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FORMATIONS OF FORMATIONS: HIERARCHY AND STABILITY

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Abstract— **In this paper we will consider a hierarchy of vehicle formations exchanging information among themselves with the intention of completing a given mission at pre-specified coordinates in space. The objective is to understand how the communication infrastructure can be used to construct a decentralized controller and how it influences the stability of hierarchical vehicle formations. Our main result is that stability at higher levels of the hierarchy is guaranteed for certain vehicle communication infrastructures.**

I. INTRODUCTION

Recently the theory of vehicle formations has been explored in great detail. Vicsek et al. in [14] consider self-propelled particles using a neighbor-averaging law that leads to a common heading. In [3], [4] conditions based on a Nyquist criterion are developed that allow us to predict when the dynamical system describing a vehicle formation will be stable. Glavaski et al. in [6], [7] explore the convergence to formation in the case where the transmission connections between vehicles break randomly. In [10], [11] the convergence to formation is looked at when the communication graph depends on the relative physical position of agents. One of the applications of interest to us is being able to analyse stability of multiple formations of uninhabited autonomous vehicles (UAV's) by considering the communication among them while performing a given mission.

In this work we are concerned with the following problem: Given a group of homogeneous vehicle formations, starting at random positions, with given dynamics and the ability to transmit, or receive, information to, or from, each other, how can a communication graph be found so that all the vehicles achieve and maintain a pre-specified overall formation? Also, can the vehicles simultaneously converge to a desired fixed position or heading? In this context, a natural first step is to investigate if, given the initial vehicle formations, a communication graph that guarantees stability of the entire system is possible. The tools used for the stability analysis are graph theoretic and involve the spectral properties of the communication graph. For ease of understanding, the initial formations are called "subformations" and in our examples the given vehicle subformations are in R^2, however this can be extended to higher dimensions.

The paper is organized as follows. In Section 2 we provide the necessary background in algebraic graph theory, in Section 3 we introduce the setup for the hierarchical formations problem and analyze the stability of the formation, in Section 4 we extend this concept by considering convergence to a pre-specified position or trajectory.

II. ALGEBRAIC GRAPH THEORY

The information exchange between vehicles can be modelled by a directed graph, where the arrows show the direction in which position and velocity information is received by a vehicle from its "neighbor". We refer to vehicles as being neighbors in the information exchange graph sense, and not necessarily in the physical sense.

Definition 2.1: A directed graph G consists of a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$. For an edge $e = (u, v) \in E(G)$, u is called the head vertex of e and v is called the tail vertex of e.

If $(u, v) \in E(G)$ for all $(v, u) \in E(G)$, then we call the graph undirected. We call the graph simple if there are no edges of the form (u, u) for $u \in V(G)$. Let G be a graph representing the communication links between a set of vehicles. The properties of such a communication graph are captured by its adjacency matrix $A_d(G)$ defined by:

Definition 2.2: The adjacency matrix of a graph G, denoted $A_d(G)$, is a square matrix of size $|V(G)| \times |V(G)|$ defined as follows:

$$A_d(G)_{ij} = \begin{cases} 1 & \text{if } (u_i, u_j) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

where $u_i, u_j \in V(G)$.

Definition 2.3: The indegree of a vertex u is the number of edges that have u as their head vertex. The indegree matrix of a graph G, denoted $D(G)$, is a diagonal matrix of size $|V(G)| \times |V(G)|$ defined as follows:

$$D(G)_{ii} = \text{indegree}(u_i)$$

where $u_i \in V(G)$.

We analogously define the outdegree of a vertex u for a graph G, as the number of edges having u as the tail vertex.

Definition 2.4: Given a graph G, the Laplacian matrix associated with it is given by

$$L = D(G) - A_d(G)$$

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where $D(G)$ is the indegree matrix and $A_d(G)$ is the adjacency matrix associated with the graph.

Clearly, the diagonal entry d_{ii} of the Laplacian matrix is then the indegree of vertex i and the negative, non-diagonal entries in row i correspond to the neighbors of vertex i. Therefore the row sums of a Laplacian matrix are always zero and hence zero is always an eigenvalue. The eigenvalues of a Laplacian matrix for undirected graphs have special properties [2], [8], some of which are that all eigenvalues are non-negative; the smallest eigenvalue of the Laplacian, λ_1, is zero; and its multiplicity equals the number of connected components of the graph. The second eigenvalue, λ_2, is directly related to the connectivity of the graph [5]. In the case of directed graphs, the non-zero eigenvalues of the Laplacian can be complex values and their role is not as clear.

Definition 2.5: Given a directed graph G, $P = (u_1,\ldots,u_k)$ is a directed path in G if for every $1 \le i < k$ there exists edge $(u_i,u_{i+1}) \in E(G)$.

Definition 2.6: Given a directed graph G, we call G connected if there exists a path from at least one vertex to all other vertices.

Definition 2.7: Given a directed graph G, we call G strongly connected if there exists a directed path between any two vertices in $V(G)$.

All graphs in this paper are connected directed graphs, unless mentioned otherwise. This is a more general class of graphs than the class of undirected graphs. One property of the eigenvalues for the Laplacian matrix of such a graph can be derived from Gershgorin's Theorem. Since all diagonal entries of the Laplacian are the indegrees of the corresponding vertex, it follows that all its eigenvalues will be located in the disc centered at $d = max_i(indegree(i))$ of radius d, so for any eigenvalue λ of the Laplacian

$$|\lambda| \le 2d$$

In this paper it will become necessary to work with Laplacian matrices from the set:

$$\mathcal{S} = \{I_n \otimes L_1 + L_2 \otimes One\}$$

where $I_n \in R^{n\times n}$ is identity matrix, $L_1 \in R^{m\times m}$ is a Laplacian matrix, $L_2 \in R^{n\times n}$ is a Laplacian matrix and $One \in R^{m\times m}$ is a matrix with the first entry equal to 1 and all other entries equal to 0.

For a matrix $L \in \mathcal{S}$ its eigenvalues depend on the Laplacian matrices L_1 and L_2. In the case of undirected graphs, some bounds for the eigenvalues of the matrix L are derived as follows. First, notice that the graph G, corresponding to the Laplacian L, has maximum vertex indegree

$$d = \max\{d_1, indegree(1) + d_2\}$$

where d_1 is the maximum indegree of the graph G_1 corresponding to Laplacian L_1, $indegree(1)$ is the indegree of the vertex labelled 1 in G_1 and d_2 is the maximum vertex indegree of the graph G_2, corresponding to the Laplacian L_2. So for all eigenvalues λ of L:

$$|\lambda| \le 2\max\{d_1, indegree(1) + d_2\}$$

Other bounds can also be found, the reader is directed to [1], [5], [8], [9]. In the case of directed graphs, we derive a bound for the eigenvalues of L as follows.

Lemma 2.8: Let $L_1 \in R^{m\times m}$ and $L_2 \in R^{n\times n}$ be Laplacian matrices of two directed graphs and suppose L_1 has its first diagonal entry equal to zero and all other diagonal entries positive. If $L = I_n \otimes L_1 + L_2 \otimes One \in \mathcal{S}$, then the eigenvalues of L are those of L_2 and the non-zero eigenvalues of L_1 repeated n times.

Proof: : We first notice that since the first entry of L_1 is zero, the indegree of vertex 1 is zero, so the first row of L_1 must have only zero entries. Also, due to the assumptions on the graph and since no other diagonal entries are zero, the multiplicity of eigenvalue zero of L_1 is 1. Then the matrix L_1 has the following form:

$$L_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \star & & & \\ \vdots & & \tilde{L}_1 & \\ \star & & & \end{pmatrix}$$

The non-zero eigenvalues of L_1 are the eigenvalues of $\tilde{L}_1$. Linear transformation can be used to show that L is similar to a matrix of the form:

$$\begin{pmatrix} L_2 & 0 & \cdots & 0 \\ \star & \tilde{L}_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \star & 0 & 0 & \tilde{L}_1 \end{pmatrix}$$

that has $n+1$ blocks on the diagonal and whose eigenvalues are the eigenvalues of L_2 and the eigenvalues of $\tilde{L}_1$ repeated n times. We notice that if the n L_1 matrices were in fact distinct, then the eigenvalues of L would be the set of all non-zero eigenvalues of the matrices replacing the matrices L_1 and the eigenvalues of matrix L_2. ■

Definition 2.9: Given n graphs, we call $G = G_1 \times G_2 \times \ldots \times G_n$ their hierarchical product if the vertices of G_{i+1} are replaced by a copy of G_i such that only the vertex labelled 1 from each G_i replaces each of the vertices of G_{i+1} for all $1 \le i \le n-1$.

Figure 1 shows an example of a two-layer hierarchical communication graph. The adjacency and the indegree matrices of the $G1 \times G2$ graph are given by:

$$A = I_4 \otimes A_1 + A_2 \otimes One$$

$$D = I_4 \otimes D_1 + D_2 \otimes One$$

where A_i and D_i are the adjacency and indegree matrices, respectively, of the graph G_i and One is a 4×4 matrix with only the first entry equal to 1, as defined prior. The Laplacian L is given by:

$$L = I_4 \otimes L_1 + L_2 \otimes One$$

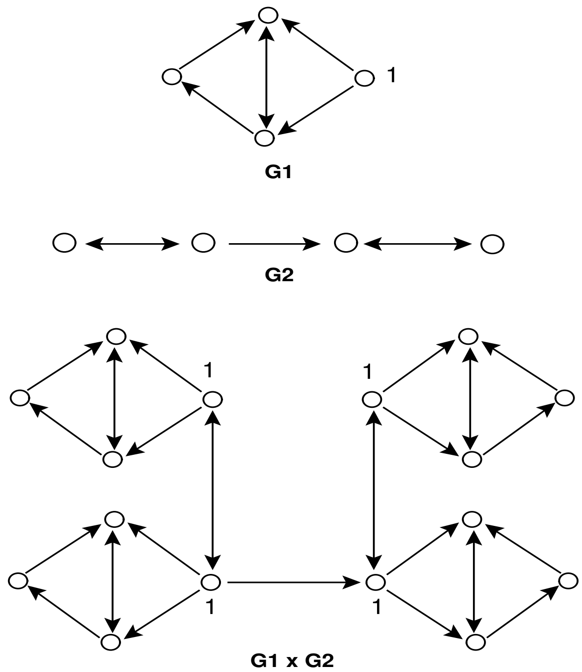

Fig. 1. Hierarchical communication graph example

where L_i is the Laplacian matrix for the graph G_i. In our model (see next Section), we allow a more general hierarchy structure, and for this purpose we introduce the following concept of a *sum of graphs*:

Definition 2.10: Given graphs G_1,...,G_n and G with vertex sets $V_i = V(G_i) = \{1_{G_i}, ..., n_{G_i}\}$ distinct for all i and $V(G) = \{1, ..., n\}$, and edge sets $E_i = E(G_i)$ and E, we call a graph Γ the sum $[G_1 + ... + G_n]_G$ if $V(\Gamma) = \cup_{i=1}^{n} V_i$ and there exist maps $f : V(\Gamma) \to \cup_{i=1}^{n} V_i$ and $\tilde{f} : \cup_{i=1}^{n} \{1_{G_i} | i = 1, \ldots, n\} \to G$ such that v_i and v_k are adjacent in Γ if either (1) $f(v_i)$ and $f(v_k)$ are adjacent in some graph G_l or (2) $f(v_i) = 1_{G_i}$ and $f(v_k) = 1_{G_k}$ and $\tilde{f}(1_{G_i})$ and $\tilde{f}(1_{G_k})$ are adjacent in G.

The graph Γ can be thought of as a graph with vertices v_i replaced by the vertices labelled 1_{G_i} of graphs G_i.

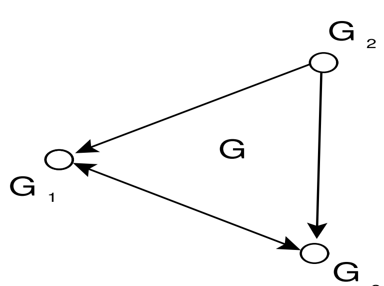

Fig. 2. Here the graph Γ is the sum of G_1, G_2, and G_3 for the given G.

III. Problem Formulation

We assume given N homogeneous vehicles with the following discrete-time dynamics:

$$x_i(k+1) = A_{veh} x_i(k) + B_{veh} u_i(k)$$

where $i = 1, 2, ..., N$ and the entries of x_i represent the v configuration variables for vehicle i and their derivatives, and u_i is the control input for vehicle i.

Definition 3.1: A group of N vehicles is said to be in formation if at all times they are the vertices of a pre-specified geometric shape.

The pre-specified formation shape can be any polygon, or it can be linked to a probability function that describes the desired area coverage for the formation. The formation shape is given by an offset vector $h = h_p \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in R^{2vN}$, where the entries of h_p represent the positions of the N vehicles. See [13] for a more detailed explanation of the offset vector.

Definition 3.2: A group of N vehicles is said to be in a *hierarchical formation* if they are in formation and the vehicles' communication graph is a sum of hierarchical products of graphs.

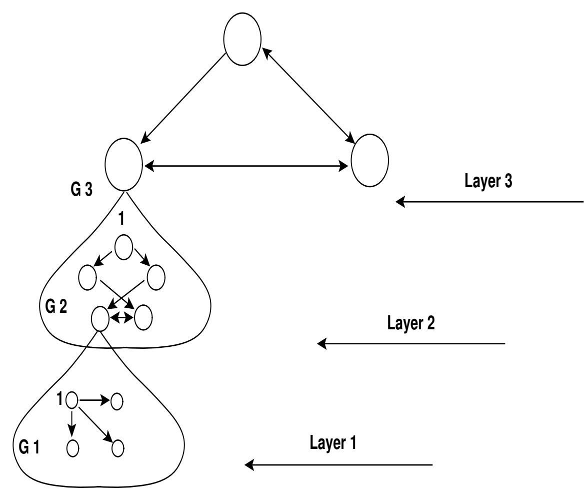

Fig. 3. Example of a three level hierarchy

At each level of the hierarchy the vertices represent a subformation. Each subformation may be constructed of other subformations, but we also allow the subformation at level 1 to be constructed of individual vehicles. Since the communication between vehicles is done via a sum of hierarchical products of graphs, the vehicle labelled 1 from a subformation is the only one that can receive or transmit information from or to other subformations, or vehicles. We will call vehicle 1 the subformation "leader". For the remainder of this paper we assume that the vertices corresponding to the non-leader vehicles have positive indegrees. Practically, this is interpreted as follows: the non-leader subformations receive information from at least one other subformation in their group. In the subformation graph this is captured by having the first diagonal entry of the laplacian equal to zero and all other diagonal entries positive.

We start our analysis by considering a two-level formation hierarchy with n vehicles at the subformation level and m subformations. For simplicity we assume that the subformation dynamics are identical, however this is not necessary. Let each subformation be represented by a system of the form:

$$\begin{aligned} x(k+1) &= A_s x(k) + B_s u(k) \\ z(k) &= L_s(x(k) - h_s) \\ u(k) &= F_s z(k) \end{aligned}$$

where the entries of the vector x represent the n vehicles' states (e.g. positions and velocities), A_s and B_s are the individual subformation dynamics, L_s is the communication graph Laplacian, F_s is the subformation control feedback matrix and h_s is the corresponding subformation offset vector. The system of all $N = nm$ vehicles becomes:

$$\begin{aligned} x^1(k+1) &= A_s x^1(k) + B_s F_s L_s(x^1(k) - h_s^1) \\ &\vdots \\ x^m(k+1) &= A_s x^m(k) + B_s F_s L_s(x^m(k) - h_s^m) \end{aligned}$$

Theorem 3.3: Suppose we have a two-level formation hierarchy, with n vehicles at the subformation level and m identical subformations, as above. With F_s such that each subformation is stable, if G is a graph on m vertices whose Laplacian eigenvalues λ_G have the property that $A_s + \lambda_G B_s F_s$ has eigenvalues with negative real part, then G can be used as the communication infrastructure for the second level of the formation so that the entire system converges to a new overall stable formation.

Proof: Let h be the vector of offsets that specify the overall formation and L_f be the Laplacian for the entire communication graph. Consider the system Σ of all $N = nm$ vehicles:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ z(k) &= L(x(k) - h) \\ u(k) &= Fz(k) \end{aligned}$$

where $A = I_m \otimes A_s$, $B = I_m \otimes B_s$, $L = L_f \otimes I_{2\nu N}$ and F is a new feedback matrix. As in the previous Section, the overall Laplacian L_f takes the following form:

$$L_f = I_n \otimes L_s + L_G \otimes One$$

where L_G is the Laplacian for the graph G. For the purpose of analyzing stability, we consider the case when $h = 0$. Let $F = I_m \otimes F_s$. Then

$$u(k) = \begin{pmatrix} \boxed{F_s} & & 0 \\ & \ddots & \\ 0 & & \boxed{F_s} \end{pmatrix} L \begin{pmatrix} x^1(k) \\ \vdots \\ x^m(k) \end{pmatrix}$$

We will show that with this feedback control matrix F, all vehicles achieve a stable hierarchical formation, or equivalently the system

$$x(k+1) = (A + BFL)x(k)$$

is stabilizable with a "decentralized" control. Let U be an invertible matrix such that $\tilde{L_f} = U^{-1} L_f U$ is an upper triangular matrix. Then, given the block structure of the matrices A, B and F we have:

$$(U^{-1} \otimes I_m)(A + BFL)(U \otimes I_m) = I_m \otimes A_s + \tilde{L_f} \otimes B_s F_s$$

The right side of this equation is upper triangular with diagonal blocks of the form $A_s + \lambda B_s F_s$, where λ is an eigenvalue of L_f. Therefore, the eigenvalues of $A + BFL$ are given by the eigenvalues of $A_s + \lambda B_s F_s$, where λ is an eigenvalue of L_f.

From Lemma 2.8, the set of eigenvalues of L_f is equal to the set $\{\lambda_s$, non-zero eigenvalues of $L_s\} \cup \{\lambda_G$, eigenvalues of $L_G\}$. Clearly the eigenvalues of $A_s + \lambda_s B_s F_s$ have negative real part for the chosen matrix F and for each nonzero eigenvalue λ_s. By hypothesis the eigenvalues of $A_s + \lambda_G B_s F_s$ also have negative real part for λ_G nonzero. Therefore the matrix $A + \lambda BF$ is stable for each nonzero eigenvalue of L_f and all vehicles converge to a hierarchical formation [13]. ■

Example 3.4: In Figure 4 level 1 has four vehicles and level 2 has five subformations. The twenty vehicles achieve a stable hierarchical formation.

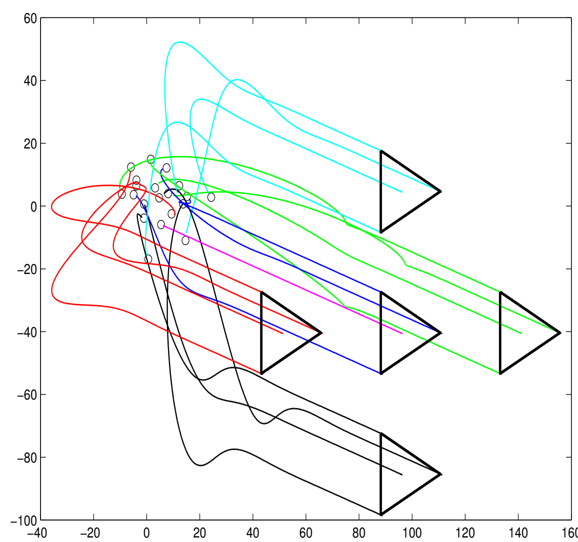

Fig. 4. Twenty vehicle starting from random positions and converging to formation. In this case the communication graph at level 1 is a directed star and at level 2 is a directed cycle.

We note that when converging to formation, the vehicles achieve consensus on where the center of the formation is [4]. This consensus is achieved at both the level 2, i.e. the subformation leaders, as well as at the level 1. This is done independently, in the sense that the relative position of each subformation vehicle only depends on the positions of the vehicles in its group.

We note also that although Theorem 3.3 assumes identical subformations, the proof is readily extended to the case where subformations have distinct communication graphs and dynamics. The choice of L_G should be such that each of $A_s + \lambda B_s F_s$ should be stable. An example of such is the case when at all levels, the information exchange graphs can be modelled using directed tree graphs. In essence, Theorem 3.5 shows that, given a local feedback controller that makes the subformations stable, there exists an entire class of communication infrastructures at the next hierarchy level for which we can guarantee stability for the entire formation.

Conjecture 3.5: The assumptions that the subformations have a leader is sufficient, but not necessary.

Corollary 3.6: Theorem 3.4 can be extended to an m-level hierarchy.

Proof: : We use strong mathematical induction as follows: Suppose levels 1, 2, ..., k have information graphs given by $G_1, G_2, \ldots, G_k$ and are stable, altogether forming a system Σ. Then at level $k+1$ we have stable subformations given by Σ. We construct F as above and choose a communication graph G that satisfies the criteria from Theorem 3.3. Then, the entire system is stable. ■

This allows us to construct a communication structure and indirectly a decentralized control for the next level of the hierarchy based on the feedback controls used at the previous level. We can thus build as large a hierarchy as required.

IV. Convergence to a Pre-Specified Position or Heading

In Section 3 we showed that communication exchange graphs exist so that the stability of the hierarchical formation is ensured. In this Section we address the problem of the position and the heading of the final formation in space. This position information could be given to any one or more vehicles in the formation. If no vehicle has this information, we introduce a virtual vehicle. Such a vehicle could also be representing a pre-computed desired trajectory or a target. The virtual vehicle represents the knowledge that some vehicles have and in the information exchange graph it would be connected to those vehicles as transmitting its position and velocity information at each time step to one or more of the vehicles. Then this vehicle becomes the “leader” for the entire group of vehicles.

A control will be designed such that the vehicles converge to formation at the specified position in space. Suppose for simplicity that the vehicles are supposed to converge to formation at a certain position in R^2, given by the vector h. Then the system of N vehicles is:

$$x(k+1) = Ax(k) + Bu^1(k) + Bu^2(k)$$

where the control u^1 depends on the communication exchange between the vehicles and the formation offsets h and the control u^2 depends on the desired final position (or heading) x_f for the formation. Without loss of generality we will give the final position or heading information to a vehicle which we will call vehicle 1. Since vehicle 1 is the only one that has that information, u^2 will only influence the dynamics of vehicle 1.

Theorem 4.1: Given a leader hierarchy of m stable subformations of n vehicles there exist controls u^1 and u^2 such that the vehicles achieve formation and position (or heading) stability.

Proof: : Let x_f represent the desired position (or heading) of the formation. Let:

$$\begin{aligned} u^1 &= FL(x(k) - h) \\ u^2 &= -Q(x(k) - x_f) \end{aligned}$$

Where F and L are the matrices defined in the previous Section and $Q = One \otimes Q_{veh1}$ for an appropriate matrix Q_{veh1}. For the purpose of analyzing stability, let $h = 0$ and $x_f = 0$. Then the system becomes:

$$x(k+1) = (A + BFL - BQ)x(k) \tag{1}$$

Let vehicle 1 be the formation leader. Then the first row of the Laplacian matrix L has only zero entries and essentially system (1) is composed of two systems: vehicle 1 and all other vehicles:

$$\begin{aligned} x^1(k+1) &= (A_{veh} - B_{veh}Q_{veh1})x^1(k) \\ \tilde{x}(k+1) &= (\tilde{A} + \tilde{B}\tilde{F}\tilde{L})\tilde{x}(k) \end{aligned}$$

So vehicle 1 is the leader of the formation and the rest of the vehicles have a control that depends on vehicle 1's position and velocity. Our choice of Q_{veh1} is such that the vehicle 1 system is position (or heading) stable [12]. We showed in Section 3 that we can choose communication graphs that make the formation stable. Therefore the system (1) converges to the pre-specified formation and position. ■

In figures 5 and 6 we show vehicles achieving formation while converging to a fixed position or a fixed heading in R^2. The individual vehicles' dynamics model used in our simulations is a double integrator vehicle model. There are several methods for designing a control that takes vehicle 1 to its desired trajectory, however for the purpose of this paper we chose a linear control [12]. Other controls may be used as well.

V. Conclusion

This work shows how we can use the structure of a communication graph to ensure stability of a hierarchy of formations. We have allowed the information exchange graph to be as general as possible, our only assumption being subformation leaders and that all non-leader vehicles should be able to receive information from at least one other vehicle. Since mission completion was also one of the objectives, where the mission requires the vehicles to go to a particular location or follow a specific trajectory,

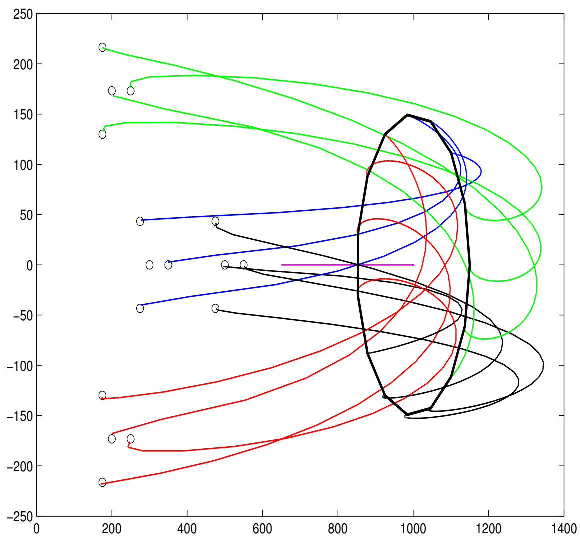

Fig. 5. Four subformations converging to a circular hierarchical formation and fixed position. The communication graphs are directed stars.

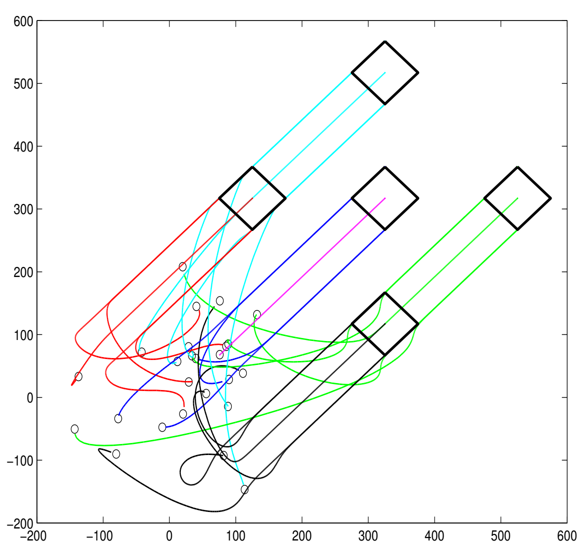

Fig. 6. 20 vehicles starting at random positions and achieving hierarchical formation and common heading.

we show how this is possible in the case that one of the vehicles is given this information.

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