\left(\frac{x_1}{r}\right)\right]\frac{\tilde x_1^2}{r^2} \\
& + \left[\frac{2C}{r^4} + \frac{1}{r^3}\right]\tilde x_2^2 + C\sum_{j=3}^{n}\frac{\hat x_j^2}{r^{2j}}. \quad (4.3)
\end{aligned}$$

2) Let

$$\begin{aligned}
\tilde x_1 &= \hat x_1 \\
\tilde x_{i+1} &= \hat x_{i+1} + \kappa_i r\tilde x_i,\ i=1,\cdots,n-1 \\
\tilde x_{n+1} &= u + \kappa_n r\tilde x_n, \quad (4.4)
\end{aligned}$$

where $\kappa_1,\cdots,\kappa_n$ are some positive constants to be determined by the subsequent recursive procedure. This coordinate transformation puts the system (3.1) into the following

$$\begin{aligned}
\dot{\tilde x}_1 &= \zeta_1(\tilde x_1,\tilde x_2,\epsilon_1,r) \\
&\vdots \\
\dot{\tilde x}_n &= \zeta_n(\tilde x_1,\cdots,\tilde x_{n+1},\epsilon_1,r) \quad (4.5)
\end{aligned}$$

where

$$\begin{aligned}\zeta_1(\tilde{x}_1,\tilde{x}_2,\epsilon_1,r) &= \tilde{x}_2-\kappa_1 r\tilde{x}_1+a_1r^2\epsilon_1\\ \zeta_i(\tilde{x}_1,\cdots,\tilde{x}_{i+1},\epsilon_1,r) &= \tilde{x}_{i+1}-\kappa_i r\tilde{x}_i+a_ir^{i+1}\epsilon_1\\ &\quad +\kappa_{i-1}r\zeta_{i-1}(\tilde{x}_1,\cdots,\tilde{x}_i,\epsilon_1,r)\\ &\quad +\kappa_{i-1}\dot{r}\tilde{x}_{i-1},\ i=2,\cdots,n.\end{aligned}$$

After each inductive step, we will make the system satisfy two properties as illustrated by the following lemma.

Lemma 4.1: Under assumptions A2.1 and A2.2, and conditions (3.10) and (3.11), by appropriate choice of the function $\varpi(\cdot)$ in (3.5), there exist positive numbers $\kappa_1,\cdots,\kappa_n$, such that, for $i=1,\cdots,n$, the following two properties hold.

P1: There exist non-negative constants $D_{(i,j)}$, $j=0,\cdots,i$, such that,

$$\begin{aligned}&|\zeta_i(\tilde{x}_1,\cdots,\tilde{x}_{i+1},\epsilon_1,r)+\kappa_i r\tilde{x}_i|\\ \le\ &|\tilde{x}_{i+1}|+\sum_{j=1}^{i-1}D_{(i,j)}\left[r^{i-j+1}+r^{i-j}\varpi\left(\frac{x_1}{r}\right)\right]|\tilde{x}_j|\\ &+D_{(i,i)}r|\tilde{x}_i|+D_{(i,0)}r^{i+1}|\epsilon_1|,\end{aligned}$$

where $D_{(i,j)}$, $j=1,\cdots,i$, depend only on $\kappa_1,\cdots,\kappa_{i-1}$, and $D_{(i,0)}$ only on $\kappa_1,\cdots,\kappa_{i-1},a_1,\cdots,a_i$.

P2: The derivative of the function

$$V_i(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_i)=V_0(\epsilon_0,\epsilon,r)+\sum_{j=1}^{i}\frac{p}{L_jr^{2j}}\tilde{x}_j^2,$$

for some positive number L_j, $j=1,\cdots,i$, along the trajectories of (3.9) and (4.5), satisfies

$$\begin{aligned}&\frac{dV_i(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_i)}{dt}\\ \le\ &-\|\epsilon_0\|^2-\frac{(n+1-i)r}{2(n+1)}\|\epsilon\|^2-\sum_{j=1}^{i}\frac{n-(i-j)}{n}\frac{1}{L_jr^{2j}}\\ &\left[r+\varpi\left(\frac{x_1}{r}\right)\right]\tilde{x}_j^2+\left[\frac{2C}{r^{2(i+1)}}+\frac{1}{r^{2i+1}}\right]\tilde{x}_{i+1}^2\\ &+\sum_{j=i+2}^{n}\frac{C\hat{x}_j^2}{r^{2j}}.\end{aligned}$$

Proof: The proof is omitted due to the space limit. ∎

A. Stability Analysis

Theorem 4.1: Under assumptions A2.1 and A2.2 and conditions (3.10) and (3.11), there exists an output feedback controller composed of (3.1), (3.5), and

$$\begin{aligned}u &= -\kappa_n r\tilde{x}_n\\ \tilde{x}_{i+1} &= \hat{x}_{i+1}+\kappa_i r\tilde{x}_i,\ i=1,\cdots,n-1,\\ \tilde{x}_1 &= \hat{x}_1,\end{aligned}\tag{4.6}$$

which solves the global stabilization problem of system (2.1) in the sense that, for all initial states $z(0)\in\Re^{n_z}$, $x(0)\in\Re^n$, $\hat{x}(0)\in\Re^n$, and $r(0)\ge 1$, the states of the closed-loop system are bounded for all $t\ge 0$, and $\lim_{t\to\infty}\mathrm{col}(z(t),x(t),\hat{x}(t))=0$.

Proof: Consider the controller (4.6) where $\kappa_1,\cdots,\kappa_n$ are designed in Lemma 4.1, and r is generated by (3.5). Let

$$V_n(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_n)=V_0(\epsilon_0,\epsilon,r)+\sum_{i=1}^{n}\frac{p}{L_ir^{2i}}\tilde{x}_i^2,$$

then the property P2 in Lemma 4.1 with $i=n$ becomes

$$\begin{aligned}&\frac{dV_n(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_n)}{dt}\\ \le\ &-\|\epsilon_0\|^2-\frac{r\|\epsilon\|^2}{2(n+1)}-\sum_{j=1}^{n}\frac{j}{n}\frac{1}{L_jr^{2j}}\left[r+\varpi\left(\frac{x_1}{r}\right)\right]\tilde{x}_j^2.\end{aligned}\tag{4.7}$$

Next we will prove that $r(t)$ is bounded, that is, $\lim_{t\to\infty}r(t)=r_\infty<\infty$. To this end, define

$$\bar{V}_n(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_n)=K_1\int_0^{V_n(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_n)}S_1(K_2\tau)d\tau$$

where $S_1(\cdot)$ is a nondecreasing function satisfying $S_1(\tau)>0$, $\forall\tau\ge 0$, and K_1, K_2 are positive numbers satisfying

$$\begin{aligned}K_1\left[\frac{\|\epsilon_1\|^2}{2(n+1)}+\frac{\tilde{x}_1^2}{nL_1r^2}\right] &\ge \frac{x_1^2}{r^2}\\ K_2\left[\lambda_{max}(P)\|\epsilon_1\|^2+\frac{p\tilde{x}_1^2}{L_1r^2}\right] &\ge \frac{x_1^2}{r^2},\end{aligned}$$

with $\lambda_{max}(P)$ being the maximal eigenvalue of P. Then

$$\begin{aligned}&\frac{d\bar{V}_n(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_n)}{dt}\\ \le\ &K_1S_1(K_2V_n)\Big\{-\|\epsilon_0\|^2-\frac{r}{2(n+1)}\|\epsilon\|^2\\ &-\sum_{j=1}^{n}\frac{j}{n}\frac{1}{L_jr^{2j}}\left[r+\varpi\left(\frac{x_1}{r}\right)\right]\tilde{x}_j^2\Big\}\\ \le\ &-S_1\left(K_2\left[\lambda_{max}(P)\|\epsilon_1\|^2+\frac{p\tilde{x}_1^2}{L_1r^2}\right]\right)\\ &\times K_1\left[\frac{\|\epsilon_1\|^2}{2(n+1)}+\frac{\tilde{x}_1^2}{nL_1r^2}\right]\\ \le\ &-S_1\left(\frac{x_1^2}{r^2}\right)\frac{x_1^2}{r^2}.\end{aligned}$$

On the other hand, let $r^*=2\varpi(0)$, and

$$V_r(r)=p\left[r-r^*-r^*\log(\frac{r}{r^*})\right],$$

which is continuously differentiable, proper, and non-negative in $(0,+\infty)$ [10]. We will prove that

$$\frac{dV_r(r)}{dt}\le\left[\varpi\left(\frac{x_1}{r}\right)-\varpi(0)\right]^2.\tag{4.8}$$

It suffices to consider the following two cases:

(i) $r>2\varpi\left(\frac{x_1}{r}\right)$: Clearly, (4.8) holds since $\frac{dV_r(r)}{dt}=0$.

(ii) $r \le 2\varpi\left(\frac{x_1}{r}\right)$: The inequality (4.8) holds from the following calculation,

$$\begin{aligned}\frac{dV_r(r)}{dt} &= r\left[\varpi\left(\frac{x_1}{r}\right)-\frac{r}{2}\right]-r^*\left[\varpi\left(\frac{x_1}{r}\right)-\frac{r}{2}\right]\\ &= (r-r^*)\left[\varpi\left(\frac{x_1}{r}\right)-\varpi(0)-\frac{r-r^*}{2}\right]\\ &\le -\frac{(r-r^*)^2}{2}+(r-r^*)\left[\varpi\left(\frac{x_1}{r}\right)-\varpi(0)\right]\\ &\le -\frac{(r-r^*)^2}{4}+\left[\varpi\left(\frac{x_1}{r}\right)-\varpi(0)\right]^2\\ &\le \left[\varpi\left(\frac{x_1}{r}\right)-\varpi(0)\right]^2. \qquad (4.9)\end{aligned}$$

From above, we obtain that

$$\begin{aligned}&\frac{d\left\{\bar{V}_n(\epsilon_0,\cdots,\epsilon_n,r,\tilde{x}_1,\cdots,\tilde{x}_n)+V_r(r)\right\}}{dt}\\ \le\ & -S_1\left(\frac{x_1^2}{r^2}\right)\frac{x_1^2}{r^2}+\left[\varpi\left(\frac{x_1}{r}\right)-\varpi(0)\right]^2\le 0\end{aligned}$$

by appropriate choice of $S_1(\cdot)$. As a result, $r(t)$ is bounded, hence, all states of the closed-loop system are bounded.

Now, let us return to (4.7). From the fact that $1 \le r(t) \le r_\infty$, we have

$$\begin{aligned}&\underline{\alpha}_0(\|\mathrm{col}(\epsilon_0,\epsilon)\|)+\sum_{i=1}^{n}\frac{p}{L_i r_\infty^{2i}}\tilde{x}_i^2\\ \le\ & V_n(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_n)\\ \le\ & \bar{\alpha}_0(\|\mathrm{col}(\epsilon_0,\epsilon)\|)+\sum_{i=1}^{n}\frac{p}{L_i}\tilde{x}_i^2,\end{aligned}$$

and

$$\begin{aligned}&\frac{dV_n(\epsilon_0,\epsilon,r,\tilde{x}_1,\cdots,\tilde{x}_n)}{dt}\\ \le\ & -\|\epsilon_0\|^2-\frac{1}{2(n+1)}\|\epsilon\|^2-\sum_{j=1}^{n}\frac{j}{n}\frac{1}{L_j r_\infty^{2j}}\tilde{x}_j^2.\end{aligned}$$

As a result, the closed-loop system (3.9)+(4.5), hence, (3.9)+(3.1), and hence, (2.1)+(3.1), with $u=-\kappa_n r\tilde{x}_n$, for any observer gain function $r(t)$ generated by (3.5), is globally asymptotically stable (In fact, globally exponentially stable). That is, $\lim_{t\to\infty}\mathrm{col}(z(t),x(t),\hat{x}(t))=0$. The proof is completed. ∎

B. Controller Design Algorithm

From the above procedure of the observer-controller design, the overall controller consists of (3.1), (3.5), and (4.6). We now determine the parameters $\kappa_1,\ldots,\kappa_n$, which can be shown to satisfy the following inequalities,

$$D_{(1,0)}=a_1,\ D_{(1,1)}=0,\ L_1=2p$$

$$\kappa_1 \ge \frac{n+1}{2}a_1^2+\frac{1}{p}+\frac{1}{4},$$

and for $I=1,\cdots,n-1$,

$$\begin{aligned}D_{(I+1,0)} &= a_{I+1}+\kappa_I D_{(I,0)}\\ D_{(I+1,j)} &= \kappa_I D_{(I,j)},\ j=1,\cdots,I-1\\ D_{(I+1,I)} &= \max\left\{\frac{\kappa_I}{2p}+\kappa_I^2+\kappa_I D_{(I,I)},\frac{\kappa_I}{p}\right\}\\ D_{(I+1,I+1)} &= \kappa_I,\end{aligned}$$

$$\begin{aligned}L_{I+1} &\ge \max\left\{2p,\frac{np^2D_{(I+1,j)}^2}{2}L_j, j=1,\cdots,I\right\}\\ \kappa_{I+1} &\ge D_{(I+1,I+1)}+\frac{n+1}{2}D_{(I+1,0)}^2\\ &\quad+\frac{3I+2+L_{I+1}}{2p}+\frac{1}{4}.\end{aligned}$$

And the function $\varpi(\cdot)$ satisfies

$$\begin{aligned}\varpi(s) &\ge 2\|P\|\sqrt{n}c(s)+(n/\delta+1)\|P\|^2\left[c(s)\right]^2+4c_z\\ \varpi(s) &\ge 2C(1+2\kappa_1^2)p\\ \varpi(s) &\ge 2C\left(1+\kappa_{I+1}^2\right)L_{I+1},\ I=1,\cdots,n-1\end{aligned}$$

where

$$\begin{aligned}C &\ge \delta+4c_z\\ c(s) &\ge \max\nolimits_{j=0}^{n} c^j(s),\end{aligned}$$

for any $\delta>0$ (δ is introduced to optimize the controller, for convenience, let $\delta=1$), and the functions $c^j(\cdot)$'s are continuous non-negative satisfying

$$c^0\left(\frac{x_1}{r}\right)\ge\sum_{i=1}^{n}\frac{c_i^0(x_1)}{r^{i-1}},\quad c^j\left(\frac{x_1}{r}\right)\ge\sum_{i=j}^{n}\frac{c_i^j(x_1)}{r^{i-j}}\quad j=1,\cdots,n.$$

It can be proved that all the above inequalities are not conflicted and are explicitly solvable.

V. An Example

Example 5.1: Consider the system

$$\begin{aligned}\dot{z} &= -z+0.1x_1\\ \dot{x}_1 &= x_2\\ \dot{x}_2 &= u+\mu_1(t)\sin(x_1)z+\mu_2x_1^2\\ y &= x_1 \qquad (5.1)\end{aligned}$$

where $\mu_1(t):[0,\infty)\mapsto[-0.1,0.1]$ and $\mu_2\in[-0.1,0.1]$.

Since the system contains uncertainties $\mu_1(t)$ and μ_2, and nonlinear growing term x_1^2, the global output feedback stabilization problem cannot be solved by the existing results, e.g., [10] and [12]. But, it is verified the assumption A2.1 holds for

$$\begin{aligned}&c_1^0(x_1)=c_1^1(x_1)=0\\ &c_2^0(x_1)=0.1, c_2^1(x_1)=0.1|x_1|, c_2^2(x_1)=0.\end{aligned}$$

Assumption A2.2 holds for $V_z(z)=z^2$ and $c_z=0.01$. And it is easy to verify that the conditions (3.10) and (3.11) are satisfied. Then by Theorem 4.1, the global output feedback stabilization problem can be solved by the method given in this paper.

Specifically, the controller can be explicitly constructed by

$$\begin{aligned} u &= -\kappa_2 r(\hat{x}_2 + \kappa_1 r \hat{x}_1) \\ \dot{\hat{x}}_1 &= \hat{x}_2 + a_1 r(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= u + a_2 r^2 (y - \hat{x}_1) \\ \dot{r} &= \frac{r}{p}\text{ramp}\left[\varpi\left(\frac{x_1}{r}\right) - \frac{r}{2}\right],\ r(0) = 1. \end{aligned} \quad (5.2)$$

Pick $a_1 = 0.1$, $a_2 = 0.3$ then

$$A = \begin{bmatrix} -0.1 & 1 \\ -0.3 & 0 \end{bmatrix}$$

is Hurwitz and the solution of the Lyapunov equation (3.2) is

$$P = \begin{bmatrix} 6.5 & -0.5 \\ -0.5 & 21.8333 \end{bmatrix},$$

with $\|P\| = 21.9496$. Since

$$P(B+I) + (B+I)P = \begin{bmatrix} 13 & -1.5 \\ -1.5 & 87.3333 \end{bmatrix},$$

it is sufficient to pick $p = 2$. And then other parameters are designed following the control design algorithm as $\kappa_1 = 0.765$, $\kappa_2 = 7$, $\varpi(s) = 110s^2 + 300$. The computer simulation results, e.g., $x_1, \hat{x}_1, x_2, \hat{x}_2$, are depicted in Figures 1-2 when $\mu_1(t) = 0.1 \sin t, \mu_2 = 0.08$, $(z(0), x_1(0), x_2(0)) = (3, 2, 3)$, and $(\hat{x}_1(0), \hat{x}_2(0)) = (0, 0)$. ∎

VI. Conclusion

In this paper, we have solved the global stabilization problem by output feedback for a class of uncertain systems with output dependent incremental rate. The method can be extended to solve the global robust servomechanism problem by output feedback control for the same class of uncertain nonlinear systems, and a preliminary investigation of this problem has been given in an accompanying paper [2].

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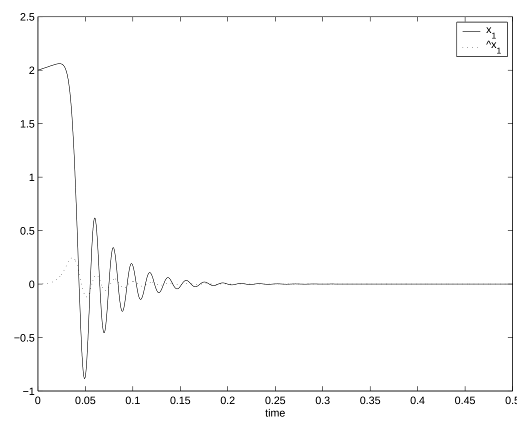

Fig. 1. Profile of states x_1 and $\hat{x}_1$

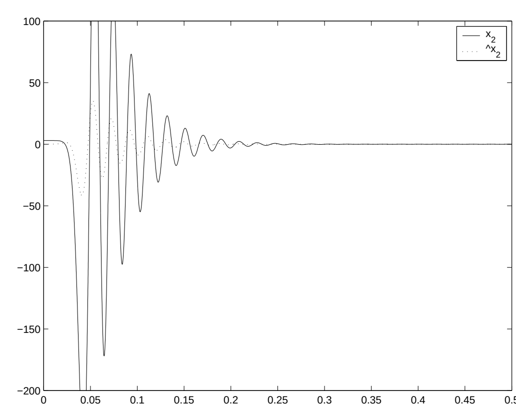

Fig. 2. Profile of states x_2 and $\hat{x}_2$