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A convex approach for robust state feedback control of discrete-time systems with state delay

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Abstract— **In this paper, uncertain discrete-time systems with state delay are investigated. The uncertainty is supposed to belong to a known convex polytope. Linear matrix inequality conditions are given for the robust stability of the system, encompassing quadratic stability based results. Then, convex conditions assuring the existence of a robust state feedback gain are derived, assuring the delay independent quadratic stability of the closed-loop system (thus allowing to deal with time-varying uncertain systems) or, in the time-invariant case, guaranteeing the robust stability irrespective of the value of the delay. Moreover, the feedback control law can also include a term depending on the delayed state which, if the value of the delay is known, can be used to improve the control design. Numerical examples illustrate the effectiveness of the proposed techniques.**

I. INTRODUCTION

Linear systems with time-delay are an important topic in control systems, as it can be inferred by the large number of publications dealing with this subject (see, for instance, the text books [1], [2], [3] and references therein).

Concerning specifically discrete-time systems with time delays, a simple solution can be adopted, in the case of known delays, by including the delayed state in an augmented state vector [4]. However, among other difficulties (such as the need of structural constraints to be imposed on the Lyapunov matrices used to compute a memoryless control design) it is not possible in this augmented representation to design a control law which takes into account the delayed state if one is interested on computing control gains for the original system. Moreover, this strategy no longer applies for the study of delay-independent stability conditions (i.e. stability with respect to unknown and unbounded time-delays), since an infinite sequence of augmented systems should be tested.

The concept of quadratic stability [5] has been used to derive robust delay-independent stability conditions in several papers, some of them also presenting an extension to cope with control design. Following this methodology, norm-bounded uncertainties are considered in [6], where nonconvex necessary and sufficient conditions for quadratic stability and quadratic stabilization are given for fixed and known delays. A similar problem is addressed in [7] by means of linear matrix inequalities (LMIs) and scaling parameters, including a quadratic cost function and also resulting in nonconvex conditions for control design. In [8], both continuous and discrete-time cases are investigated in the context of $\mathcal{H}_\infty$ state feedback control. Sufficient conditions for the design of an $\mathcal{H}_\infty$ state feedback control are given in terms of LMIs and a scaling parameter, but only norm-bounded uncertainties are allowed and the delay is supposed to be fixed. Sufficient algebraic conditions for the robust stability of discrete-time systems with delays are given in [9] (singular systems), [10], [11] and [12]. Modified Riccati equations have been proposed in [13], based on the results of [14], to compute a robust state feedback $\mathcal{H}_\infty$ control gain, but the uncertainty (norm-bounded) is limited to the dynamic matrix only. Other results include [15] (defining an augmented descriptor system), [16], [17], [2] and [18] where some nonconvex strategies are proposed for control design. All of these results are based on a fixed Lyapunov matrix that can provide, in some cases, a robust control gain by means of nonconvex conditions. Extensions to deal with polytopic uncertainties or decentralized control do not seem to be immediate. Moreover, it is well known that quadratic stability (fixed Lyapunov matrix) can lead to conservative results in the evaluation of time-invariant uncertainty domains.

Although there have been recent results using parameter dependent Lyapunov functions to investigate robust stability [19], [20], [21], and to compute state feedback gains for uncertain linear discrete-time systems in polytopic domains [22] as well as new results to deal with discrete time-varying systems [23], to the best of the authors' knowledge there exists no extension of these results to deal with the presence of delayed states.

In this paper, LMI conditions for the robust stability of linear uncertain discrete-time systems with state delays are given. The uncertainties are supposed to belong to convex bounded (polytopic) domains, affecting all the system matrices. The robust stability is assured, independently of the size of the time-delays (which can be unknown), by means of a Lyapunov-Krasovskii functional with parameter dependent matrices, encompassing the results based on fixed matrices (quadratic stability). Thanks to some extra parameter dependent matrix variables, less conservative robust stability evaluations are obtained. Then, convex conditions for the existence of a robust stabilizing state feedback control law are given. The closed-loop stability is assured by means of parameter dependent Lyapunov-Krasovskii matrices which are not directly used to compute the robust control gains,

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allowing to extend the results to cope with decentralized control without constraining the matrices used to assess stability. By imposing the Lyapunov-Krasovskii matrices to be fixed, the robust stabilizability conditions can also be applied in the time-varying case. When the delayed state is available for feedback (i.e. the delay value is precisely known), a term based on past values of the state can be included in the control law, providing robust stabilization in situations where a memoryless stabilizing control gain may not exist. As discussed in [24], the use of past values of the state can provide an interesting and useful degree of freedom, even for discrete-time systems without delay. Finally, the results are illustrated by means of examples.

The notation used in this paper is quite standard. $\mathbb{R}$ is the set of real numbers. $\mathbf{I}$ and $\mathbf{0}$ denotes, respectively, the identity matrix and the null matrix of appropriate dimensions. $M > \mathbf{0}$ ($< \mathbf{0}$) means that matrix M is positive (negative) definite. M' is the transpose of M. The symbol $\star$ stands for symmetric blocks in the LMIs.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider the uncertain discrete-time system given by [24]

$$x(k+1) = A(\alpha)x(k) + A_d(\alpha)x(k-d) + B(\alpha)u(k) + B_d(\alpha)u_d(k) \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, d is a nonnegative integer representing the time delay, $u(k) \in \mathbb{R}^{m_1}$ and $u_d(k) \in \mathbb{R}^{m_2}$ represent the control inputs. Matrices $A(\alpha)$, $A_d(\alpha)$, $B(\alpha)$ and $B_d(\alpha)$ are supposed fixed but not precisely known, that is, $(A, A_d, B, B_d)(\alpha)$ belong to the polytope $\mathcal{D}$ given by

$$\mathcal{D} = \Big\{(A,A_d,B,B_d)(\alpha) \ : \ (A,A_d,B,B_d)(\alpha) = \sum_{j=1}^{N} \alpha_j (A,A_d,B,B_d)_j, \sum_{j=1}^{N} \alpha_j = 1, \ \alpha_j(t) \geq 0, \ j = 1,\ldots,N\Big\} \quad (2)$$

It is clear that any $(A,A_d,B,B_d)(\alpha) \in \mathcal{D}$ can be written as a convex combination of the N vertices of $\mathcal{D}$. Throughout the paper, the vertices of $\mathcal{D}$ are referred as $(A,A_d,B,B_d)_j$ or A_j, A_{dj}, B_j and B_{dj}, $j = 1,\ldots,N$. Remarks along the paper discuss where the results also apply for time-varying uncertain systems and/or unknown time-delays. Two problems are addressed:

Problem 1: Determine if the uncertain discrete-time autonomous system with delayed state given by (1)-(2) with $u(k) = u_d(k) = 0$ is robustly stable irrespective of the value d of the time delay.

Problem 2: Find, if possible, robust state feedback control gains $K \in \mathbb{R}^{m_1 \times n}$ and $K_d \in \mathbb{R}^{m_2 \times n}$ yielding the control laws

$$u(k) = Kx(k); \quad u_d(k) = K_d x(k-d) \quad (3)$$

which assure that the uncertain closed-loop system

$$x(k+1) = \tilde{A}(\alpha)x(k) + \tilde{A}_d(\alpha)x(k-d) \quad (4)$$

with

$$\tilde{A}(\alpha) \triangleq A(\alpha) + B(\alpha)K; \quad \tilde{A}_d(\alpha) \triangleq A_d(\alpha) + B_d(\alpha)K_d \quad (5)$$

is robustly stable irrespective of the value d of the time delay.

In the sequel, sufficient conditions for solving Problems 1 and 2 are given in terms of LMIs. In both cases, solutions with parameter dependent as well as fixed (constant) Lyapunov-Krasovskii matrices are provided.

III. ROBUST STABILITY ANALYSIS

A solution to Problem 1 can be obtained from the use of Lyapunov-Krasovskii functionals. Next Lemma presents four equivalent conditions based on the existence of parameter dependent matrices.

Lemma 1: The uncertain discrete-time system (4) with $(A,A_d,B,B_d)(\alpha) \in \mathcal{D}$ given by (2) is robustly stable irrespective of the time delay d if there exist symmetric positive definite parameter dependent matrices $P(\alpha) \in \mathbb{R}^{n\times n}$ and $S(\alpha) \in \mathbb{R}^{n\times n}$ such that any one of the following equivalent conditions holds for all $\alpha \in \mathbb{R}^N$ such that $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$:

a) The Lyapunov-Krasovskii functional $V(x(k))$ given by

$$V(x(k)) = x(k)'P(\alpha)x(k) + \sum_{j=0}^{d} x(k-j)'S(\alpha)x(k-j) \quad (6)$$

is such that

$$V(x(k)) > 0 \quad \text{and} \quad V(x(k+1)) - V(x(k)) < 0 \quad (7)$$

for all $(x(k), x(k-d)) \neq 0$.

b)

$$\Theta(\alpha) \triangleq \begin{bmatrix} \Theta_{11}(\alpha) & \Theta_{12}(\alpha) \\ \star & \Theta_{22}(\alpha) \end{bmatrix} > \mathbf{0} \quad (8)$$

where

$$\Theta_{11}(\alpha) \triangleq P(\alpha) - \tilde{A}(\alpha)'((S(\alpha)+P(\alpha))\tilde{A}(\alpha)$$
$$\Theta_{12}(\alpha) \triangleq -\tilde{A}(\alpha)'((S(\alpha)+P(\alpha))\tilde{A}_d(\alpha)$$
$$\Theta_{22}(\alpha) \triangleq S(\alpha) - \tilde{A}_d'(\alpha)((S(\alpha)+P(\alpha))\tilde{A}_d(\alpha)$$

c)

$$\Upsilon(\alpha) \triangleq \begin{bmatrix} \Upsilon_{11}(\alpha) & \Upsilon_{12}(\alpha) & \Upsilon_{13}(\alpha) \\ \star & P(\alpha) & \mathbf{0} \\ \star & \star & S(\alpha) \end{bmatrix} > \mathbf{0} \quad (9)$$

where

$$\Upsilon_{11}(\alpha) \triangleq S(\alpha) + P(\alpha) \ ; \ \Upsilon_{12}(\alpha) \triangleq -(S(\alpha)+P(\alpha))\tilde{A}(\alpha)$$
$$\Upsilon_{13}(\alpha) \triangleq -(S(\alpha)+P(\alpha))\tilde{A}_d(\alpha)$$

d) There exist parameter dependent matrices $F(\alpha) \in \mathbb{R}^{n\times n}$, $G(\alpha) \in \mathbb{R}^{n\times n}$ and $H(\alpha) \in \mathbb{R}^{n\times n}$ such that

$$\mathcal{M}(\alpha) = \begin{bmatrix} \Gamma_{11}(\alpha) & \Gamma_{12}(\alpha) & \Gamma_{13}(\alpha) \\ \star & \Gamma_{22}(\alpha) & \Gamma_{23}(\alpha) \\ \star & \star & \Gamma_{33}(\alpha) \end{bmatrix} > \mathbf{0} \quad (10)$$

where

$$\Gamma_{11}(\alpha) \triangleq -(F(\alpha)+F(\alpha)'+P(\alpha)+S(\alpha))$$
$$\Gamma_{12}(\alpha) \triangleq F(\alpha)\tilde{A}(\alpha)-G(\alpha)'; \Gamma_{13}(\alpha) \triangleq F(\alpha)\tilde{A}_d(\alpha)-H(\alpha)'$$
$$\Gamma_{22}(\alpha) \triangleq P(\alpha)+G(\alpha)\tilde{A}(\alpha)+\tilde{A}(\alpha)'G(\alpha)'$$
$$\Gamma_{23}(\alpha) \triangleq G(\alpha)\tilde{A}_d(\alpha)+\tilde{A}(\alpha)'H(\alpha)'$$
$$\Gamma_{33}(\alpha) \triangleq S(\alpha)+H(\alpha)\tilde{A}_d(\alpha)+\tilde{A}_d(\alpha)'H(\alpha)' \qquad (11)$$

Proof: Developing (7), using (4) and taking into account (9) one has

$$V(x(k+1))-V(x(k)) = x(k+1)'(S(\alpha)+P(\alpha))x(k+1)$$
$$-x(k)'P(\alpha)x(k)-x(k-d)'S(\alpha)x(k-d) =$$
$$= -\begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix}' \Theta(\alpha) \begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix} < 0 \qquad (12)$$

and clearly with $P(\alpha) > \mathbf{0}$ and $S(\alpha) > \mathbf{0}$ *a)* is verified if and only if *b)* holds. The equivalence between *b)* and *c)* is established straightforwardly by Schur complement. Finally, note that if *c)* holds, then (10) is verified for the particular choice $F(\alpha) = F(\alpha)' = -(P(\alpha)+S(\alpha))$, $G(\alpha) = H(\alpha) = \mathbf{0}$ and, conversely, if *d)* holds, then $\Theta(\alpha) = T(\alpha)'\mathcal{M}(\alpha)T(\alpha) > \mathbf{0}$ with

$$T(\alpha) = \begin{bmatrix} \tilde{A}(\alpha)' & \mathbf{I} & \mathbf{0} \\ \tilde{A}_d(\alpha)' & \mathbf{0} & \mathbf{I} \end{bmatrix}' \qquad (13)$$

■

Note that the robust stability of the uncertain system (1) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ given by (2) with $u(k) = u_d(k) = 0$ can also be verified through the conditions of Lemma 1, which assure the robust stability irrespective of the value d of the delay for time-invariant (i.e. unknown but fixed) uncertain matrices $(A, A_d, B, B_d) \in \mathcal{D}$. The four equivalent sufficient conditions for the robust stability of (4) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ given by (2) are provided in terms of parameter dependent matrices $P(\alpha)$, $S(\alpha)$, $F(\alpha)$, $G(\alpha)$ and $H(\alpha)$, and must be verified for all $\alpha \in \mathbb{R}^N$ such that $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$. These conditions could also be used to assure the robust stability of $(A, A_d, B, B_d)(\alpha)$ belonging to other compact sets, but their usefulness is very limited since they must be verified for an infinite number of real values of vector α. In order to test the existence of a solution to Problem 1 with a finite number of LMIs, certain choices on the structure of the parameter dependent matrices must be done. For instance, a simple way to extend the results to cope with time-varying uncertain matrices is provided by the choice of parameter independent (fixed) matrices P and S in the Lyapunov-Krasovskii functional (6), yielding the so called quadratic stability condition.

An interesting remark about Lemma 1 is that the extra matrix variables appearing in condition *d)* can be used to provide less conservative robust stability evaluations, as well as sufficient conditions for control design, as it will be clear in next section. For instance, following the ideas of [25], the existence of affine parameter dependent matrices satisfying condition *d)* in Lemma 1 can be investigated by means of a finite number of LMIs.

Theorem 1: If there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $S_i \in \mathbb{R}^{n\times n}$ and matrices $F_i \in \mathbb{R}^{n\times n}$, $G_i \in \mathbb{R}^{n\times n}$ and $H_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,N$ such that

$$\mathcal{M}_i \triangleq \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \star & \Psi_{22} & \Psi_{23} \\ \star & \star & \Psi_{33} \end{bmatrix} > \mathbf{I}; \; i = 1,\ldots,N \qquad (14)$$

where

$$\Psi_{11} \triangleq -(F_i+F_i'+P_i+S_i) \; ; \; \Psi_{12} \triangleq -G_i'+F_i\tilde{A}_i$$
$$\Psi_{13} \triangleq -H_i+F_i\tilde{A}_{di} \; ; \; \Psi_{22} \triangleq P_i+G_i\tilde{A}_i+\tilde{A}_i'G_i'$$
$$\Psi_{23} \triangleq G_i\tilde{A}_{di}+\tilde{A}_i'H_i' \; ; \; \Psi_{33} \triangleq S_i+H_i\tilde{A}_{di}+\tilde{A}_{di}'H_i'$$

$$\mathcal{M}_{ir} \triangleq \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \star & \Omega_{22} & \Omega_{23} \\ \star & \star & \Omega_{33} \end{bmatrix} > \frac{-1}{(N-1)^2}\mathbf{I}; \quad \begin{matrix} i,r = 1,\ldots,N \\ i \neq r \end{matrix} \qquad (15)$$

where

$$\Omega_{11} \triangleq -2(P_i+S_i+F_i+F_i')-(P_r+S_r+F_r+F_r')$$
$$\Omega_{12} \triangleq -2G_i'-G_r'+F_i\tilde{A}_i+F_i\tilde{A}_r+F_r\tilde{A}_i$$
$$\Omega_{13} \triangleq -(2H_i+H_r)+F_i\tilde{A}_{di}+F_i\tilde{A}_{dr}+F_r\tilde{A}_{di}$$
$$\Omega_{22} \triangleq 2P_i+P_r+G_i\tilde{A}_i+G_i\tilde{A}_r+G_r\tilde{A}_i+\tilde{A}_i'G_i'+\tilde{A}_i'G_r'+\tilde{A}_r'G_i'$$
$$\Omega_{23} \triangleq G_i\tilde{A}_{di}+G_i\tilde{A}_{dr}+G_r\tilde{A}_{di}+\tilde{A}_i'H_i'+\tilde{A}_i'H_r'+\tilde{A}_r'H_i'$$
$$\Omega_{33} \triangleq 2S_i+S_r+H_i\tilde{A}_{di}+H_i\tilde{A}_{dr}+H_r\tilde{A}_{di}$$
$$+\tilde{A}_{di}'H_i'+\tilde{A}_{di}'H_r'+\tilde{A}_{dr}'H_i'$$

$$\mathcal{M}_{irf} \triangleq \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \star & \Xi_{22} & \Xi_{23} \\ \star & \star & \Xi_{33} \end{bmatrix} > \frac{-6}{(N-1)^2}\mathbf{I},$$
$$i = 1,\ldots,N-2; \quad r = i+1,\ldots,N-1; \quad f = r+1,\ldots,N \qquad (16)$$

where

$$\Xi_{11} \triangleq -2(P_i+P_r+P_f+S_i+S_r+S_f$$
$$+F_i+F_r+F_f+F_i'+F_r'+F_f')$$
$$\Xi_{12} \triangleq -2(G_i'+G_r'+G_f')+F_f\tilde{A}_i+F_i\tilde{A}_f+F_i\tilde{A}_r$$
$$+F_r\tilde{A}_i+F_r\tilde{A}_f+F_f\tilde{A}_r$$
$$\Xi_{13} \triangleq -2(H_i+H_r+H_f)+F_f\tilde{A}_{di}+F_i\tilde{A}_{df}+F_i\tilde{A}_{dr}$$
$$+F_r\tilde{A}_{di}+F_r\tilde{A}_{df}+F_f\tilde{A}_{dr}$$
$$\Xi_{22} \triangleq 2(P_i+P_r+P_f)+G_f\tilde{A}_i+G_i\tilde{A}_f+G_i\tilde{A}_r+G_r\tilde{A}_i+G_f\tilde{A}_r$$
$$+G_r\tilde{A}_f+\tilde{A}_f'G_i'+\tilde{A}_i'G_f'+\tilde{A}_f'G_r'+\tilde{A}_r'G_f'+\tilde{A}_i'G_r'+\tilde{A}_r'G_i'$$
$$\Xi_{23} \triangleq G_f\tilde{A}_{di}+G_i\tilde{A}_{df}+G_i\tilde{A}_{dr}+G_r\tilde{A}_{di}+G_f\tilde{A}_{dr}+G_r\tilde{A}_{df}$$
$$+\tilde{A}_f'H_i'+\tilde{A}_i'H_f'+\tilde{A}_i'H_r'+\tilde{A}_r'H_i'+\tilde{A}_f'H_r'+\tilde{A}_r'H_f'$$
$$\Xi_{33} \triangleq 2(S_i+S_r+S_f)+H_f\tilde{A}_{di}+H_i\tilde{A}_{df}+H_i\tilde{A}_{dr}+H_r\tilde{A}_{di}$$
$$+H_f\tilde{A}_{dr}+H_r\tilde{A}_{df}+\tilde{A}_{df}'H_i'+\tilde{A}_{di}'H_f'+\tilde{A}_{di}'H_r'+\tilde{A}_{dr}'H_i'$$
$$+\tilde{A}_{df}'H_r'+\tilde{A}_{dr}'H_f'$$

then (10) holds with

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i;\ S(\alpha) = \sum_{i=1}^{N} \alpha_i S_i;\ F(\alpha) = \sum_{i=1}^{N} \alpha_i P_i;$$
$$G(\alpha) = \sum_{i=1}^{N} \alpha_i S_i;\ H(\alpha) = \sum_{i=1}^{N} \alpha_i H_i;$$
$$\sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0, \quad i = 1,\ldots,N \tag{17}$$

implying that the uncertain discrete-time system (4) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ given by (2) is robustly stable irrespective of the value d of the time delay.

Proof: Clearly, $P_i > \mathbf{0}$, $S_i > \mathbf{0}$, $i = 1,\ldots,N$ imply that $P(\alpha)$ and $S(\alpha)$ given by (17) are positive definite matrices. Since $\mathcal{M}(\alpha)$ in (10) can be written as

$$\mathcal{M}(\alpha) = \sum_{i=1}^{N} \alpha_i^3 \mathcal{M}_i + \sum_{i=1}^{N} \sum_{r=1; i\neq r}^{N} \alpha_i^2 \alpha_r \mathcal{M}_{ir} + \sum_{i=1}^{N-2} \sum_{r=i+1}^{N-1} \sum_{f=r+1}^{N} \alpha_i \alpha_r \alpha_f \mathcal{M}_{irf} \tag{18}$$

and $\alpha_i \geq 0$, $i = 1,\ldots,N$, conditions (14)-(16) from Theorem 1 assure that $\mathcal{M}(\alpha) < 0$ for all $\alpha \in \mathbb{R}^N$ such that $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$. ■

Particular choices of the extra matrices in Theorem 1 can provide less numerically involving (but more conservative) sufficient conditions. For instance, imposing $F_i = F$, $G_i = G$ and $H_i = H$, $i = 1,\ldots,N$ one has the following result.

Corollary 1: If there exist N symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $S_i \in \mathbb{R}^{n\times n}$ and matrices $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \star & \Phi_{22} & \Phi_{23} \\ \star & \star & \Phi_{33} \end{bmatrix} > \mathbf{0}; \quad i = 1,\ldots,N \tag{19}$$

where

$$\Phi_{11} \triangleq -(F + F' + P_i + S_i)\ ;\ \Phi_{12} \triangleq -G' + F\tilde{A}_i$$
$$\Phi_{13} \triangleq -H + F\tilde{A}_{di}\ ;\ \Phi_{22} \triangleq P_i + G\tilde{A}_i + \tilde{A}_i' G'$$
$$\Phi_{23} \triangleq G\tilde{A}_{di} + \tilde{A}_i' H'\ ;\ \Phi_{33} \triangleq S_i + H\tilde{A}_{di} + \tilde{A}_{di}' H'$$

then (10) holds with $F(\alpha) = F$, $G(\alpha) = G$, $H(\alpha) = H$ and $P(\alpha)$, $S(\alpha)$ as in (17), implying that the uncertain discrete-time system (4) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ given by (2) is robustly stable irrespective of the value d of the time delay.

Moreover, the quadratic stability condition (i.e. constant matrices $P(\alpha)$ and $S(\alpha)$ in Lemma 1) can be recovered from the particular choice $P_i = P$, $S_i = S$, $F = F' = -(S + P)$ and $G = H = \mathbf{0}$ in (19), allowing to test the robust stability of time-varying systems (including unknown time-varying delays d) as well, as stated in the following corollary.

Corollary 2: If there exist symmetric positive definite matrices $P \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} P + S & -(P + S)\tilde{A}_i & -(P + S)\tilde{A}_{di} \\ \star & P & \mathbf{0} \\ \star & \star & S \end{bmatrix} > \mathbf{0};\ i = 1,\ldots,N \tag{20}$$

then (6)-(7) hold with $P(\alpha) = P$ and $S(\alpha) = S$, implying that the uncertain discrete time-varying system (4) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ given by (2) is robustly stable irrespective of the value d of the time delay.

The LMI conditions presented here, formulated in terms of the closed-loop matrices $\tilde{A}(\alpha)$ and $\tilde{A}_d(\alpha)$, allow a simple solution to Problem 1 by means of specialized polynomial time algorithms [26]. Note that similar robust stability conditions could be obtained by simply replacing $\tilde{A}(\alpha)$ by $\tilde{A}(\alpha)'$ and $\tilde{A}_d(\alpha)$ by $\tilde{A}_d(\alpha)'$ (i.e. by testing the robust stability of the dual of system (4)). The extra degree of freedom provided by the parameter dependent matrices $F(\alpha)$, $G(\alpha)$ and $H(\alpha)$ in the condition *d)* of Lemma 1 can be used to derive a solution to Problem 2, as shown in next section.

IV. ROBUST STABILIZATION

Consider the uncertain discrete-time system given by (1) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ given by (2) (time-invariant case) and suppose that the state as well as the delayed state are available for feedback. A solution to Problem 2 is given by the following theorem.

Theorem 2: If there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$ and $S_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,N$, and matrices $F \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{m_1\times n}$ and $Z_d \in \mathbb{R}^{m_2\times n}$ such that

$$\begin{bmatrix} -(F + F' + P_i + S_i) & FA_i' + Z'B_i' & FA_{di}' + Z_d'B_{di}' \\ \star & P_i & \mathbf{0} \\ \star & \star & S_i \end{bmatrix} > \mathbf{0};\quad i = 1,\ldots,N \tag{21}$$

then the robust state feedback gains K and K_d respectively given by

$$K = Z(F')^{-1}; \quad K_d = Z_d(F')^{-1} \tag{22}$$

are such that the closed-loop system (5) is robustly stable irrespective of the value d of the time delay. Moreover, $P(\alpha)$ and $S(\alpha)$ given by (17) are such that the conditions of Lemma 1 hold.

Proof: First, note that since $P_i > \mathbf{0}$ and $S_i > \mathbf{0}$, $-(F + F') > \mathbf{0}$ and thus F is nonsingular. Then, using (22) the LMIs of (21) can be rewritten as

$$\begin{bmatrix} -(F + F' + P_i + S_i) & F(A_i + B_iK)' & F(A_{di} + B_{di}K_d)' \\ \star & P_i & \mathbf{0} \\ \star & \star & S_i \end{bmatrix} > \mathbf{0};\quad i = 1,\ldots,N \tag{23}$$

which, by Corollary 1, with the particular choice $G = H = \mathbf{0}$, assures that the dual of the closed-loop system (4) (and thus also (4)) is robustly stable irrespective of the value d of the time delay. ∎

Using the results of Theorem 2, a solution to Problem 2 can be investigated by means of a feasibility test in a set of N LMIs described only at the vertices of the uncertainty polytope $\mathcal{D}$. The closed-loop stability is guaranteed by means of a parameter dependent Lyapunov-Krasovskii functional given by (6), with $P(\alpha)$ and $S(\alpha)$ as in (17) solving Lemma 1. These results apply for uncertain time-invariant systems, but next Corollary states sufficient conditions for the existence of robust feedback gains solving Problem 2 for the time-varying case, providing constant matrices P and S assuring the robust stability of system (4) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ given by (2) irrespective of the time delay d.

Corollary 3: If there exist symmetric positive definite matrices $P \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$ and matrices $F \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{m_1\times n}$ and $Z_d \in \mathbb{R}^{m_2\times n}$ such that

$$\begin{bmatrix} -(F+F'+P+S) & FA_i' + Z'B_i' & FA_{di}' + Z_d'B_{di}' \\ \star & P & \mathbf{0} \\ \star & \star & S \end{bmatrix} > \mathbf{0};$$
$$i = 1,\ldots,N \quad (24)$$

then the robust state feedback gains K and K_d given by (22) are such that the time-varying closed-loop system (5) is quadratically stable irrespective of the value d of the time delay. Moreover, condition (7) holds with $P(\alpha) = P$ and $S(\alpha) = S$ in (6).

Proof: Similar to the proof of Theorem 2. Note that with the control gains (22), the closed-loop system (5) is such that Corollary 2 holds with P and S. ∎

Theorem 2 and Corollary 3 explore the extra variables defined in Lemma 1, *d)*, to provide convex LMI conditions to solve Problem 2. The results of Theorem 2 and Corollary 3 can also be used to compute decentralized control gains, by imposing a decentralized structure to matrices

$$F = F_D = \text{block-diag}\{F^1,\ \ldots,\ F^M\} \quad (25)$$
$$Z = Z_D = \text{block-diag}\{Z^1,\ \ldots,\ Z^M\} \quad (26)$$
$$Z_d = Z_{dD} = \text{block-diag}\{Z_d^1,\ \ldots,\ Z_d^M\} \quad (27)$$

with M being the number of subsystems, yielding the block-diagonal stabilizing feedback gains

$$K_D = Z_D(F_D')^{-1} \quad ; \quad K_{dD} = Z_{dD}(F_D')^{-1} \quad (28)$$

Note that in this case no structural constraint is imposed to matrices $P(\alpha)$ and $S(\alpha)$ used in the Lyapunov-Krasovskii functional. Moreover, it is also possible to design state feedback control laws that use only $x(k)$ or $x(k-d)$ for feedback, decentralized or not, by simply fixing $Z = \mathbf{0}$ or $Z_d = \mathbf{0}$ in the LMIs.

V. EXAMPLES

Example 1: The first example is borrowed from [6], where the system has been found to be quadratically stable for a fixed delay $d = 2$ and norm-bounded uncertainties. A polytopic representation of the uncertain discrete-time system with state delay is given by (4) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ as in (2) with vertices

$$A_1 = \begin{bmatrix} -0.545 & -0.43 \\ 0.185 & -0.61 \end{bmatrix}; A_{d1} = \begin{bmatrix} 0.24 & 0.07 \\ -0.12 & 0.09 \end{bmatrix} \quad (29)$$

$$A_2 = \begin{bmatrix} -0.455 & -0.37 \\ 0.215 & -0.59 \end{bmatrix}; A_{d2} = \begin{bmatrix} 0.36 & 0.13 \\ -0.08 & 0.11 \end{bmatrix} \quad (30)$$

The conditions of Corollary 2 provide a feasible solution, implying that the uncertain system is quadratically stable irrespective of the value d of the time-delay (not only for $d = 2$ as in [6]), which can in addition be time-varying.

Suppose now that the uncertain polytopic system given by (29)-(30) is perturbed by a fixed positive value ρ, yielding a polytope described by the vertices $(\rho A_1, \rho A_2, \rho A_{d1}, \rho A_{d2})$ with $\rho = 4$ and input matrices given by

$$B_1 = B_2 = B_{d1} = B_{d2} = \begin{bmatrix} 0 & 1 \end{bmatrix}' \quad (31)$$

This uncertain discrete-time system with time delay d is not quadratically stabilizable, but the LMIs of Theorem 2 provide a feasible solution yielding the stabilizing feedback gains

$$K = \begin{bmatrix} 1.8042 & 4.4983 \end{bmatrix}; \quad K_d = \begin{bmatrix} -1.2807 & -0.9686 \end{bmatrix}$$

as a solution to Problem 2. In this case, d can be any nonnegative integer value, but the exact knowledge of $x(k-d)$ is crucial for the robust stabilization of the system. In fact, no feasible solution is obtained if Z or Z_d are fixed to zero.

Example 2: The second example is also from [6], where for a fixed delay $d = 2$ and norm-bounded uncertainties a quadratic stabilizing feedback gain has been computed from nonconvex conditions. A polytopic model is given by (4) with $(A, A_d, B, B_d)(\alpha) \in \mathcal{D}$ as in (2) with vertices

$$A_1 = \begin{bmatrix} 0.998 & -0.603 \\ 0.398 & 0.497 \end{bmatrix}; \; A_{d1} = \begin{bmatrix} 0.498 & 0.199 \\ 0.598 & 0.399 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1.002 & -0.597 \\ 0.402 & 0.503 \end{bmatrix}; \; A_{d2} = \begin{bmatrix} 0.502 & 0.201 \\ 0.602 & 0.401 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.1 & 0.05 \\ -0.2 & -0.05 \end{bmatrix}; \; B_2 = \begin{bmatrix} 0.3 & 0.35 \\ 0.2 & 0.25 \end{bmatrix}$$

Using the constraint $Z_d = \mathbf{0}$ in Corollary 2, a quadratically stabilizing memoryless feedback gain is obtained

$$K = \begin{bmatrix} 4.6907 & -0.4151 \\ -6.2837 & 0.6122 \end{bmatrix}$$

Differently from the gain given in [6], this control gain assures the closed-loop quadratic stability irrespective of the value d of the delay, which can be unknown and/or time-varying.

Example 3: The third example, randomly generated, is given by (1)-(2) with unknown delay d and $N = 3$ vertices

$$A_1 = \begin{bmatrix} 0.266 & 0.126 & 0.343 & 0.175 \\ 0.35 & 0.021 & 0.182 & 0.238 \\ 0.105 & 0.105 & 0.028 & 0.049 \\ 0.364 & 0.14 & 0.245 & 0.112 \end{bmatrix}$$

$$A_{d1} = \begin{bmatrix} 0.114 & 0.054 & 0.147 & 0.075 \\ 0.15 & 0.009 & 0.078 & 0.102 \\ 0.045 & 0.045 & 0.012 & 0.021 \\ 0.156 & 0.06 & 0.105 & 0.048 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.024 & 0.198 & 0.108 & 0.198 \\ 0.342 & 0.096 & 0.078 & 0.168 \\ 0.096 & 0.072 & 0.09 & 0.27 \\ 0.012 & 0.072 & 0.192 & 0.342 \end{bmatrix}$$

$$A_{d2} = \begin{bmatrix} 0.016 & 0.132 & 0.072 & 0.132 \\ 0.228 & 0.064 & 0.052 & 0.112 \\ 0.064 & 0.048 & 0.06 & 0.18 \\ 0.008 & 0.048 & 0.128 & 0.228 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.168 & 0.004 & 0.168 & 0.036 \\ 0.192 & 0.012 & 0.032 & 0.084 \\ 0.172 & 0.116 & 0.076 & 0.12 \\ 0.064 & 0.048 & 0.132 & 0.124 \end{bmatrix}$$

$$A_{d3} = \begin{bmatrix} 0.252 & 0.006 & 0.252 & 0.054 \\ 0.288 & 0.018 & 0.048 & 0.126 \\ 0.258 & 0.174 & 0.114 & 0.18 \\ 0.096 & 0.072 & 0.198 & 0.186 \end{bmatrix}$$

Both Corollaries 1 and 2 (quadratic stability) fail to provide a feasible solution, but the more complex LMIs of Theorem 1 assure that the system is robustly stable irrespective of the value d of the delay.

VI. CONCLUSION

Convex delay independent LMI conditions have been given for the robust stability and robust stabilizability of discrete-time systems with uncertain parameters in polytopic domains and state delay. The use of parameter dependent matrices in the Lyapunov-Krasovskii functional and extra matrices in the conditions allows less conservative evaluations of stability domains as well as the convex procedure design of state feedback gains, encompassing previous results based on quadratic stability. Additional constraints can be incorporated in the feedback gains without imposing a particular structure to the Lyapunov-Krasovskii matrices, allowing for instance to cope with decentralized control.

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