Interconnection and damping assignment passivity–based control of mechanical systems with underactuation degree one

José Ángel Acosta, Romeo Ortega and Alessandro Astolfi

***Abstract*— We consider the problem of (asymptotic) stabilization of mechanical systems with underactuation degree one. A state–feedback design is derived applying the Interconnection and Damping Assignment Passivity-Based Control methodology. Its application relies on the possibility of solving a set of partial differential equations that identify the energy functions that can be assigned to the closed–loop. The following results are established: 1) identification—in terms of some *algebraic inequalities*—of a subclass of these systems for which the partial differential equations are trivially solved; 2) characterization of all systems which are feedback–equivalent to this subclass; and 3) introduction of a suitable parametrization of the assignable energy functions that provides the designer with a handle to address transient performance and robustness issues. An additional feature of our developments is that the open–loop system need not be described by a port–controlled Hamiltonian (or Lagrangian) model, a situation that arises often in applications due to model reductions or preliminary feedbacks that destroy the structure. The new result is applied to obtain an (almost) globally stabilizing controller for the inertia wheel pendulum, a controller for the chariot with pendulum system that can swing–up the pendulum from any position in the upper half plane and stop the chariot at any desired location, and an (almost) globally stabilizing scheme for the vertical takeoff and landing aircraft with strong input coupling. In all cases we obtain very simple and intuitive solutions that do not rely on, rather unnatural and technique–driven, linearization or decoupling procedures but instead endows the closed–loop system with a Hamiltonian structure with desired potential and kinetic energy functions.**

I. INTRODUCTION

In [1] we introduced a controller design technique, called Interconnection and Damping Assignment Passivity–Based Control (IDA–PBC), that achieves stabilization for underactuated mechanical systems invoking the physically motivated principle of *energy shaping*. IDA–PBC endows the closed–loop system with a Port–Controlled Hamiltonian (PCH) structure where the kinetic and potential energy functions have some desirable features, a minimal requirement being to have a minimum at the desired operating point to ensure its stability. Similar techniques have been reported for general PCH and Lagrangian systems in [2], [3] and [4], [5], respectively. The success of these methods relies on the possibility of solving a set of partial differential equations (PDEs) that identify the energy functions that can be assigned to the closed–loop. In spite of many interesting developments, e.g. [6], [7], [4], [8], the need to solve the PDEs remains the main stumbling block for a wider applicability of these methods.

In this paper we show that for a class of mechanical systems with *underactuation degree one* it is possible to trivially solve the PDEs of the IDA-PBC design method, we provide a characterization of those systems which are feedback equivalent to systems in the considered class, and we show that the set of assignable energy functions can be simply parameterized.

For illustration we present an (almost) globally stabilizing scheme for the inertia wheel pendulum, an (almost) globally stabilizing scheme for the vertical takeoff and landing aircraft with strong input coupling, and a controller for the chariot with the pendulum that can swing–up the pendulum from any position in the (open) upper half plane and stop the chariot at any desired location.

This is an abridged version of the full paper which is available, upon request, from the authors.

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II. THE PDES FOR A CLASS OF MECHANICAL SYSTEMS WITH UNDERACTUATION DEGREE ONE

The class of systems that we consider is given by

$$\begin{aligned} \dot q &= M^{-1}(q_r)p \\ \dot p &= s(q_r) + G(q_r)u, \end{aligned} \tag{1}$$

where q_r, with r an integer taking values in the set $\{1, \ldots, n\}$, is a distinguished element of $q \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $u \in \mathbb{R}^{n-1}$ are the control inputs, the matrix $M(q_r)$ is symmetric positive definite and bounded, and $s(q_r), G(q_r)$ are analytic functions of q_r, and we assume that $G(q_r)$ is full column rank. Notice that the system has underactuation degree one.

The control objective is to stabilize an equilibrium $(q_*, 0)$. In IDA–PBC of mechanical systems [1] this is achieved assigning to the closed–loop the total energy

$$H_d(q,p) = \frac{1}{2}\, p^\top M_d^{-1}(q_r)\; p + V_d(q) \tag{2}$$

with $M_d(q_r) = M_d^\top(q_r) > 0$ and $V_d(q)$, with $q_* = \arg\min V_d(q)$, the (to be defined) closed–loop inertia matrix and potential energy function, respectively. For, we endow the system with a PCH structure of the form

$$\begin{bmatrix} \dot q \\ \dot p \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_dM^{-1} & J_2 - GK_vG^\top \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \quad (3)$$

where $K_v = K_v^\top > 0$, $J_2(q_r, p) = -J_2^\top(q_r, p)$ are *free* matrices.

As shown in [1] the assignable energy functions are characterized by a set of PDEs. We now show that for the considered class of systems the PDEs take a special and simple form, and can be trivially solved. This paves the way for a constructive state feedback stabilization result.

Proposition 1: For the system (1) with desired energy function (2) the PDE from the IDA-PBC method takes the form

$$G^\perp M_d(q_r)M^{-1}(q_r)e_r M_d'(q_r) = -2\mathcal{J}\mathcal{A}^\top(q_r) \quad (4)$$

$$G^\perp \left[s(q_r) + M_d(q_r)M^{-1}(q_r)\nabla_q V_d(q)\right] = 0 \quad (5)$$

where $\mathcal{J}(q_r) \in \mathbb{R}^{n\times n_o},\ n_o \triangleq \frac{n}{2}(n-1)$, is a *free* matrix, $e_r \in \mathbb{R}^n$ is the r–th vector of the standard Euclidian basis, and

$$\mathcal{A}(q_r) \triangleq \left[W_1^\top \left(G^\perp(q_r)\right)^\top, \ldots, W_{n_o}^\top \left(G^\perp(q_r)\right)^\top\right] \in \mathbb{R}^{n\times n_o} \quad (6)$$

where $W_i \in \mathbb{R}^{n\times n},\ i = 1, \ldots, n_o$, are some constant, skew–symmetric matrices which can be explicitely constructed.

Remark 1: An $n \times n$ skew–symmetric matrix contains *at most* n_o non–zero different terms. Hence, the proposed $J_2(q_r, \tilde p)$ contains all skew–symmetric matrices which are linear in $\tilde p$, that is, all matrices of the form $\sum_{i=1}^n \Omega_i(q_r)\tilde p_i,\ \Omega_i(q_r) = -\Omega_i^\top(q_r)$, and the parametrization is done without loss of generality.

All assignable energy functions of the form (2) are characterized by the solutions of (4) and (5). Typically in IDA–PBC we start with (4), which is a set of *nonlinear* ODEs in the unknown matrix $M_d(q_r)$, with $\mathcal{J}(q_r)$ a free matrix to be chosen by the designer. Then, plugging in $M_d(q_r)$ in (5), we solve the PDE for $V_d(q)$. It is important to recall that, to comply with the stability requirements, we also have to satisfy the additional constraints of positivity of $M_d(q_r)$ and the minimum condition $q_* = \arg\min V_d(q)$.

Even though we have full freedom in the selection of $\mathcal{J}(q_r)$ finding a solution of (4) is nontrivial because the matrix $\mathcal{A}(q_r)$ is not full rank—in particular we have that

$$G^\perp(q_r)\mathcal{A}(q_r) = 0. \quad (7)$$

III. Main result

There are no systematic methods for the solution of nonlinear ODEs. In spite of this it is possible to show that with a suitable parametrization of the desired inertia matrix, the PDE (4) is obviated; with the additional advantage that (5) becomes a trivial linear PDE that we can explicitly solved. This result is used to give stabilization conditions in terms of a set of *algebraic inequalities*. Thus, for the class (1) considered, the complete solution of the PDE and the IDA-PBC controller are given in an explicit formula.

Proposition 2: Consider the system (1). Assume there exists matrices $\Psi(q_r)$ and $M_d^0 = (M_d^0)^\top > 0$ and $G^\perp(q_r)$, with $G^\perp(q_r)G(q_r) = 0$, such that

Assumption A.1

$$\rho \triangleq G^\perp(q_{r*})M_d(q_{r*})M^{-1}(q_{r*})e_r \neq 0,$$

where e_r is the r–th vector of the Euclidean basis and

$$M_d(q_r) = \int_{q_{r*}}^{q_r} G(\mu)\Psi(\mu)G^\top(\mu)d\mu + M_d^0. \quad (8)$$

Assumption A.2

$$-\frac{1}{\rho}\left(\frac{d}{dq_r}\{G^\perp s\}\right)(q_{r*}) > 0.$$

Under these conditions:

- The IDA–PBC takes the form
$$\begin{aligned} u \;&=\; A_1(q)PS(q-q_*) + \begin{bmatrix} p^\top A_2(q_r)p \\ \vdots \\ p^\top A_n(q_r)p \end{bmatrix} + \\ &+\; A_{n+1}(q_r) - K_vG^\top(q_r)M_d^{-1}(q_r)p \end{aligned} \quad (9)$$
where $P = P^\top > 0$ and $S \in \mathbb{R}^{(n-1)\times n}$ is obtained removing the r–th row from the n–dimensional identity matrix.
- The total energy function (2) is defined with (8) and
$$\begin{aligned} V_d(q) \;&=\; -\frac{1}{\rho}\int_0^{q_r} G^\perp(\mu)s(\mu)d\mu + \\ &+\; \frac{1}{2}\left[z(q)-z(q_*)\right]^\top P\left[z(q)-z(q_*)\right] \end{aligned} \quad (10)$$
where $z(q)$, is an $n-1$ dimensional vector with elements
$$z_i \triangleq q_i - \frac{1}{\rho}\int_0^{q_r} G^\perp(\mu)M_d(\mu)M^{-1}(\mu)e_i d\mu \quad (11)$$
- $(q_*, 0)$ is a *stable* equilibrium with Lyapunov function $H_d(q,p)$.

The equilibrium $(q_*, 0)$ is *asymptotically* stable if furthermore

Assumption A.3 There exists at least one column of the matrix $G(q_r) = \{G_{ij}\}$ such that the following two conditions *cannot happen simultaneously*: (i) $G_{rj} \equiv 0$; and (ii) there exists constants c_i such that $\sum_{i=1, i\neq r}^n G_{ij}c_i \equiv 0$.

Remark 2: Assumption A.1 is needed to trivialize the solution of the PDEs. Although this (pointwise) assumption is generically satisfied, the computation of the controller involves a division by $G^\perp(q_r)M_d(q_r)M^{-1}(q_r)e_r$. From

(10) we see that Assumption A.2 ensures that the potential energy attains its minimum at the desired point.[1]

Remark 3: The set of *assignable energy functions* of the form (2) that lead to a stabilizing controller is parameterized by all triplets $\{\Psi, M_d^0, \rho\}$ that satisfy the conditions of Proposition 2. Moreover, it can be shown that the second term in (10) can be any differentiable function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $z(q_*) = \arg\min \Phi(z)$.

IV. Characterization of the class

A natural question that arises at this point is what is the class of underactuation degree one mechanical systems that can be transformed, via canonical change of coordinates and state feedback, into the form (1).[2] We say then that the mechanical system is feedback–equivalent to (1). A complete answer to this question is provided in the proposition below. For brevity we present only the case where $M(q_r) = I$, the general case is discussed in Remark 4.

Proposition 3: Consider the classical underactuation degree one (simple) mechanical system

$$\tilde{D}\ddot{y} + \dot{\tilde{D}}\dot{y} - \frac{1}{2}\nabla_y \left(\dot{y}^\top \tilde{D}\dot{y}\right) + \nabla_y V(y) = \begin{bmatrix} I_{n-1} \\ 0 \cdots 0 \end{bmatrix} \tilde{w}$$

where $y \in \mathbb{R}^n$ are the generalized coordinates, $\tilde{w} \in \mathbb{R}^{n-1}$ the controls, $\tilde{D}(y) = \tilde{D}^\top(y) > 0$ the inertia matrix (the argument is omitted) and $V(y)$ the potential energy function. The system is *globally feedback equivalent* to (1) with $M(q_r) = I$ *if and only if* there exists a function $\psi : \mathbb{R}^n \to \mathbb{R}^n$, which is a global diffeomorphism, solution of the set of second order homogeneous PDE's [3]

$$\sum_{i=1}^{n-1} d_i(\psi)\nabla_q^2\psi_i + \nabla_q^2\psi_n + \nabla_q^\top \psi F_1(\psi)\nabla_q\psi = 0 \qquad (12)$$

such that the algebraic equations

$$\begin{aligned} \big[d_1(\psi)\ \cdots\ d_n(\psi)\ 1\big]\nabla_q\psi G(q_r) &= 0 \\ \big[d_1(\psi)\ \cdots\ d_n(\psi)\ 1\big]\nabla_q\psi s(q_r) &= -F_0(\psi) \end{aligned} \qquad (13)$$

are satisfied for some integer $r \in \{1, \cdots, n\}$, and some functions $s(q_r), G(q_r)$, where the scalar functions $d_i(\psi(q)), F_0(\psi(q))$ and the matrix $F_1(\psi(q))$ are determined by $\tilde{D}(y), V(y)$.

Remark 4: For the more general case where $M(q_r) \neq I$, the algebraic constraints remain unaltered. However, the matrix $M(q_r)$ provides a new degree of freedom that appears in the PDE (12) in the form of a *free term* which is linear in $\nabla_q\psi$.

[1] Assumption A.2 is sufficient for injectivity of $G^\perp(q_r)s(q_r)$ which, from Brockett's condition, is *necessary* for stabilization.

[2] A change of coordinates for a mechanical systems is canonical if it maps positions into positions.

[3] The argument of $\psi(q)$ is omitted for compactness.

V. Examples

A. The inertia wheel pendulum

The dynamic equations can be written in the simplified description

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \begin{bmatrix} m_3 \sin(q_1) \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \end{aligned}$$

where $q, p \in \mathbb{R}^2,\ u \in \mathbb{R}$, and $I_1,\ I_2$ are the respective angles and moments of inertia of the pendulum disk, where $m_3 \triangleq mgl$ with m the pendulum mass, l its length, g the gravity constant, and for simplicity, we have taken $I_1 = I_2 = 1$. The equilibrium to be stabilized is the upward position with the inertia disk aligned, which corresponds to $q_{1*} = q_{2*} = 0$.

Assumption A.3 is obviously verified. We will select now the terms of the triplet $\{\Psi, M_d^0, \rho\}$ that satisfy Assumptions A.1–A.2. In this simple case we can take the desired inertia matrix to be constant, henceforth we set the first parameter $\Psi = 0$ to yield

$$M_d = M_d^0 = \begin{bmatrix} m_{11}^0 & m_{12}^0 \\ m_{12}^0 & m_{22}^0 \end{bmatrix}.$$

The (non–trivial) left annihilators of G are given by $G^\perp = \eta[1,\ 1]$, with $\eta \neq 0$. To satisfy Assumption A.1 we impose

$$m_{11}^0 + m_{12}^0 \neq 0. \qquad (14)$$

The positivity condition for the inertia matrix is clearly

$$m_{11}^0 > 0, \qquad m_{11}^0 m_{22}^0 > \left(m_{12}^0\right)^2. \qquad (15)$$

We have that

$$\eta = \frac{\rho}{m_{11}^0 + m_{12}^0},$$

for some number ρ. Now, as $G^\perp s(q_1) = \frac{m_3\rho}{m_{11}^0 + m_{22}^0}\sin(q_1)$, we need to verify the inequality

$$m_{11}^0 + m_{12}^0 < 0, \qquad (16)$$

to assign the minimum to $V_d(q)$, because ρ cancels. Finally, we select $\Phi(z(q)) = \frac{P}{2}z^2(q)$, where

$$z(q) = q_2 - \rho\frac{m_{12}^0 + m_{22}^0}{m_{11}^0 + m_{12}^0}q_1$$

is directly computed from (11), to obtain from (10)

$$V_d(q) = \frac{m_3}{m_{11}^0 + m_{12}^0}\cos q_1 + \frac{P}{2}\left(q_2 - \rho\frac{m_{12}^0 + m_{22}^0}{m_{11}^0 + m_{12}^0}q_1\right)^2$$

The conditions on the m_{ij}^0 coefficients (14)–(16) exactly coincide, for $\rho = 1$, with those of [9] where the almost globally stabilizing controller derived for this example requires four pages of (painful) computations.

B. Pendulum on a cart

The dynamic equations can be put in the desired form

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= a \sin q_1 e_1 + \begin{bmatrix} -b\cos q_1 \\ 1 \end{bmatrix} u \end{aligned} \tag{17}$$

where $q, p \in \mathbb{R}^2,\ u \in \mathbb{R},\ a = \frac{g}{l},\ b = \frac{1}{l}$, with l the length of the pendulum. Notice that $G^{\perp}(q_1) = \eta(q_1)[1,\ b\cos q_1]$, where $\eta(q_1)$ is a function to be defined. The equilibrium to be stabilized is the upward position of the pendulum with the cart placed in *any desired location*, corresponding to $q_{1*} = 0$ and arbitrary q_{2*}.

It is possible to show that, to satisfy the conditions of Proposition 2, $\eta(q_1)$ *cannot be a constant*. Hence, we compute $M_d(q_1)$ from (8) to get

$$\begin{aligned} G^{\perp}(q_1) \quad & M_d(q_1)e_1 = \eta(q_1)\big[b^2\big(\int_{q_{1*}}^{q_1} \Psi(\mu)\cos^2\mu d\mu \\ - \cos q_1 \quad & \int_0^{q_1} \Psi(\mu)\cos\mu d\mu\big) + m_{11}^0 + m_{12}^0 b\cos q_1\big]. \end{aligned}$$

We have to select a function $\Psi(q_1)$ so that the term in brackets (evaluated at zero) is bounded away from zero (Assumption A.1) and can be explicitly integrated. The first condition allows to define $\eta(q_1)$ such that $G^{\perp}(q_1)M_d(q_1)e_1 = \rho$, while the second one is needed to compute the control law. It is easy to see that $\Psi(q_1) = const$ is, unfortunately, *not adequate*. We propose then $\Psi(q_1) = -k\sin q_1$, with $k > 0$ a parameter to be determined, and select $m_{11}^0 = \frac{kb^2}{3}$, $m_{12}^0 = -\frac{kb}{2}$. This leads to

$$M_d = \begin{bmatrix} \frac{kb^2}{3}\cos^3 q_1 & -\frac{kb}{2}\cos^2 q_1 \\ -\frac{kb}{2}\cos^2 q_1 & k(\cos q_1 - 1) + m_{22}^0 \end{bmatrix} \tag{18}$$

This matrix is positive definite and bounded for all $q_1 \in (\frac{-\pi}{2}, \frac{\pi}{2})$, provided $m_{22}^0 > k$. Also, we can take

$$\eta(q_1) = -\frac{6\rho}{kb^2\cos^3 q_1}.$$

Assumption A.2 is verified noting that

$$\frac{1}{\rho}G^{\perp}(q_1)s(q_1) = -\frac{6a}{kb^2}\frac{\sin q_1}{\cos^2 q_1}.$$

Finally, Assumption A.3 is obviously satisfied.

Proposition 4: A set of energy functions of the form (2) assignable via IDA–PBC to system (17) is characterized by the *locally* positive definite and bounded inertia matrix (18), with $m_{22}^0 > k$, and the potential energy function

$$V_d(q) = \frac{3a}{kb^2\cos^2 q_1} + \frac{P}{2}z^2(q)$$

where

$$z(q) = q_2 - q_{2*} + \frac{3}{b}\ln(\sec q_1 + \tan q_1) + \frac{6m_{22}^0}{kb}\tan q_1.$$

The IDA–PBC's with full state–feedback (9) ensure *asymptotic stability* of the desired equilibrium $(0, q_{2*}, 0, 0)$, with a domain of attraction of the former containing the set $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^3$.

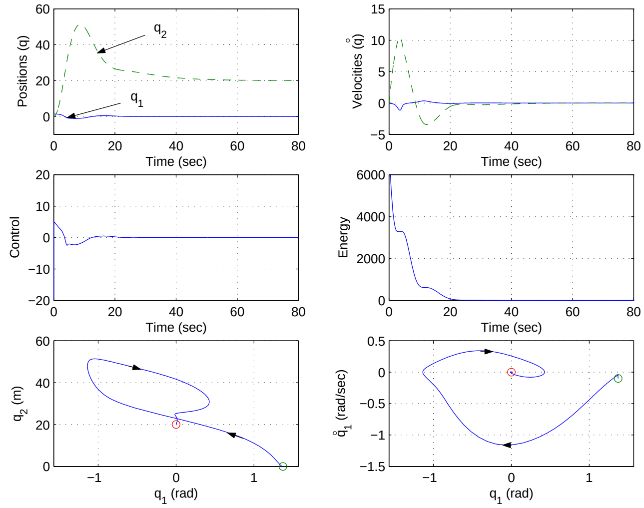

Fig. 1. Trajectories with the pendulum starting near the horizontal $[q(0), p(0)] = [\pi/2 - 0.2, -0.1, 0.1, 0]$, full state feedback.

Simulations were made with $a = b = 1$, $K_v = 0.01$, $m_{22}^0 = k = 0.01$ and $P = 1$. We tested a set of "limiting" initial conditions with the pendulum starting near the horizontal $[q(0), p(0)] = [\pi/2 - 0.2, -0.1, 0.1, 0]$ and the desired position for the cart $q_{2*} = 20$. The result for *full state–feedback* (Fig. 1) shows an excellent performance.

C. Vertical takeoff and landing aircraft

The dynamics may be written as

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \frac{g}{\epsilon}\sin q_3 e_3 + G(q_3)u \end{aligned} \tag{19}$$

where $q, p \in \mathbb{R}^3,\ u \in \mathbb{R}^2$, and we defined the matrix

$$G(q_3) \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{\epsilon}\cos q_3 & \frac{1}{\epsilon}\sin q_3 \end{bmatrix}.$$

The control requirement is the asymptotic stabilization of all equilibria of the form $(q_{1*}, q_{2*}, 0, 0, 0, 0)$.

Proposition 5: A set of energy functions of the form (2) assignable via IDA–PBC to system (19) is characterized by the *globally* positive definite and bounded inertia matrix

$$M_d = \begin{bmatrix} k_1\epsilon\cos^2 q_3 + k_3 & k_1\epsilon\cos q_3\sin q_3 & k_1\cos q_3 \\ k_1\epsilon\cos q_3\sin q_3 & -k_1\epsilon\cos^2 q_3 + k_3 & k_1\sin q_3 \\ k_1\cos q_3 & k_1\sin q_3 & k_2 \end{bmatrix}$$

with $k_1 > 0,\ k_3 > 5k_1\epsilon,\ k_1 > k_2\epsilon > \frac{k_1}{2}$, and the potential energy function

$$V_d(q) = -\frac{g}{k_1 - k_2\epsilon}\cos q_3 + \frac{1}{2}z^{\top}(q)Pz(q)$$

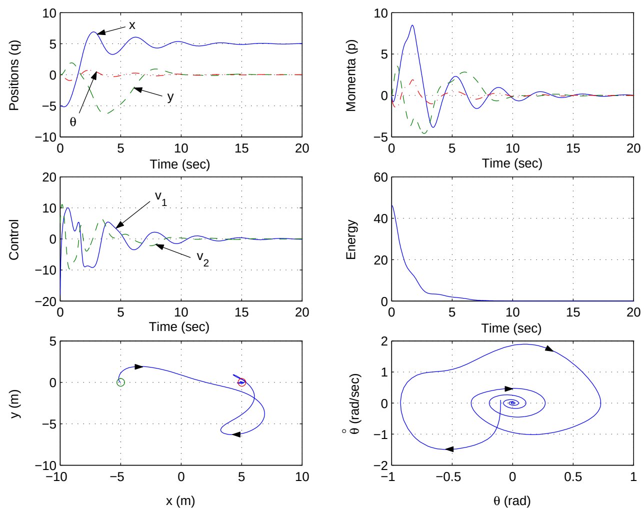

Fig. 2. Badly tuned controller for VTOL. Initials conditions $[q(0), p(0)] = [-5, 0, 0.1, -0.1, -0.1, 0.1]$. References $q_{1*} = 5$ and $q_{2*} = q_{3*} = 0$.

where

$$z(q) = \begin{bmatrix} q_1 - q_{1*} - \frac{k_3}{k_1 - k_2\epsilon} \sin q_3 \\ q_2 - q_{2*} + \frac{k_3 - k_1\epsilon}{k_1 - k_2\epsilon}(\cos q_3 - 1) \end{bmatrix}.$$

The full–state feedback IDA–PBC's ensure *almost global asymptotic stability* of the desired equilibrium $(q_{1*}, q_{2*}, 0, 0, 0, 0)$.

Simulations were carried out with a twofold objective, first to show how the energy shaping controller proposed in this paper ensures a satisfactory response for strong coupling coefficients $\epsilon > 0$, and second to illustrate the tuning flexibility provided by the design parameters. All simulations are made with a strong value of coupling $\epsilon = 1$. The damping injection matrix was fixed to

$$K_v = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}.$$

The normal conditions of maneuvering for the VTOL aircraft is to keep an accurate lateral motion near the ground. This problem has been normally solved in two steps (see for instance [10]): decoupling the altitude output from the lateral motion and rolling moment by means of a pre-feedback control law and then, designing a control law for the new decoupled system; this procedure renders satisfactory results for small enough ϵ. Now, with the energy shaping controller independently of the value of ϵ it is possible to "virtually decouple" the outputs using the weighting matrix P in the potential energy (10). To illustrate this point two simulations were made, first with a "bad" potential energy taking P diagonal and the weights equal to 1 and $1/10$. This simulation for a lateral motion is shown in Fig. 2. The same simulation was made for a "good" potential energy taking the P again diagonal but with the weights now $1/2$ and 1, with the response shown in Fig. 3—notice the different scales in the graphs. The posture of the VTOL aircraft along the trajectory for both cases is shown (at the same scale) in Fig. 4. It can be seen that, for the first case, the altitude ($q_2 = y$) makes very large excursions to drive the VTOL to rest, while in the second one a simple slow amplitude rocking motion achieves the objective.

The simulations, depicted in Fig. 5 and Fig. 6, show the time behavior an posture for the VTOL respectively, in an aggressive maneuver, from a limit upside down position for the roll angle (q_3), and a great step on the lateral motion (q_1) and altitude (q_2).

Remark 5: [Robustness] Recent results [11] show that the full–state feedback IDA–PBC controller ensures *almost global asymptotic stability* even with dynamics friction in the model (19). With a suitable selection of the controller gains $k_i, \ i = 1, 2, 3,$ the controller is able to dominate the undesirable friction effects.

VI. Conclusions

In this paper we have identified a class of underactuated mechanical systems for which the IDA–PBC design methodology gives a complete solution to the full–state feedback stabilization problem—without the need to solve any PDE. The main assumptions made on the system are that it has underactuation degree one and that, roughly speaking, the dynamics are determined by only one generalized coordinate. A complete characterization of all mechanical systems which are feedback equivalent to this class is also given. This class contains several practically interesting benchmark examples some of which are studied in the paper. Besides ensuring asymptotic stability the IDA–PBC methodology provides the designer with some degrees of freedom to improve the transient performance and the robustness. These degrees of freedom are given in

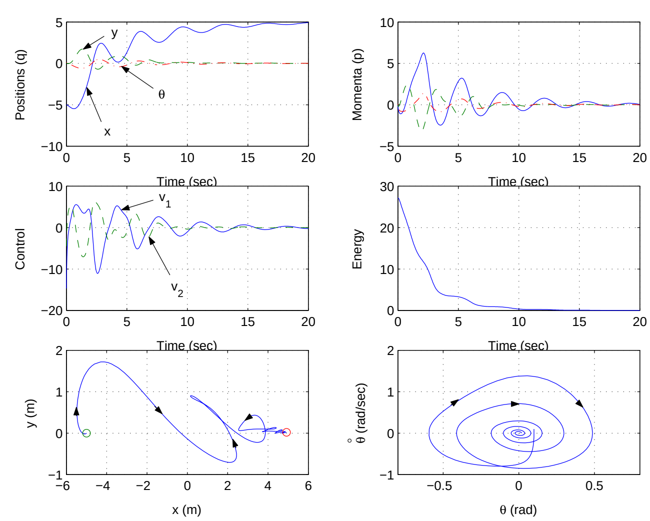

Fig. 3. Well tuned controller for VTOL; $q = (x, y, \theta)$. Initials conditions $[q(0), p(0)] = [-5, 0, 0.1, -0.1, -0.1, 0.1]$. References $q_{1*} = 5$ and $q_{2*} = q_{3*} = 0$.

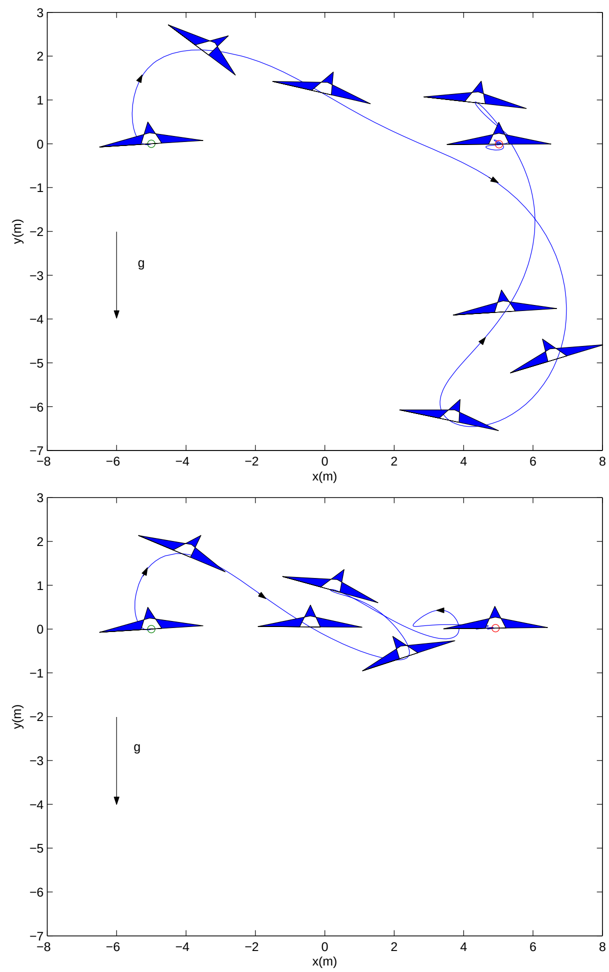

Fig. 4. Posture of the VTOL along the trajectory; $q = (x, y, \theta)$. *Badly* tuned controller (top) and well tuned controller (bottom).

terms of parameterized expressions of the assignable energy functions.

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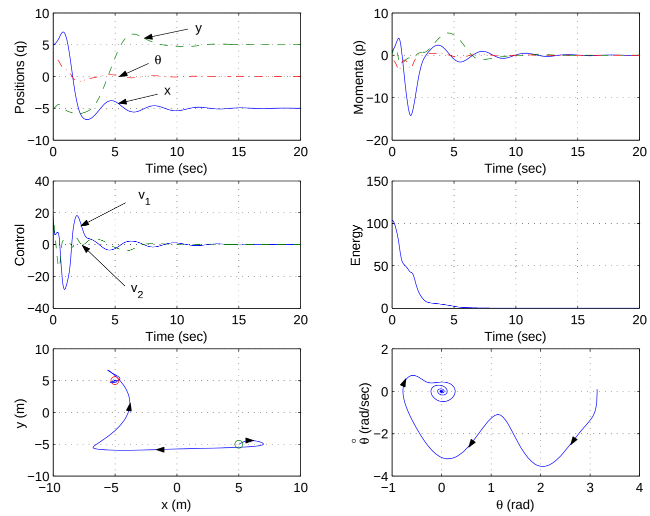

Fig. 5. Upside down simulation; $q = (x, y, \theta)$. Initials conditions $[q(0), p(0)] = [5, -5, \pi, 0.1, -0.1, 0.1]$. References $q_{1*} = -5$ and $q_{2*} = 5$ and $q_{3*} = 0$.

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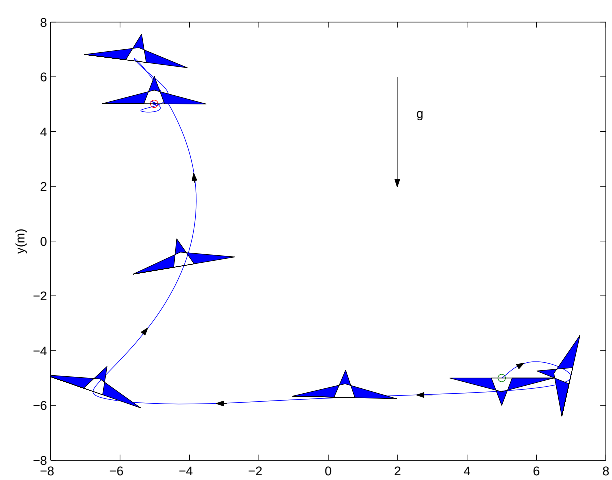

Fig. 6. Upside down simulation. Posture of the VTOL along the trajectory.