Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Global Stability of Internet Congestion Controllers with Heterogeneous Delays

Lei Ying, G. E. Dullerud and R. Srikant
Emails: {lying,dullerud,rsrikant}@uiuc.edu

Abstract— **In this paper, we study the problem of designing globally stable, scalable congestion control algorithms for the Internet. Prior work has primarily used linear stability as the criterion for such a design. Global stability has been studied only for single node, single source problems. Here, we obtain conditions for a general topology network accessed by sources with heterogeneous delays. We obtain a sufficient condition for global stability in terms of the increase/decrease parameters of the congestion control algorithm and the price functions used at the links.**

I. Introduction

Design of congestion control for the Internet has received much attention since the work of Kelly et al [10]. Originally control theory, especially Lyapunov techniques, were used to analyze the stability property of congestion control algorithms in the absence of delay. The goal of the delay-free analysis was to show that the congestion controllers asymptotically lead to fair resource allocation [10], [14], [17], [30], [21], [29], [2]. However, such techniques do not provide insight into how congestion control parameters should be chosen in the presence of feedback delays. A series of papers provided such design guidelines by considering a linearized system and using frequency-domain techniques: first for a single link [11], [20], [8], [13] and then, for general network topologies with an arbitrary number of sources and heterogeneous delays [9], [19], [25], [26], [27], [22], [14], [15], [16], [23]; see [24] for a comprehensive survey. A significant open challenge is to verify whether the design criteria obtained from such linear analysis ensures global stability, or at least, whether they ensure convergence from a large region of attraction around the equilibrium. In the one-link case, using Razumikhin's theorem, global stability and region of attraction results were obtained in [7], [3] for the case of congestion managements mechanisms with a dynamic source algorithm and static link law, the so-called primal algorithms. Analogous results were obtained for the case of static source and dynamics algorithms, the so-called dual algorithms, in [28]. The extension of these to the network case has proved to be very difficult, except in the case of very small feedback delays [5], [1].

In this paper, we study the global stability of Internet congestion controllers following the work of [3] . The key idea in [3] is to first show that the source rates are both upper and lower bounded, and then use these bounds in Razumikhin's theorem to derive conditions for global stability. However, a stumbling block in extending the results in [3] to a general network is the difficulty in obtaining reasonable bounds on the source rates and in finding an appropriate Lyapunov-Razumikhin function. In this paper, we take a significant step in this direction by finding a Lyapunov-Razumikhin function that provides global stability conditions for a general topology network with heterogenous delays.

For our purpose, we consider a version of the scalable TCP algorithm suggested in [26]. For this congestion control algorithm, we show that one can obtain conditions for global stability that relate the parameters of the congestion algorithm to the parameters of the price functions used at the links of the network.

L. Ying and R. Srikant are with the Coordinated Science Laboratory and Department of Electrical and Computer Engineering and G. E. Dullerud is with the Coordinated Science Laboratory and Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign.

II. Preliminaries

In this section, we will present some basic definitons and the Razumkhin's theorem that will be used to prove our main result.

In the sequel, given an interval $\mathbb{I}$ of the real line $\mathbb{R}$, we will use $C(\mathbb{I},\mathbb{R}^n)$ to denote the set of continuous functions mapping $\mathbb{I}$ to $\mathbb{R}^n$, equipped with the sup norm. We define the following important object.

Definition 1: Suppose $d>0$ and that S is a bounded subset of $C=C([-d,0],\mathbb{R}^n)$. Given $x\in C([-d,\infty),\mathbb{R}^n)$ and any $t\geq 0$, let $x_t\in C$ be defined by $x_t(\theta)=x(t+\theta)$ for $\theta\in[-d,0]$. Given a locally Lipschitz mapping $f:S\to\mathbb{R}^n$, we say the equation

$$\dot{x}(t)=f(x_t),$$

defined for $t\geq 0$, is a retarded differential equation (RFDE) with domain S. ⋄

Given an initial condition $\phi\in S$, we say that x is a corresponding solution if $x_0=\phi$ and $x_t\in S$ for all $t\geq 0$. If the RFDE has a unique solution for all initial conditions in S, then it is said to be well-posed.

The version of Razumkhin's theorem that will play a central role in this paper is now stated; it follows directly from [18, Theorem 7.3.1].

Theorem 2: Suppose that the RFDE is well-posed and $\alpha^2>1$. If there exists a continuous function $W:\mathbb{R}^n\to[0,\infty)$ which takes the value zero only at $\hat{x}$ and which satisfies the inequality

$$\lim\sup_{a\to 0^+}\frac{1}{a}\{W(x(t+a))-W(x(t))\}<0$$

at each t satisfying $\alpha^2 W(x(t)) > \max_{r\in[t-d,t]} W(x(r))$, then every bounded solution $x(t)$ converges asymptotically to $\hat{x}$. ◇

Consider a Lyapunov function $V(t) = W(x(t))$. To ensure global asymptotic stability, Lyapunov theory requires $V(t)$ to decrease for all time t till the equilibrium is reached. Razhumikhin's theorem relaxes this condition and only requires $V(t)$ to decrease at every t such that

$$\alpha^2 V(t) > \max_{t-d\le r\le t} V(r). \quad (1)$$

Thus, as shown in Figure 1, the Lyapunov function can increase occasionally as long it decreases whenever it is larger than $1/\alpha^2$ times its maximum value over a time interval of duration d. This completes the preliminaries.

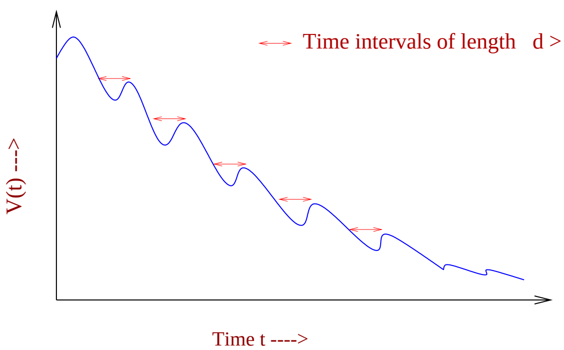

Fig. 1. A plot illustrating the behavior of $V(t)$ under the conditions of Razumikhin's theorem

III. GLOBAL STABILITY

We consider a general topology network consisting of an arbitrary number of sources and links. Each source is assumed to use a fixed route (a collection links) from its origin to its destination and therefore, we will use the same notation for both a source and its route. In other words, we may use the index i to denote either source i or the set of links used by source i. Let $d_f(i,l)$ be the forward delay from source i to link l, and $d_r(i,l)$ be the reverse delay from link l to source i. Denote by $T_i = d_f(i,l) + d_r(i,l)$ the round trip time (RTT) for source i. An illustration of d_f and d_r is provided in Figure 2.

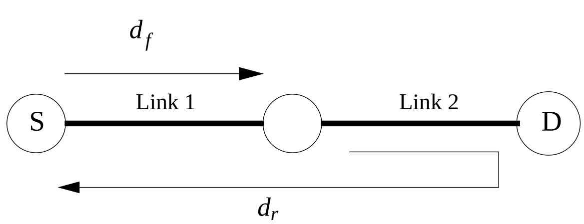

Fig. 2. A network with two links

We consider the following TCP-like congestion control algorithm suggested in [26]:

$$\dot{x}_i(t) = \kappa_i x_i(t-T_i)\left(\frac{1}{x_i^n(t)} - q_i(t)x_i^m(t)\right), \quad (2)$$

where

$$q_i(t) = \sum_{l\in i} p_l(t - d_r(i,l)),$$

$$p_l(t) = f_l(y_l)),$$

$$f_l(y) = \begin{cases} \left(\frac{y}{c_l}\right)^h, & \text{if } 0 < y \le Y_l; \\ \left(\frac{Y_l}{c_l}\right)^h & \text{if } y > Y_l; \end{cases}, \quad (3)$$

and

$$y_l(t) = \sum_{k:l\in k} x_k(t - d_f(i,l)).$$

Here m, n and h are real numbers and we assume that $h > 0$ and $m + n > 0$. In the above set of equations, x_i is the rate at which source i transmits data, y_l is the arrival rate at link l, p_l is price of link l, q_i is the price of source i's route and $f_l(y)$ is the function of the link arrival rate which is used to compute p_l. The price of a route is simply the sum of the prices of the links along its path.

The above algorithm can be used to model the increase/decrease behavior of today's versions of TCP (such as Reno and NewReno) as well as versions of TCP that have been suggested for scalable data transmission over very high-speed links and over large RTTs [24].

Associated with the model is the initial condition given by continuous functions ϕ_i on $[-T_i, 0]$; when each of these functions is strictly positive we say the network model has a *nonzero initial condition*. We will restrict our attention to this situation. It is possible to show that there is a unique positive equilibrium point for the above set of equations; namely there is only one positive constant solution $x(t) = \hat{x}$; see for instance [24, Chapter 3]. We assume that the constant Y_l in (3) is chosen to be greater than $\hat{y}_l$.

The fact that $f_l(y)$ defined in (3) has an upper bound for each l, implies that p_l and q_r are also upper-bounded for all l, r. Thus by the following result, $x(t)$ is always positive.

Proposition 3: Suppose the network model in (2) has a nonzero initial condition and that the route prices for all routes are less than some constant γ, (i.e., $q_i(t) < \gamma$ for all $t \ge 0$ and all i.) Then the inequality

$$x_i(t) \ge \min\left(x_i(0), \gamma^{-1/(m+n)}\right)$$

holds for each source i and all $t \ge 0$.

Proof: Suppose on the contrary that the inequality condition is false. Then by continuity of $x_i(t)$ there must exist $t_0 \ge 0$ such that $x_i(t)$ is non-negative on $[-T_i, t_0]$ with both

$$\dot{x}_i(t_0) < 0 \text{ and } x_i(t_0) = \min\left(x_i(0), \gamma^{-1/(m+n)}\right) \le \gamma^{-1/(m+n)}$$

satisfied. Now by assumption $0 \le q_i(t) < \gamma$ and so the latter condition above yields

$$x_i^m(t_0)\left(\frac{1}{x_i^{n+m}(t_0)} - q_i(t_0)\right) > 0.$$

Considering (2) with $t = t_0$, we see that the previous inequality and $\dot{x}_i(t_0) < 0$ lead to the conclusion that $x_i(t_0 - T_i) < 0$. But this contradicts the fact that $x_i(t)$ is non-negative on $[-T_i, t_0]$, and so no such time t_0 exists. ■

We now define the functions

$$W_j(t) = \frac{1}{2}\left(\log x_j(t) - \log \hat{x}_j\right)^2 \tag{4}$$

which by the above result are well-defined provided that the network model has a nonzero initial condition. Also define

$$W(t) = \max_j W_j(t). \tag{5}$$

Recall that, to apply Razumikhin's theorem, we are interested in those time instants t at which

$$\alpha^2 W(t) > \max_{t-d \le r \le t} W(r) \tag{6}$$

for some $\alpha > 1$ and $d = \max_i T_i$. The next lemma shows that, if the Lyapunov function defined in (4-5) satisfies the condition (6) at some time instant t, then this naturally imposes upper and lower bounds on $x_j(r)$ for all j and $r \in [t-d, t]$.

Lemma 4: Suppose the network model has a nonzero initial condition and suppose that the Lyapunov function (5) satisfies the condition (6) at some time instant t. Let i be the index of a user (or route) such that

$$W(t) = W_i(t).$$

Then,

$$B_i^{-\beta} \hat{x}_j < x_j(r) < B_i^{\beta} \hat{x}_j, \tag{7}$$

for all j and all $r \in [t-d, t]$, where $B_i = \frac{x_i(t)}{\hat{x}_i}$ and $\beta = \text{sgn}(\log B_i)\alpha$.

Proof: First note that since $W_i(t) \neq 0$ that $B_i \neq 1$. By the definition on $W_i(t)$ we have for each index j and $r \in [t-d, t]$ that

$$\frac{1}{2}\alpha^2 (\log B_i)^2 = \alpha^2 W(t) > W_j(t) = \frac{1}{2}\left(\log \frac{x_j(t)}{\hat{x}_j}\right)^2.$$

Note that above inequality can be rewritten as

$$(\log B_i^{\alpha})^2 > \left(\log \frac{x_j(t)}{\hat{x}_j}\right)^2$$

which implies that

$$-\text{sgn}(\log B_i) \log B_i^{\alpha} < \log \frac{x_j(t)}{\hat{x}_j} < \text{sgn}(\log B_i) \log B_i^{\alpha}.$$

From this, we get the inequalities

$$\hat{x}_j B_i^{-\beta} < x_j(r) < \hat{x}_j B_i^{\beta}.$$

■

The next lemma then shows that the route prices are also upper and lower bounded as a consequence of the previous lemma.

Lemma 5: Under the conditions of Lemma 4, the prices of all routes are bounded as given by the following expression:

$$B_i^{-h\beta} \hat{x}_i^{-m-n} < q_i(t) < B_i^{h\beta} \hat{x}_i^{-m-n}.$$

Proof: Recall that the arrival rate at link l is defined as

$$y_l(t) = \sum_{k: l \in k} x_k(t - d_f(i,l))$$

and from the equilibrium condition

$$\hat{q}_i = \hat{x}_i^{-m-n}.$$

Thus, from the definition of the price function in (3), the result is obvious if $Y_l = \infty$ for all l. We now show that the bounds on the route prices also hold for finite $Y_l > \hat{y}_l$. Let $\bar{y}_l(r) = \min(y_l(r), Y_l)$, so

$$p_l(r) = \left(\frac{\bar{y}_l(r)}{c_l}\right)^h.$$

First we have

$$\bar{y}_l(r) \le y_l(r) < \hat{y}_l B_i^{\beta}.$$

Furthermore $\hat{y}_l B_i^{-\beta} < \hat{y}_l < Y_l$ implies

$$\hat{y}_l B_i^{-\beta} < \bar{y}_l(r).$$

So we have

$$\left(\frac{\hat{y}_l B_i^{-\beta}}{c_l}\right)^h < \left(\frac{\bar{y}_l(r)}{c_l}\right)^h < \left(\frac{\hat{y}_l B_i^{\beta}}{c_l}\right)^h,$$

and thus,

$$\hat{q}_i B_i^{-\beta h} < q_i(t) < \hat{q}_i B_i^{\beta h},$$

where $\hat{q}_i = \sum_{l \in i} \left(\frac{\hat{y}_l}{c_l}\right)^h$. ■

Corollary 6: If $m+n > h$, the supposition of Lemma 4 holds and $\alpha = (m+n+h)/2h$ then $\dot{W}_i(t) < 0$.

Proof: The derivative $\dot{W}_i(t)$ is given by

$$\dot{W}_i(t) = \frac{\dot{x}_i(t)}{x_i(t)} \log\left(\frac{x_i(t)}{\hat{x}_i}\right) = \frac{\dot{x}_i(t)}{x_i(t)} \log B_i,$$

and thus to prove the result we need to show that $\dot{x}_i(t) \log B_i < 0$. From (2) we see that

$$\dot{x}_i(t) = \kappa_i \frac{x_i(t - T_i)}{x_i^n(t)} \left(1 - q_i(t) x^{m+n}(t)\right).$$

Since $x_i(t - T_i)$ and $x_i(t)$ are both positive it is sufficient to show that

$$\left(1 - q_i(t) x^{m+n}(t)\right) \log B_i < 0. \tag{8}$$

We can show this using Lemma 5 by considering the following two cases. Suppose first that $B_i > 1$, then we have to show that

$$q_i(t) x_i^{m+n}(t) > 1.$$

From the lower bound in Lemma 5, we have

$$q_i(t) x_i^{m+n}(t) > B_i^{-h\alpha} \hat{x}_i^{-m-n} x_i^{m+n}(t) = B_i^{-h\alpha+m+n}$$

which is greater than 1 since $\alpha h < m+n$. Next, suppose that $B_i < 1$. Then, we have to show that

$$q_i(t)x_i^{m+n}(t) < 1.$$

From the upper bound in Lemma 5,

$$q_i(t)x_i^{m+n}(t) < B_i^{-h\alpha}\hat{x}_i^{-m-n}x_i^{m+n}(t) = B_i^{-h\alpha+m+n}$$

which is now less than 1 since $\alpha h < m+n$. ■

This brings us to the main result of the paper.

Theorem 7: If $m+n > h > 0$, then the network model in (2) is globally asymptotically stable.

Proof: By the hypothesis $\alpha = \frac{m+n+h}{2h} > 1$. Furthermore, we have already shown that $x(t)$ is lower bounded and it is easy to show that $x(t)$ is upper bounded too. Now invoking Theorem 2, with $W(t)$ and $W_j(t)$ as defined above, it is enough to prove for every $t \geq 0$ that

$$\text{limsup}_{a\to 0^+} \frac{1}{a}\{W(t+a) - W(t)\} < 0 \quad (9)$$

whenever $\alpha^2 W(t) > \max_{t-d\leq r\leq t} W(r)$. Fix any such t and let $\mathscr{I}$ denote the set of indices i satisfying $W_i(t) = W(t)$. Since each function $W_j(t)$ is continuous there exists a neighborhood of zero such that for every a in this neighborhood

$$\max_j W_j(t+a) = \max_{i\in\mathscr{I}} W_i(t+a) \text{ holds.}$$

Thus for every positive a in this neighborhood

$$\frac{1}{a}\{W(t+a) - W(t)\} = \max_{i\in\mathscr{I}} \frac{1}{a}\{W_i(t+a) - W_i(t)\}.$$

By Lemma 6 each $\dot{W}_i(t) < 0$ and so there exists $\varepsilon > 0$ such that

$$\max_{i\in\mathscr{I}} \frac{1}{a}\{W_i(t+a) - W_i(t)\} \leq -\varepsilon$$

for all $a > 0$ sufficiently near to zero. This implies (9) as required.

■

IV. Local Stability

The global stability condition in the previous section imposes no constraint on $\{\kappa_i\}$, but rather imposes a condition on the increase/decrease parameters of TCP and the price function. This is in contrast to the results in [26] where κ_i is required to be smaller than some constant times the RTT on route i. We now show that the condition derived in [26] is more conservative than what is necessary to ensure locally stability when $m+n > h$. We will illustrate it only for case $n = 0$ and $m = 1$. In this case, the network of TCP users governed by the differential equations (2) is shown to be locally in [26] if

$$\kappa_i T_i h \leq \frac{\pi}{2}. \quad (10)$$

But from the result of above, we know that the network is globally, asymptotically stable if $h < 1$. It means that, for the case $h < 1$, the local stability condition (10) is very conservative. In this section, we will take a look at how the local stability condition is derived in [26], and show that the derivation in [26] can be modified to ensure local stability for all κ_i. Consider the congestion controller

$$\dot{x}_i(t) = \kappa_i x_i(t - T_i)(1 - q_i(t)x_i(t)) \quad (11)$$

After linearizing the system atround the equilibrium point and then taking the Laplace transforms, the return ratio of the system is given as

$$\text{diag}\left(\kappa_i T_i h \frac{e^{-sT_i}}{sT_i + \kappa_i T_i}\right)\text{diag}\left(\frac{\hat{x}_i}{\hat{q}_i}\right) R^T(-s)\text{diag}\left(\frac{\hat{p}_l}{\hat{y}_l}\right) R(s)$$

where

$$R_{il} = \begin{cases} e^{-sd_f(i,l)}, & \text{if source } i \text{ uses link } l; \\ 0, & \text{otherwise.} \end{cases}.$$

Furthermore, it is shown in[26] that

$$0 \leq \sigma\left(\text{diag}\left(\frac{\hat{x}_i}{\hat{q}_i}\right) R^T(-s)\text{diag}\left(\frac{\hat{p}_l}{\hat{y}_l}\right) R(s)\right) \leq 1,$$

where $\sigma(A)$ denotes the spectrum of a square matrix A, and

$$\frac{\pi}{2}\frac{e^{-j\omega T_i}}{j\omega T_i + \kappa_i T_i}$$

always crosses the real axis to the right of the point > -1. So according to the generalized Nyquist stability criterion [4], if

$$\kappa_i T_i h \leq \frac{\pi}{2},$$

the network is locally stable. This condition tells us that the choice of κ_i depends on the round trip delay T_i. But this is only a sufficient condition for general h, if we consider the special case of $h \leq 1$, then we have

$$\begin{aligned} \left|\kappa_i T_i h \frac{e^{-j\omega T_i}}{j\omega T_i + \kappa_i T_i}\right| &= h\frac{\left|e^{-j\omega T_i}\right|}{\left|j\frac{\omega}{\kappa_i} + 1\right|} \\ &= h\frac{1}{\sqrt{1 + \left(\frac{\omega}{\kappa_i}\right)^2}} \\ &\leq 1. \end{aligned}$$

So we can conclude from the local analysis that the network is locally stable for any $\kappa_i > 0$ when $h \leq 1$.

V. Simulation

In Section III, we have shown that, if $m+n > h$, the network is globally stable for any $\kappa_i > 0$. However, the choice of κ_i may still influence the transient performance of the congestion control algorithm. In this section, we will study the effect on $\{\kappa_i\}$ on the system performance using simulations.

Consider a network with 4 routes and 5 links network as shown in Figure 3.

The network parameters used in the simulations are:

1) Propagation delay on each link is 10ms;
2) Capacity of each link is 1 Gbps;
3) There are 50 users on every route;

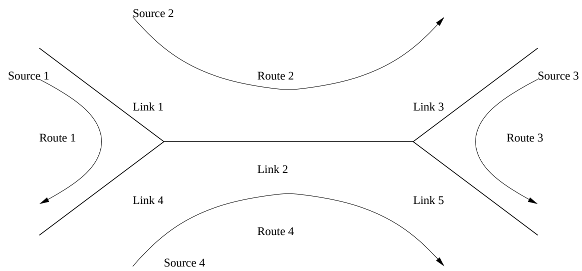

Fig. 3. The network used in the simulation

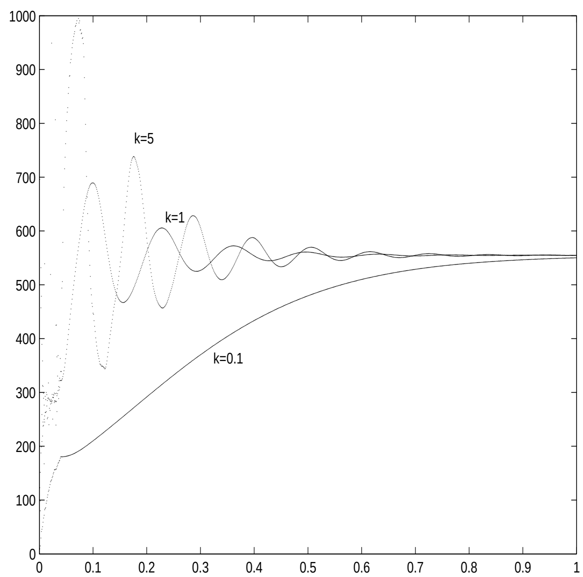

Fig. 4. The rate of user 1 for $\kappa_1 = 0.1$, $\kappa_1 = 1$ and $\kappa_1 = 5$

4) The route-link incidence matrix R is

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

In other words, $R_{il} = 1$ indicates that route i uses link l.

We use the congestion control algorithm

$$\dot{x}_i(t) = \kappa_i x_i(t - T_i)\left(\frac{100}{x_i^n(t)} - q_i(t)x_i^m(t)\right),$$

and

$$p_l(t) = \left(\frac{y_l(t - d_r(i,l))}{c_l}\right)^h,$$

where $n = 0.1$, $m = h = 1$, and $c_l = 10^6$.

We run the simulations for three different κ_is: $\kappa_i = 0.1$, $\kappa_i = 1$ and $\kappa_i = 5$ and plot the evolution of $x_1(t)$ for these three different cases. The simulation results are shown in Fig. 4. We notice that when $\kappa_1 = 0.1$, the rate of convergence is quite slow; when $\kappa_1 = 5$, the oscillation is quite big. So the best choice among these three values of κ_1 seems to be $\kappa_1 = 1$. For this value of κ_1, the rate allocated to user 1 converges fast and the oscillation is small. This suggests that, even though the network may be stable for many value of $\{\kappa_r\}$, these parameters have to be chosen carefully to provide a tradeoff between transient performance and the rate at which the equilibrium is reached.

VI. Conclusions

An important open problem in the study of Internet congestion control has been the design of congestion controllers which are globally, asymptotically stable in a network with heterogeneous feedback delays. In this paper, we have established the global stability of a class of congestion management algorithms, i.e., a combination of congestion controllers at the sources and congestion signalling mechanisms at the routers. The proof of global stability is obtained by placing certain restrictions on the increase/decrease parameters of TCP and the parameters of the link price functions. A significant open question for further research is the following: for congestion management algorithms that are outside the class considered in this paper, is it possible to stabilize the system using a *slow-start* procedure to bring the system close to its equilibrium and choosing the control parameters to ensure local stability?

References

[1] T. Alpcan and T. Başar. Global stability analysis of end-to-end congestion control schemes for general topology networks with delay, 2003. Preprint.

[2] T. Alpcan and T. Başar. A utility-based congestion control scheme for internet-style networks with delay. In *Proceedings of IEEE Infocom*, San Francisco, California, March-April 2003.

[3] S. Deb and R. Srikant. Global stability of congestion controllers for the internet. *IEEE Transactions on Automatic Control*, 48(6):1055–1060, June 2003.

[4] C. A. Desoer, and Y. T. Wang. On the generalized Nyquist stability criterion. In *IEEE Transactions on Automatic Control*, 1980.

[5] X. Fan, M. Arcak, and J. Wen. l_p stability and delay robustness of network flow control, 2003. Preprint.

[6] J. Hale and S. M. V. Lunel. *Introduction to Functional Differential Equations*. 2nd edition, Springer Verlag, New York, NY, 1991.

[7] C. V. Hollot and Y. Chait. Nonlinear stability analysis for a class of TCP/AQM schemes. In *Proceedings of the IEEE Conference on Decision and Control*, December 2001.

[8] C.V. Hollot, V. Misra, D. Towsley, and W. Gong. On designing improved controllers for AQM routers supporting TCP flows. In *Proceedings of IEEE Infocom*, Anchorage, Alaska, April 2001.

[9] R. Johari and D. Tan. End-to-end congestion control for the Internet: Delays and stability. *IEEE/ACM Transactions on Networking*, 9(6):818–832, December 2001.

[10] F. P. Kelly, A. Maulloo, and D. Tan. Rate control in communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49:237–252, 1998.

[11] F. P. Kelly. Models for a self-managed Internet, *Philosophical Transactions of the Royal Society*, A358:2335-2348, 2000.

[12] H. Khalil. *Nonlinear Systems*. 2nd edition, Prentice Hall, Upper Saddle River, NJ, 1996.

[13] S. Kunniyur and R. Srikant. Analysis and design of an adaptive virtual queue algorithm for active queue management, *Proceedings of ACM Sigcomm*, August 2001, San Diego, CA, 123–134.

[14] S. Kunniyur and R. Srikant. A time-scale decomposition approach to adaptive ECN marking. *IEEE Transactions on Automatic Control*, 47(6):882–894, June 2002.

[15] S. Kunniyur and R. Srikant. Stable, Scalable, Fair Congestion Control and AQM Schemes that Achieve High Utilization in the Internet. *IEEE Transactions on Automatic Control*, 2003, pp. 2024-2029.

[16] S. Kunniyur and R. Srikant. End-to-end congestion control: utility functions, random losses and ECN marks. In *IEEE/ACM transactions on Networking*, 2004. To appear.
[17] S. H. Low and D. E. Lapsley. Optimization flow control, I: Basic algorithm and convergence. *IEEE/ACM Transactions on Networking*, pages 861–875, December 1999.
[18] V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities*, vol. 2. Academic Press, 1969.
[19] L. Massoulie. Stability of distributed congestion control with heterogenous feedback delays. *Technical Report, Microsoft Research, Cambridge, UK*, 2000.
[20] V. Misra, W. Gong, and D. Towsley. A fluid-based analysis of a network of AQM routers supporting TCP flows with an application to RED. In *Proceedings of ACM Sigcomm*, Stockholm, Sweden, September 2000.
[21] F. Paganini. A global stability result in network flow control. *Systems and Control Letters*, 46(3):153–163, 2002.
[22] F. Paganini, J. Doyle, and S. Low. Scalable laws for stable network congestion control. In *Proceedings of the IEEE Conference on Decision and Control*, December 2001.
[23] F. Paganini, Z. Wang, J. Doyle, and S. Low. A new TCP/AQM for stable operation in fast networks. In *Proceedings of the IEEE Infocom*, San Francisco, CA, April 2003.
[24] R. Srikant. *The Mathematics of Internet Congestion Control*. Birkhauser, 2004.
[25] G. Vinnicombe. On the stability of end-to-end congestion control for the Internet, 2001. University of Cambridge Technical Report CUED/F-INFENG/TR.398. Available at http://www.eng.cam.ac.uk/˜gv.
[26] G. Vinnicombe. On the stability of networks operating TCP-like congestion control. In *Proceedings of the IFAC World Congress*, Barcelona, Spain, 2002. University of Cambridge Technical Report CUED/F-INFENG/TR.398. Available at http://www.eng.cam.ac.uk/˜gv.
[27] G. Vinnicombe. Robust congestion control for the Internet, 2002. University of Cambridge Technical Report. Available at http://www.eng.cam.ac.uk/˜gv.
[28] Z. Wang and F. Paganini. Global stability with time-delay in network congestion control. In *Proceedings of the IEEE Conference on Decision and Control*, 2002.
[29] J.T. Wen and M. Arcak. A unifying passivity framework for network flow control. In *Proceedings of IEEE Infocom*, April 2003.
[30] H. Yaiche, R. R. Mazumdar, and C. Rosenberg. A game-theoretic framework for bandwidth allocation and pricing in broadband networks. *IEEE/ACM Transactions on Networking*, 8(5):667–678, October 2000.