Continuous Swarm Optimization Technique with Stability Analysis

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***Abstract*— A new continuous-time particle swarm optimization (PSO) algorithm is introduced as opposed to the well-known discrete-time PSO. A compact notation for the classical PSO is first introduced, then the new PSO is presented. The proposed PSO has two forms, one can be proved to be Lyapunov stable and the other can be proved to be globally asymptotically stable. Effect of the parameters of the new PSO is studied. Pertinent simulation results are provided to illustrate its major characteristics.**

***Index Terms*— Particle Swarm, Optimization, Stability**

I. Introduction

Deterministic techniques were dominating the optimization field for several decades. Their inability to find global minima, and their complexity were the motivation for stochastic and heuristic algorithm. Most of the heuristic algorithm are based on modeling natural optimization processes such as simulated annealing, crystal growth and genetic evolution These algorithms are based on populations of individuals with a specific behavior depending on the natural process modeled.

Among the first heuristic algorithms, random search (RS) is a simple technique based on random test moves of one of the individuals in the search space [1]. The individual is actually moved to the new position, if the cost function in the new position is better. Genetics and Memetics algorithms are other more recent examples. As their name imply genetic algorithm (GA) models the natural genetic evolution. GA is based on exchanging parts of a string of coded information (gene) between two individuals (parents). Memetics algorithm models the exchange of ideas among a group of persons [2].

Recently, the concept of particle swarm optimization (PSO) is proposed in [3]. It is motivated by the zoologist models of the movement of individuals among a group (school of fishes, flock of birds or swarm of insects). It has been noticed that group members share information about the best positions found during their search for food. Hence, the PSO model incorporate both individual and social experience in the search. PSO has been successfully applied, among other fields, in electrical power system applications [4] and in electromagnetic field calculations [5]. The basic PSO can be modified to include bad memories to be avoided [6]. Parameters effect on the PSO has been investigated in [7]. From systems point of view, standard PSO algorithm is discrete in nature: particle motions are updated every time step.

In this paper a new continuous time model for PSO is developed. The continuous PSO (CPSO) model is motivated by the fact that natural PSO have continuous motions (they do not jump). Moreover, from the systems point of view, discretization has a de-stabilizing effect. A review of the original PSO algorithm is presented in section 2 with a new compact notation. The developed CPSO model is presented in section 3. In section 4, CPSO is shown to be Lyapunov stable. To ensure asymptotic stability, a modified CPSO is presented in section 5, and its global asymptotic stability is proven in section 6. Section 7 compares the parameters of the PSO with those of the CPSO. Simulation results presented in section 8 shows that the developed model has a better performance than classical discrete PSO.

II. Discrete Swarm Algorithm

As inspired form the natural swarms, the PSO technique conducts the search for an optimum using a group (population) of particles which are traveling in the feasible region. The position of each particle (bird) at any given time represents a candidate solution. Every period of time, each bird evaluates the cost function at its current position and compares it with the self-best that it 'remembers'. Of course, the self-best is the position at which the cost function had the best value during the bird's search. The birds exchange information about their individual self-bests to determine the best position they have ever discovered, the global best. The birds direction of motion includes both social-only and cognition only components. The social-only component pulls up the birds towards the global best, while the cognition only component pulls each bird towards its self-best. Hence the speed of each bird has three components, namely, a momentum component trying to keep the same search direction, a component towards the birds self-best and a component toward the global swarm best. The PSO update equations for the i^{th} bird are hence given by the vectorial equation [7]:

$$v_i(k+1) = \alpha_d v_i(k) + \beta_d(x_{lb_i} - x_i(k)) + \gamma_d(x_{gb} - x_i(k)) \quad (1)$$

$$x_i(k+1) = x_i(k) + v_i(k+1) \quad (2)$$

where x_i represents the i^{th} bird position, v_i its speed, x_{lb_i} its local best, x_{gb} the swarm global best, α_d, β_d and γ_d are positive non zero reals. As the previous equation suggest, the basic PSO algorithm is discrete-time in nature, where each bird has a simple set of motion equations. These equations can be rearranged to describe the whole swarm in a compact manner.

We consider, without loss of generality, the problem of minimizing a cost function and define the symbols throughout the remaining of this work: d denotes the problem dimension, $\Omega \subseteq \mathcal{R}^d$ represents the feasible region, n denotes the number of birds, $f : \Omega \to \mathcal{R}$ is the function to be minimized. In addition, for the sake of compactness, the following vectors and matrices are defined as follows:

- $X \triangleq \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \in (\Omega \times \mathcal{R}^n)$ the position matrix
- $V \triangleq \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \in (\mathcal{R}^d \times \mathcal{R}^n)$ the velocity matrix
- $X_{lb} \triangleq \begin{bmatrix} x_{lb_1} & \dots & x_{lb_n} \end{bmatrix} \in (\Omega \times \mathcal{R}^n)$ the local best position matrix
- $X_{gb} \in \mathcal{R}^d$ the global best position matrix
- $F \triangleq \begin{bmatrix} f(x_1) & \dots & f(x_n) \end{bmatrix} : (\Omega \times \mathcal{R}^n) \to \mathcal{R}^n$ is the stacked objective function raw vector
- $T \in \mathcal{R}^d$ a row vector composed of ones
- $Q_i \in \mathcal{R}^n$ is a column vector having all elements equal to zero except the i$^{\text{th}}$ element that equals one.
- $I_n \in (\mathcal{R}^n \times \mathcal{R}^n)$ is the identity matrix of size n.

Hence the proposed new elegant representation for the classic discrete time PSO algorithm is:

$$\begin{aligned} V(k+1) &= \alpha_d V(k) + \beta_d (X_{lb}(k) - X(k)) \\ &\quad + \gamma_d (X_{gb}(k) T - X(k)) \end{aligned} \tag{3}$$

$$X(k+1) = X(k) + V(k+1) \tag{4}$$

$$\begin{aligned} X_{lb}(k+1) &= \frac{1}{2}\left[X(k+1) + X_{lb}(k)\right] \\ &\quad + \frac{1}{2}\left[X(k+1) - X_{lb}(k)\right]\zeta \end{aligned} \tag{5}$$

$$\text{where: } \zeta = \text{diag}\left[\text{sgn}\left(F\left(X_{lb}(k)\right) - F\left(X(k+1)\right)\right)\right] \tag{6}$$

diag$[y]$ is a diagonal matrix with the diagonal elements given by the elements of the vector y and sgn( ) denotes the signum function, defined as:

$$\text{sgn}(y) = \begin{cases} 1 & y \geq 0 \\ -1 & y < 0 \end{cases} \tag{7}$$

III. Continuous Swarm Algorithm

Although discrete time swarm model was successfully applied into several applications, natural swarm are continuous in nature, i.e., they are not jumping at fixed time intervals, their motion is actually smooth. This fact proposes that the development of the continuous time swarm model may boost the PSO performance. Another important motivation is the fact that the discretizing phenomena has a de-stabilizing effect which may affect the learning procedure.

To develop a complete continuous model, equations (3) and (4) must be replaced by their continuous equivalent. Using the same notations as in discrete model, we can describe the motion equations as:

$$\dot{V} = -\alpha V + \beta(X_{lb} - X) + \gamma(X_{gb} T - X) \tag{8}$$

$$\dot{X} = V \tag{9}$$

where the dimensions of X, V, X_{lb}, X_{gb}, and T are as defined above; and the time notation is dropped for brevity; α, β and γ are positive non zero reals.

However, it should be noted that X and V are not the only system states, X_{lb} is also a state since it has a 'memory'. For example, the i$^{\text{th}}$ column in X_{lb}, which represents the best position the i$^{\text{th}}$ bird encountered during its previous path, remains unchanged as long as the bird does not find a better position, but once a better position is encountered, its location replaces the previous value of the i$^{\text{th}}$ column in X_{lb}. From the state model point of view, the matrix X_{lb} should be dealt with as additional states. The time derivative of these states will be zero as long as the local best position does not change and impulsive once a new best position is found to change the 'state' value in no time.

Although the implementation of such zero/impulsive derivative state is possible using standard software package, their stability analysis is not straight forward. Hence, assuming a to be a positive constant, it is proposed to approximate the evolution of x_{lb_i} for a minimization:

$$\dot{x}_{lb_i} = a\left(x_i - x_{lb_i}\right) + a\left(x_i - x_{lb_i}\right)\left[\text{sgn}\left(f\left(x_{lb_i}\right) - f\left(x_i\right)\right)\right] \tag{10}$$

This equation can be represented in matrix form as:

$$\dot{X}_{lb} = a\left(X - X_{lb}\right)\left[I_n + \text{diag}\left[\text{sgn}\left(F\left(X_{lb}\right) - F\left(X\right)\right)\right]\right] \tag{11}$$

Remark 1: To better understand equation (11), let us consider equation (10) for the following cases:

- If $f(x_{lb_i}) - f(x_i) \geq 0$, then sign$(f(x_{lb_i}) - f(x_i)) = 1$ and $\dot{x}_{lb_i} = 2a(x_i - x_{lb_i})$, which means that if the value of the objective function evaluated at the current point is less than its value at the local best, then the local best is attracted to the current point.
- If $f(x_{lb_i}) - f(x_i) < 0$, then sign$(f(x_{lb_i}) - f(x_i)) = -1$ and $\dot{x}_{lb_i} = 0$, which means that if the value of the objective function evaluated at the local best is less than its value at the current point, then the local best remains unchanged.

Remark 2: Based on the equation above, it is worth pointing out that the X_{lb} is not necessarily the local best, but rather an approximation because no impulsive derivative is used in this continuous-time algorithm

Finally, it remains to characterize the global best X_{gb}, which can be defined as:

$$X_{gb} = X_{lb} Q_j \text{ with } j = \arg\left\{\inf_{0 < i \leq n}\left(f\left(x_{lb_i}\right)\right)\right\} \tag{12}$$

From the system point of view, X_{gb} can be considered a setpoint since its only effect is to change the equilibrium point of the dynamical system defined by the swarm.

Definition 3: Using the previously defined notations, the continuous swarm is described by the following dynamical system Σ:

$$\begin{aligned}
&\dot{X}= V\\
&\dot{V}= -\alpha V + \beta(X_{lb} - X) + \gamma(X_{gb}T - X)\\
&\dot{X}_{lb}= a\,(X - X_{lb})\,[I_n + \text{diag}\,[\text{sgn}\,(F\,(X_{lb}) - F\,(X))]]\\
&X_{gb} = X_{lb}Q_j \text{ where } j = \arg\left\{\inf_{0<i\leq n}\,(f\,(x_{lb_i}))\right\}
\end{aligned} \tag{13}$$

Remark 4: Notice that the notation used above is not the standard state space notation since the state variable X, V and X_{lb} are not vectors but rather matrices of the adequate dimension previously defined. This choice is obviously motivated by the simplicity and compactness it provides without loss of clarity.

IV. Stability Analysis of the Continuous Swarm

The continuous swarm Σ can be considered as a dynamical system whose stability can be investigated using standard Lyapunov stability tools. The determination of the equilibria of the system needs further analysis. The equilibria must verify the following equations:

$$V = 0 \tag{14}$$

$$\beta(X_{lb} - X) + \gamma(X_{gb}T - X) = 0 \tag{15}$$

$$a\,(X - X_{lb})\,[I_n + \text{diag}\,[\text{sign}\,(F\,(X_{lb}) - F\,(X))]] = 0 \tag{16}$$

A close look at equation (16) shows that it can be verified for the following two cases:

1) $X = X_{lb}$, which implies by equation (15) that $X = X_{lb} = X_{gb}T$
2) $\exists$ a subset $\Gamma \subseteq \{1, 2, ..., n\}$ such that $\forall i \in \Gamma, x_i \neq x_{lb_i}$ and $f\,(x_{lb_i}) - f\,(x_i) < 0$; and $\forall j \in [\{1, 2, ..., n\} - \Gamma]$ we have $x_j = x_{lb_j}$. In this case, we necessarily have $\forall i \in \Gamma$ the following equation satisfied:

$$\beta(x_{lb_i} - x_i) = -\gamma(X_{gb} - x_i) \tag{17}$$

which can be interpreted as follows: the pulls of the local and global bests are vectorially opposite in sign for some of the birds while their speed was zero.

This phenomenon is particular to the continuous swarm and does not have a counterpart in the discrete-time swarm. Of course, the interesting equilibrium point is the first equilibrium where all the birds are located at the global best. Let us consider the stability of this equilibrium point. We will see in the next section, how the continuous swarm algorithm can be modified in order to make the equilibrium point unique.

We define the following error vectors:

$$\tilde{X} = X - X_{gb}T \tag{18}$$

$$\tilde{X}_{lb} = X_{lb} - X_{gb}T \tag{19}$$

Assuming that the global best remains constant or changes very slowly, the continuous-time swarm equations can be expressed as:

$$\begin{aligned}
&\dot{\tilde{X}}= V\\
&\dot{V}= -\alpha V + \beta(\tilde{X}_{lb} - \tilde{X}) - \gamma\tilde{X}\\
&\dot{\tilde{X}}_{lb}= a\left(\tilde{X} - \tilde{X}_{lb}\right)[I_n + \text{diag}\,[\text{sgn}\,(F\,(X_{lb}) - F\,(X))]]
\end{aligned} \tag{20}$$

Consider the following candidate Lyapunov function:

$$W = \frac{1}{2}\text{tr}\left[\gamma\tilde{X}^T\tilde{X} + V^TV + \beta\left(\tilde{X} - \tilde{X}_{lb}\right)^T\left(\tilde{X} - \tilde{X}_{lb}\right)\right] \tag{21}$$

where $\text{tr}(\cdot)$ is the trace of $(\cdot)$. The derivative of W along the solution of the continuous swarm is given by:

$$\begin{aligned}
\dot{W} &= \text{tr}\left[\gamma\tilde{X}^T\,\dot{\tilde{X}} + V^T\dot{V} + \beta\left(\tilde{X} - \tilde{X}_{lb}\right)^T\left(\dot{\tilde{X}} - \dot{\tilde{X}}_{lb}\right)\right]\\
&= \text{tr}\Big[\gamma\tilde{X}^TV - \alpha V^TV + \beta V^T(\tilde{X}_{lb} - \tilde{X}) - \gamma V^T\tilde{X}\\
&\quad +\beta\left(\tilde{X} - \tilde{X}_{lb}\right)^T V\\
&\quad -a\beta\left(\tilde{X} - \tilde{X}_{lb}\right)^T\left(\tilde{X} - \tilde{X}_{lb}\right)\varphi\Big]\\
&= \text{tr}\left[-\alpha V^TV - a\beta\left(\tilde{X} - \tilde{X}_{lb}\right)^T\left(\tilde{X} - \tilde{X}_{lb}\right)\varphi\right]
\end{aligned} \tag{22}$$

where $\varphi = I_n +$ diag[sgn $(F\,(X_{lb}) - F\,(X))]$. It can be seen immediately that all elements of the $n \times n$ matrix φ can take only two values 0 or 2, which directly implies that the derivative $\dot{W}$ is negative semi-definite, which implies that the considered equilibrium point is stable in the sense of Lyapunov.

This result even though ensures that the swarm does not 'blow up', does not guarantee that the birds converge to the global best or even to their individual local bests. Even a trial to apply invariance theorem will never prove asymptotic stability of the equilibrium point due to similar arguments investigated in the determination of the equilibria of the swarm. A modification should be made to the proposed continuous-time swarm algorithm to guarantee:

- Uniqueness of the global best equilibrium point
- Global asymptotic stability of this equilibrium.

V. Modified Continuous-Time Swarm Algorithm

Up till this section, the continuous swarm algorithm is quite inspired by the classical discrete-time swarm algorithm, however, as we have seen in the previous section, no asymptotic stability of the global best equilibrium point can be guaranteed. This basically due to the possibility of the occurrence that for one or more of the birds, the pulls of the local and global bests are equal and opposite in direction while the birds do not have any momentum. Even though this case will rarely occur practically, there is always a mathematical possibility of its occurrence. We propose in this section the addition of small damping terms which will cancel out this possibility. The modified continuous swarm algorithm Σ_m is thus defined as:

$$\begin{aligned}
&\dot{X}= V + \delta \left(X_{gb}T - X\right) \\
&\dot{V}= -\alpha V + \beta(X_{lb} - X) + \gamma(X_{gb}T - X) \\
&\dot{X}_{lb}= a \left(X - X_{lb}\right) \left[I_n + \text{diag}\left[\text{sgn}\left(F\left(X_{lb}\right) - F\left(X\right)\right)\right]\right] \\
&\qquad +\varepsilon \left(X_{gb}T - X_{lb}\right) \\
&X_{gb} = X_{lb}Q_j \text{ where } j = \arg\left\{\inf_{0<i\leq n}\left(f\left(x_{lb_i}\right)\right)\right\}
\end{aligned} \tag{23}$$

where δ and ε are small positive real constants. The addition of these damping terms will have the effect of attracting both the bird position and their individual bests to the global best. It is worth pointing out that this limits the mobility of the birds, i.e. their ability to eventually discover new local bests and thus updating the global best. However, this negative effect is minimized by choosing δ and ε are chosen very small. Simulations results provided in the subsequent sections show how their effects are negligeable.

VI. Modified CPSO Stability Analysis

As we have proceeded in the continuous-time swarm, let us first verify that the modification introduced makes the global best the unique equilibrium point. The equilibria must verify the following equations:

$$V + \delta \left(X_{gb}T - X\right) = 0 \tag{24}$$

$$-\alpha V + \beta(X_{lb} - X) + \gamma(X_{gb}T - X) = 0 \tag{25}$$

$$a \left(X - X_{lb}\right) \varphi + \varepsilon \left(X_{gb}T - X_{lb}\right) = 0 \tag{26}$$

Obviously $V = 0$, $X = X_{lb} = X_{gb}T$ is an equilibrium point, but is it the unique equilibrium point? A close investigation is necessary to verify this fact. From equations (24) and (25), we obtain that:

$$\left(\alpha\delta + \gamma\right)\left(X_{gb}T - X\right) - \beta(X - X_{lb}) = 0 \tag{27}$$

Equation (26) can be rearranged after adding and substracting εX as follows:

$$\varepsilon \left(X_{gb}T - X\right) + \left(a + \varepsilon\right)\left(X - X_{lb}\right)\varphi = 0 \tag{28}$$

For each column of the matrices used in equations (27) and (28), two cases can happen according to the value of φ. The matrix defining the linear system of equation is given by :

$$A = \begin{bmatrix} \alpha\delta + \gamma & -\beta \\ \varepsilon & \lambda\left(a + \varepsilon\right) \end{bmatrix} \tag{29}$$

where $\lambda = 0$ or 2. Its determinant is given by:

$$|A| = \lambda\left(\alpha\delta + \gamma\right)\left(a + \varepsilon\right) + \beta\varepsilon \tag{30}$$

which is always positive non-zero for all possible values of λ. This implies that the only solution to the linear equations defined by (27) and (28) is $X = X_{lb} = X_{gb}T$. Thus, the unique equilibrium point of the modified continuous-time swarm is the global best.

To study the stability of this unique equilibrium point, we express its dynamical equations in the alternate coordinate frame used by (20):

$$\begin{aligned}
&\dot{\tilde{X}}= V - \delta\tilde{X} \\
&\dot{V}= -\alpha V + \beta(\tilde{X}_{lb} - \tilde{X}) - \gamma\tilde{X} \\
&\dot{\tilde{X}}_{lb}= a\left(\tilde{X} - \tilde{X}_{lb}\right)\left[I_n + \text{diag}\left[\text{sgn}\left(F\left(X_{lb}\right) - F\left(X\right)\right)\right]\right] \\
&-\varepsilon\tilde{X}_{lb}
\end{aligned} \tag{31}$$

and consider the same candidate Lyapunov function proposed in equation (21), whose derivative along the solution of Σ_m is given by:

$$\begin{aligned}
\dot{W} \quad = \quad \text{tr}\Big[&-\alpha V^TV - a\beta\left(\tilde{X} - \tilde{X}_{lb}\right)^T\left(\tilde{X} - \tilde{X}_{lb}\right)\varphi \\
&-\delta\tilde{X}^T\tilde{X} - \varepsilon\tilde{X}_{lb}^T\tilde{X}_{lb}\Big]
\end{aligned} \tag{32}$$

Because W is globally positive definite and radially unbounded and $\dot{W}$ is globally negative definite, global asymptotic stability of the global best is implied.

VII. PSO and CPSO Parameters

In order to be able to compare the behavior of the discrete time and continuous time , a method to relate the parameters in each case has to be presented. The analysis of the behavior of the discrete-time swarm has been investigated in [7]. However, it is possible to get the same results of [7] from the systems point of view, in a much simpler way. We first begin by identifying the conditions that the parameters of the discrete swarm have to fulfill in order to guarantee stability and then provide a method to map the discrete time parameters to those of the continuous time swarm.

A. Discrete Time Swarm Parameters

As it can be seen from equation (3), (4) and (5), the adjustable parameters for the discrete swarm are α_d, β_d and γ_d. α_d is the momentum term,while β_d and γ_d are the magnitude of the attractions to the local and global bests, respectively. For the sake of clarity of the argumentation, let us limit ourselves to a swarm consisting of only one bird traveling in a single dimensional space. Very similar arguments can be used in the general case. Notice that, in this case, the global best is irrelevant (it always coincides with the local best). Assuming for the purpose of this analysis that x_{lb} is constant, the equations of motion of the bird are given in a state-space form by:

$$v(k+1) = \alpha_d v(k) - \beta_d\left(x(k) - x_{lb}\right) \tag{33}$$

$$\left(x(k+1) - x_{lb}\right) = \alpha_d v(k) + \left(1 - \beta_d\right)\left(x(k+1) - x_{lb}\right) \tag{34}$$

The stability of this system depends on the roots of the characteristic equation:

$$\lambda^2 - \left(\alpha_d - \beta_d + 1\right)\lambda + \alpha_d = 0 \tag{35}$$

For the system to be stable, the solutions of (35) must be inside the unit circle. These roots are:

$$\lambda_{1d,2d} = \frac{(\alpha_d - \beta_d + 1) \pm \sqrt{\Delta}}{2} \quad (36)$$

where

$$\Delta = (\alpha_d - \beta_d + 1)^2 - 4\alpha_d \quad (37)$$

However, to increase the bird's mobility in order for it to cover as much of the space as possible before converging to the local best, the roots should be imaginary and the closest possible to the unit circle. Hence, the determinant Δ of (35) should be negative, i.e.: $\Delta < 0$ and $\alpha_d < 1$. An easy way to achieve this requirement is to choose α_d and β_d satisfying

$$\beta_d = \alpha_d + 1 \quad (38)$$

In this case the eigenvalues λ_{1d} and λ_{2d} of the discrete-time swarm are $\pm j\sqrt{\alpha_d}$. Hence, it remains to choose α_d such that $\sqrt{\alpha_d}$ is nearly unity.

B. Continuous Time Swarm Parameters

Before trying to map the parameters of the discrete and continuous time swarms, we, similarly to the previous subsection, consider the evolution of a single bird of the continuous time swarm in a single dimensional space. Its equations of motion, under the assumption that x_{lb} is constant, are given by:

$$\begin{aligned} \dot{v} &= -\alpha v - \beta(x - x_{lb}) \\ \frac{d}{dt}(x - x_{lb}) &= v \end{aligned} \quad (39)$$

The dynamics of the system are governed by the roots of the characteristic equation:

$$\lambda^2 + \alpha\lambda + \beta = 0 \quad (40)$$

which immediately implies that the eigenvalues λ_{1c} and λ_{2c} of the continuous-time swarm are:

$$\lambda_{1c,2c} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} \quad (41)$$

Using the root mapping technique from the Z-plane to the S-plane ($Z = e^{sT}$), where T is the sampling time, the roots of (35) can be mapped by:

$$\lambda_c = \frac{1}{T}\left(\ln \|\lambda_d\| + j\measuredangle\lambda_d\right) \quad (42)$$

where $\|y\|$ and $\measuredangle y$ are the norm and angle of y, respectively. Noticing that the product $\lambda_{1d}\lambda_{2d}$ is the constant term in (35), or in other words $\|\lambda_{1d}\| = \|\lambda_{2d}\| = \sqrt{\alpha_d}$ and hence:

$$\lambda_{1c} + \lambda_{2c} = -\alpha = \frac{2}{T}\ln\|\lambda_d\| = \frac{2}{T}\ln\sqrt{\alpha_d} = \frac{1}{T}\ln\alpha_d \quad (43)$$

Hence, to obtain the same dynamic response α and α_d have to be related by:

$$\alpha = -\frac{1}{T}\ln\alpha_d \quad (44)$$

On the other hand, β can be calculated as follows:

$$\begin{aligned} \beta &= \lambda_{1c}\lambda_{2c} = \left(\frac{1}{T}\ln\|\lambda_d\|\right)^2 + \left(\frac{1}{T}\measuredangle\lambda_d\right)^2 \\ &= \alpha^2 + \left(\frac{1}{T}\tan^{-1}\left(\frac{\sqrt{-\Delta}}{\alpha_d - \beta_d + 1}\right)\right)^2 \end{aligned} \quad (45)$$

If the choice proposed at the end of the last section is made, the selection of β becomes much easier and is given by:

$$\beta = \alpha^2 + \left(\frac{\pi}{2T}\right)^2 \quad (46)$$

Remark 5: It remains to determine the value of T, the sampling time. However, even though T does not have any physical meaning in the problem, it affects how close the discrete time and continuous time dynamics relate. Simulations results given in the next section show that, as it can be easily expected, the smaller T, the more damping there is in the continuous time dynamics with respect to the discrete and vice-versa.

VIII. Simulation Results

In this section, a relatively simple quadratic optimization problem is selected to emphasize the properties of the continuous swarm model as well as to compare its performance with discrete PSO. The selected cost function is given by:

$$f(y_1, y_2, y_3) = (Y - \sigma)^T (Y - \sigma) \quad (47)$$

where $\sigma = [\pi \ \ 2 \ \ 5]^T$. For this problem, the dimension of the problem is $d = 3$. The global minimum is clearly σ.

A. Comparison Between Continuous and Discrete PSO

To compare the response of continuous and discrete PSO, one bird swarm is selected in both cases, i.e. $n = 1$. The bird is initialized as follows: $X[0] = [\ 2 \ \ 4 \ \ 6\ ]^T$, $V[0] = [\ -1 \ \ 1 \ \ -1\ ]^T$. The discrete swarm parameters are set to $\alpha_d = 0.9$, and $\beta_d = 1.9$ from (38). The corresponding parameters for the continuous swarm are set according to (44) and (46), to $\alpha = 0.1054$, $\beta = 2.47$ at $T = 1$, for comparison purposes simulations are also carried out for $\alpha = 0.0527$, $\beta = 0.618$ at $T = 2$. The parameter a is set to 3.0 so that the dynamics of X_{lb} is fast compared to X .Since there is only one bird, the global and local bests are identical and there is no point in adding the global best term, i.e. both γ_d and γ are set to zero. Figure (1) shows the evolution of the first spatial component in the discrete, continuous with $T = 1$, and continuous with $T = 2$ cases. Figure (2) depicts the evolution of the value of the objective function at the best obtained local best for the cases defined above. A close look at the figures shows that:

- The transient performance of the position for the PSO and the CPSO are similar except for discretization effects. Moreover, the value of T affects the similarity between the PSOs, as expected.
- The CPSO finds the best faster than the PSO. This can be explained by the fact that the CPSO moves along all the points in the trajectory, as opposed to the PSO which computes the objective function at discrete points along the trajectory.
- This single bird swarm did not locate the global minimum of this three dimensional problem for both PSO and CPSO (see remark below).

Remark 6: Referring to figure (1), the bird converges to $x_1 = 2.71$ instead of π (the minimum). This result can be explained as follows. The motion of the bird in the discrete model from $k = 0$ to $k = 1$ is given by it initial speed at $k = 1$, either $X(0)$ or $X(1)$ will be considered as local best depending on the cost function. The speed equation is

$$v(2) = \alpha_d v(1) + 2 * \beta_d(x_{lb}(1) - x(1)) \quad (48)$$

The second term in the RHS is either zero or is equal to $-v(1)$, hence the two terms in the RHS are linearly dependent, and the new speed vector is just a scaled version of the previous speed. Using mathematical induction, it is easy to prove that this statement is valid for all k. Hence, the bird keeps moving rectilinearly and its locus is:

$$x(k) = x(0) + \rho v(0) \quad (49)$$

where ρ is some proportionality constant. The same discussion can be extended to continuous swarm model. The best cost achievable along the bird trajectory line can be found by substituting (49) into the cost function given by (47), which gives:

$$f(.) = (X(0) + \rho V(0) - \sigma)^T (X(0) + \rho V(0) - \sigma) \quad (50)$$

differentiating w.r.t. ρ, and equating to zero we get:

$$\rho = \frac{V(0)^T(X(0)-\sigma)}{V(0)^T V(0)} = -0.7139 \quad (51)$$

Substituting this value into eq(49), the best achievable position is given by $X = [\ 2.71 \quad 3.29 \quad 6.71\ ]^T$. This matches the steady state value of the bird trajectory.

B. Continuous PSO Performance

To evaluate the performance of a multi bird swarm, the cost function $f(.)$ of eq(47) is used. A swarm of 20 birds is formed with random initial positions and speeds uniformly distributed in $\left[\ 0 \quad 10\ \right]$ and $\left[\ -3 \quad 3\ \right]$. The swarm parameters are $\alpha = 0.1054$, $\beta = 2.47$ and $a = 3$. Simulation results, given in figure(3), show that the swarm settles to $X = [\ 3.141 \quad 2 \quad 5\ ]^T$, which shows that the CPSO finds the global minimum point. The information exchange between birds enables them to modify their trajectories not to be trapped to the simple line trajectory as in the single bird case. Figure (4) shows the Lyapunov function W of the swarm for the first five seconds of simulations along with the binary value that represents the occurrence of a change in X_{gb}.The Lyapunov function is generally decreasing, however it has some spikes, when X_{gb} is changed. The Lyapunov analysis presented in this paper assumes that there is no change in X_{gb}. If such change occurs, the negative definiteness of $\dot{W}$ does not hold at this point. However, as soon as X_{gb} stop changing, it starts attracting the birds in a stable manner. Notice also that the Lyapunov function does not correspond to the best cost function, an increase in the Lyapunov function is always associated with a decrease in cost function.

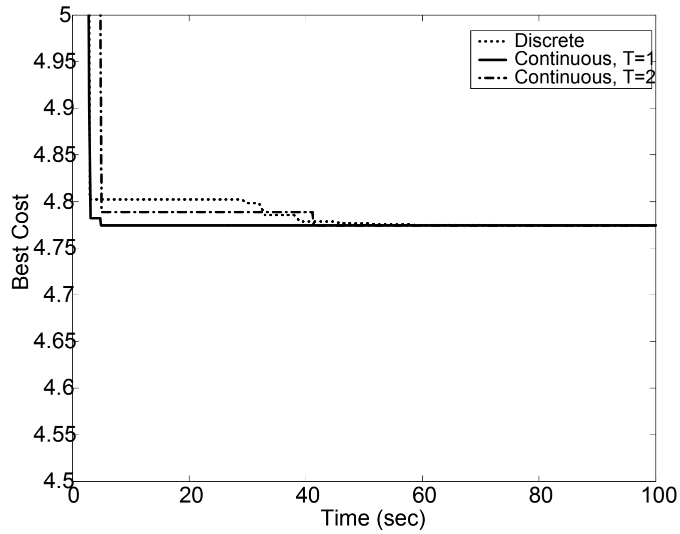

Fig. 1. PSO and CPSO First Dimesion Trajectories

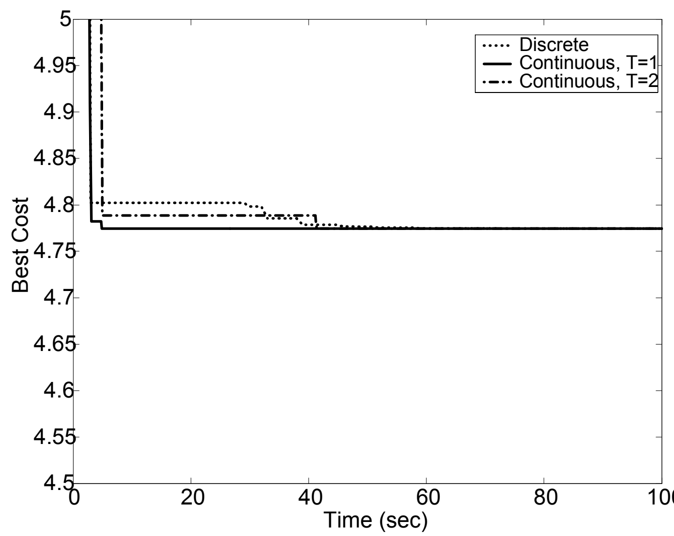

Fig. 2. Best Cost for PSO and CPSO

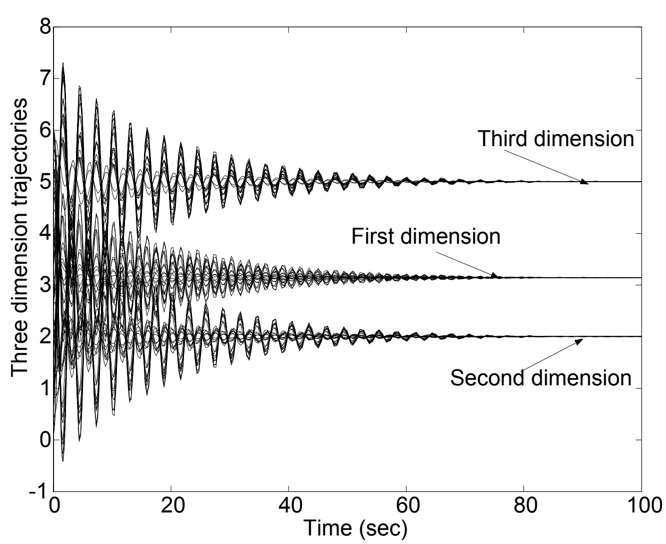

Fig. 3. CPSO Trajectories for the Three Dimension $n = 20$

Figure (5) shows the first component of the positions for a poorly tuned a parameter ($a = 0.2$ keeping α and β as previously used) versus the properly tuned at $a = 3$. The swarm converges to 4.6 instead of 5. X_{lb} does not move fast enough to capture the true local best before the birds move away from it: The swarm may be attracted to a deviated point even if the swarm trajectories passes through the global minimum.

Finally, the performance of the CPSO and the modified CPSO are compared. For figures clarity, the number of birds is decreased to $n = 5$. Other parameters are $\alpha = 0.1054$, $\beta = 2.47$ and $a = 3$. In addition to the CPSO where $\varepsilon = \delta = 0$, figure (6) depicts the response of the third dimension trajectories for $\varepsilon = \delta = 0.01$ and $\varepsilon = \delta = 1$. The CPSO and the modified CPSO with the first selection (ε and δ small) have similar responses. However, the modified CPSO with the second set (ε and δ relatively large) of parameters is attracted too soon to an incorrect value, before the birds have the chance to 'discover' the true global minimum.

IX. Conclusion

In this paper, a new continuous time PSO has been developed with two slightly different forms. The first form is quite close to the classical discrete time PSO. Stabilty study of this model shows that only Lyapunov stability can be guaranteed. The second form incorporates extra stabilzing terms that ensures global asymptotic stability of the CPSO but which can reduce the birds mobility and thus their ability to find the optimum. Detailed discussions of the parameter effects on both forms are provided and are compared with the classical PSO. Simulation results illustrate the performance of the proposed CPSO. CPSO is expected to give better solutions than classical PSO. Current research aims at comparing the response of both PSO techniques when applied to difficult optimization problems.

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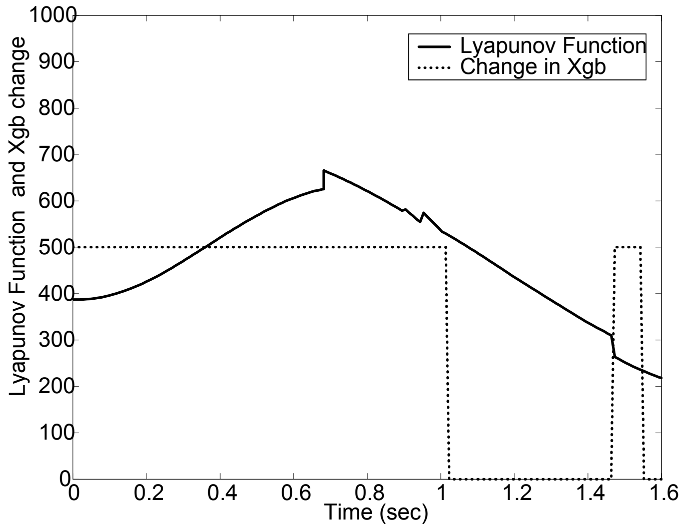

Fig. 4. Lyapunov Function Trajectory with X_{gb} Changes

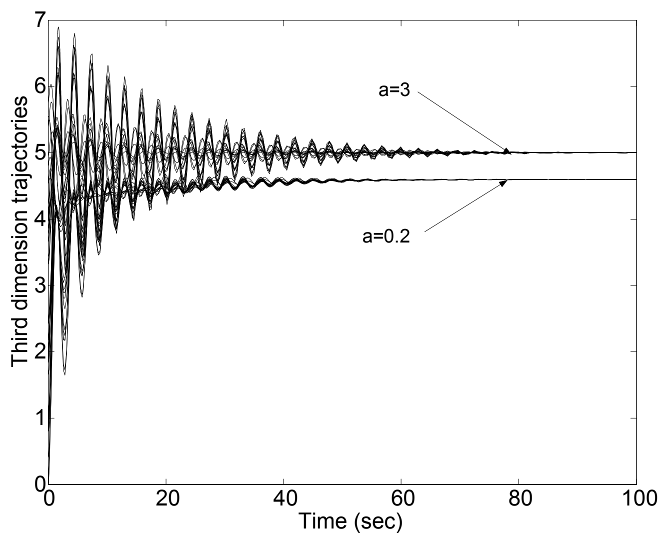

Fig. 5. Trajectory of Third Dimension With Different a Selection

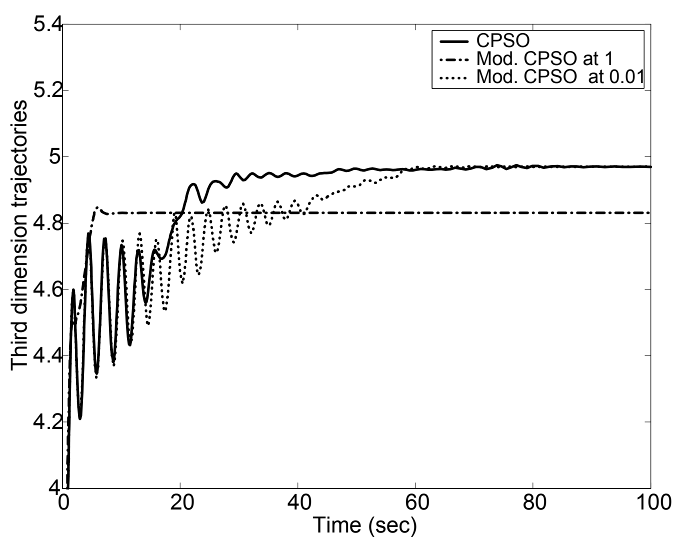

Fig. 6. Trajectory of Third Dimension With Different ε Selection