Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Common Stabilizers for Linear Control Systems in the Presence of Actuators Outage

Yew-Wen Liang and Der-Cherng Liaw

Abstract— **This paper presents common stabilizers for linear control systems when actuators happen to fail. The possible outage of actuators examined in this study are not confined to a pre-specified set. By finding common quadratic-type Lyapunov functions, we obtain sufficient conditions for the existence of common stabilizers. For cases where all the possible failed actuators belonged to a pre-specified set, the results presented in this paper agree with those obtained by Veillette in 1995. The control gain of common stabilizer for non-nested case is explicitly derived to guarantee system stability. A simplified checking condition for the existence of common stabilizers is also obtained for the extreme case when only single actuator can normally operate.**

I. INTRODUCTION

Recently, the study of reliable controls that can tolerate the failure of actuators or sensors in control systems has attracted much attention (see e.g., [1],[3]-[4],[6]-[7],[9]-[11]). However, most existing results for reliable control design are limited to systems with failure of actuators within a pre-specified subset. Among these studies, Veillette [7] also inspected, in his example, whether the designed controllers could tolerate the outage of actuators outside the pre-specified subset. In [3], although Medanic investigated the possible outage of actuators outside a pre-specified subset, it was restricted to single actuator outage. Zhao and Jiang [11] synthesized a reliable controller for dynamic systems with redundant actuators. Though their approach doesnot involve the construction of Lyapunov function, the controlled system $\dot{x} = Ax + Bu$ is required to have actuator redundancy with (A, b_i) is a controllable pair for each i, with $B = (b_1, \cdots, b_p) \in \mathbb{R}^{n\times p}$. Moreover, a pre-compensator proposed to transform the non-uniform redundancy property into uniform property might increase system order and reconfigure system structure. In this paper, the authors will extend the reliable stabilization of [7] to systems where the outage of actuators might be outside a pre-specified subset and the number of failed actuators is not restricted to one. Moreover, the control system is not assumed to possess the controllability property required in [11]. To tackle the reliable design problem, one might consider the existence of either common or noncommon Lyapunov functions with regard to faulty systems. In this paper, the authors will consider the existence of common Lyapunov functions, while an example of seeking noncommon Lyapunov functions for stabilizing switched systems may be found in [8]. Our approach is to seek a common quadratic-type Lyapunov function whose time derivative is negative for all the directions in which the controls have no contribution. A sufficient condition for common stabilizers is derived and the method of its implementation is demonstrated.

The goal of this paper is then to propose and implement a checking condition for the existence of common stabilizers for a control system experiencing the outage of actuators. The idea behind the study is to present a common stabilizer that can tolerate the outage of certain actuators without switching the control law, since switching the control law could require more control elements to sense the outage of actuators. Otherwise, the reliability of additional sensor elements would have to be considered. Potential applications of such a stabilizer include space missions or any highly dangerous area where actuators of equipment fail. This issue is important because retrieving satellites is expensive and instability of equipment in highly dangerous areas might result in disaster.

There are two main differences between the paper and those of [7]. First, the paper proposes a unified approach to determine the existence of common stabilizers regardless of whether the outage of actuators are confined within a pre-specified set, while those of [7] didnot. Moreover, it is also shown that the obtained results for the existence of common stabilizers agree with those of [7] when the outage of actuators are confined within a pre-specified set. Second, once the common stabilizer is determined to exist by the checking condition proposed in this paper, the control gain of the common stabilizers can be determined from the Routh-Hurwitz criteria to fulfill the task, while the choice of control gain in [7] was fixed to one. An example is also given to demonstrate the importance of the selection of such a control gain.

This paper is organized as follows. Section 2 introduces the problem. An example in which all the faulty systems are completely controllable does not guarantee the existence of common stabilizers is also given. It is followed by the derivation of the existence of common stabilizers. The procedure for implementing such conditions and determining the control gain that guarantees the stability of the faulty systems is also proposed. Section 4 presents an illustrative example to demonstrate the application of the results. The existence of common stabilizer for the admissible faulty systems of the given example is shown not to be obtained by Veillette's design [7]. Finally, Section 5 gives concluding remarks.

The authors are with the Department of Electrical and Control Engineering, National Chiao Tung University, Hsinchu 300, Taiwan, Republic of China `ywliang@cc.nctu.edu.tw` and `dcliaw@cc.nctu.edu.tw`

0-7803-8335-4/04/$17.00 ©2004 AACC

II. SET OF THE PROBLEM

Consider a linear control system

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^{\mathrm{n}}$, $u \in \mathbb{R}^{\mathrm{m}}$, $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ and $B \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$. Define the set of control matrices

$$\begin{aligned}\mathcal{B} \;=\; & \{B_i \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}} \mid B_i \text{ is obtained from } B \text{ by replacing} \\ & \text{some columns or no column of } B \text{ with zero column vector and } (A, B_i) \text{ is stabilizable}\}.\end{aligned} \tag{2}$$

That is, for each $B_i \in \mathcal{B}$, B_i denotes the control matrix resulting from B experiencing the outage of some actuators. Note that the set $\mathcal{B}$ contains a finite number of matrices.

Recall that the goal of this paper is to determine the existence conditions of common stabilizers for all system pairs (A, B_i) with $B_i \in \mathcal{B}$. Note that the outage of actuators considered here is not confined to be within a pre-specified set.

From linear system theory, it is known that a linear system pair (A, B) is stabilizable if the unstable subspace of A is contained in the controllability space of (A, B) (see e.g., [4]). Using this observation, one might predict that the class of systems (A, B_i) with $B_i \in \mathcal{B}$ and $\mathcal{B}$ as defined in (2) possess a common stabilizer if the intersection of the controllability space of all the system pairs (A, B_i) contains the unstable subspace of A. Unfortunately, such a prediction is generally not true. An example is given in Example 1 below.

Example 1: Consider system (1) with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{3}$$

Let

$$B_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

It is easy to check that all the system pairs (A, B) and (A, B_i) for $i = 1, 2$ are completely controllable. Thus, according to the definition in (2), we have $\mathcal{B} = \{B, B_1, B_2\}$.

Suppose that these three system pairs possess a common stabilizer $u = Kx$, where

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}. \tag{5}$$

That is, all the matrices $A + BK$ and $A + B_iK$ for $i = 1, 2$ are Hurwitz. Then, from the Routh-Hurwitz stability criteria, to provide for the stability of system pair (A, B_1) one needs to have $\mathrm{tr}(A + B_1K) = k_{11} - k_{12} + 3 < 0$ and $\det(A+B_1K) = 2k_{11} - k_{12} + 2 > 0$, where $\mathrm{tr}(\cdot)$ and $\det(\cdot)$ denote the trace and determinant of a matrix. This results in $k_{11} > 1$ and $k_{12} > 4$. Similarly, for system pair (A, B_2) one needs to have $\mathrm{tr}(A + B_2K) = k_{21} + k_{22} + 3 < 0$ and $\det(A + B_2K) = 2k_{21} + k_{22} + 2 > 0$. This means that $k_{21} > 1$ and $k_{22} < -4$. By direct calculation, for system pair (A, B) one finds:

$$\begin{aligned} det(A + BK) \;=\; & 2k_{11}(k_{22} + 1) + 2k_{21}(1 - k_{12}) - k_{12} \\ & +(k_{22} + 2). \end{aligned} \tag{6}$$

According to the stability conditions for system pairs (A, B_1) and (A, B_2) discussed above, all the terms in the right hand side of (6) are negative. It then follows that $det(A+BK) < 0$. This contradicts $u = Kx$ as a stabilizer for (A, B). Thus, the three given pairs of control systems do not possess a common stabilizer.

III. MAIN RESULTS

In this section, we will employ the Lyapunov approach to derive a condition for the existence of common stabilizers as given by Theorem 1 below. Then, we will demonstrate the implementation of the existence condition. Details are given as below:

A. Existence Condition for Common Stabilizers

Suppose the class of systems (A, B_i), $B_i \in \mathcal{B}$, possesses a common stabilizer $K \in \mathbb{R}^{\mathrm{m}\times\mathrm{n}}$ and $A + B_iK$ shares a common Lyapunov function $V(x) = x^TPx$. Then $x^TP(A + B_iK)x < 0$ for all nonzero x and for all i. This leads to the following result.

Theorem 1: Consider the class of linear control systems (A, B_i), where $B_i \in \mathcal{B}$ and $\mathcal{B}$ is defined as in (2). If there exists a symmetric positive definite matrix $P > 0$ such that

$$x^TPAx < 0 \text{ for all } x \in \cup_{B_i\in\mathcal{B}} N(B_i^TP)\backslash\{0\}, \tag{7}$$

then the class of systems (A, B_i), $B_i \in \mathcal{B}$, possess a common stabilizer. Here, $N(\cdot)$ denotes the null space of a matrix. Moreover, a common stabilizer can be chosen in the form $u = -\alpha \cdot B^TPx$ with α satisfying Condition (9) below.

Proof: By the application of optimal control design, we choose a common stabilizer candidate in the form of $u = -\alpha \cdot B^TPx$ to meet Condition (7). It is observed that, from the special structure of B_i, $B_iB^T = B_iB_i^T$ for all $B_i \in \mathcal{B}$. The time derivative of $V(x) = x^TPx$ along the trajectories of the system $\dot{x} = Ax + B_iu$ with $u = -\alpha \cdot B^TPx$ has the form

$$\dot{V} = 2 \cdot (x^TPAx - \alpha \cdot x^TPB_iB_i^TPx). \tag{8}$$

In the following, we will show the existence of α such that $\dot{V} < 0$ for all $x \neq 0$ and for all $B_i \in \mathcal{B}$. This will then imply the existence of common stabilizers.

If $x^TPAx < 0$ for all $x \neq 0$, then A must be a Hurwitz matrix [4] and $\dot{V} < 0$ for all $x \neq 0$ and for all $B_i \in \mathcal{B}$ no matter what $\alpha > 0$ is chosen. On the other hand, if $x^TPAx \geq 0$ for some $x \neq 0$, then the set $S := \{x|x^TPAx \geq 0, ||x|| = 1\}$ is a nonempty compact set. Thus, Condition (7) implies that $x^TPB_i \neq 0$ for all $x \in S$. As such, for all $B_i \in \mathcal{B}$, $\gamma_i := \min_{||x||=1, x^TPAx\geq 0} ||x^TPB_i|| > 0$. Since $\mathcal{B}$ only contains a finite number of matrices, it follows that $\gamma :=$

$\min_i \gamma_i > 0$. From the definition of γ, all the nonzero points x satisfying $x^T PAx \geq 0$ have the property that $||x^T PB_i|| = ||\frac{x^T}{||x||} PB_i|| \cdot ||x|| \geq \gamma \cdot ||x||$ for all $B_i \in \mathcal{B}$. Choose the control gain α satisfying

$$\alpha > \frac{||A^T P||}{\gamma^2} > 0. \tag{9}$$

It then follows from (8) that, if x is a nonzero point with $x^T PAx \geq 0$, then

$$\begin{aligned}\dot{V} &< 2 \cdot (x^T PAx - \frac{||A^T P||}{\gamma^2} ||x^T PB_i||^2) \\ &\leq 2 \cdot (x^T PAx - \frac{||A^T P||}{\gamma^2} \cdot \gamma^2 ||x||^2) \leq 0\end{aligned}$$

for all $B_i \in \mathcal{B}$. The conclusion of the theorem is hence provided. ■

Remark 1: Though the control gain α of the common stabilizers can be determined from Eq. (9), it is not easy to compute directly from there. However, the control gain α may be determined by employing Routh-Hurwitz criteria (see e.g., [4].

B. Existence of a Matrix P Satisfying Condition (7)

According to Theorem 1, if one can find a symmetric positive definite matrix P which satisfies Condition (7), a common stabilizer for the class of systems (A, B_i), $B_i \in \mathcal{B}$, can then be determined. In this subsection, we will derive conditions for the existence of such a matrix P. For this purpose, we define the terminology of nested subset of $\mathcal{B}$.

A subset $\mathcal{B}_1 = \{B_1, \cdots, B_k\}$ of $\mathcal{B}$ as defined in (2) is said to be nested if it has the property: Range$(B_1) \subseteq$Range$(B_2) \subseteq \cdots \subseteq$Range$(B_k)$. Under this condition, we say that B_1 corresponds to the worst case (i.e., minimum number of actuators under operation) for all system pairs (A, B_i) with $B_i \in \mathcal{B}_1$.

First, consider the case in which the outage of actuators is confined within a pre-specified set as considered by [8,9]. That is, set $\mathcal{B}$ is nested. The existence of a P satisfying Condition (7) can be guaranteed by solving the algebraic Riccati equation (ARE) associated with the worst case of $\mathcal{B}$, say B_1^*, as given below:

$$A^T P + PA - PB_1^* B_1^{*T} P + H = 0 \tag{10}$$

for any given $H > 0$. Indeed, under this case, $\cup_{B_i \in \mathcal{B}} N(B_i^T P) = N(B_1^{*T} P)$ and, from (10), $2x^T PAx = -x^T Hx < 0$ for all $x \in N(B_1^{*T} P) \backslash \{0\}$. This verifies the existence of P that satisfies Condition (7) and thus the existence of common stabilizers is guaranteed by Theorem 1. Note that the derived result agrees with that obtained by Veillette et. al. [8,9].

Next, consider the case in which the outage of actuators are not confined within a pre-specified set. Motivated by the previous case, we divide $\mathcal{B}$, as given by (2), into several nested subsets, say $\mathcal{B}_1, \cdots, \mathcal{B}_s$. Denote B_j^* the worst case of $\mathcal{B}_j$ for $1 \leq j \leq s$. We can check that Condition (7) of Theorem 1 is equivalent to the following condition:

$$x^T PAx < 0 \text{ for all } x \in \cup_{j=1}^s N(B_j^{*T} P) \backslash \{0\}. \tag{11}$$

In addition, it is not difficult to check that Condition (11) above is equivalent to Condition (12) below by letting $x = Wy$ and $W = P^{-1}$:

$$y^T AWy < 0 \text{ for all } y \in \cup_{j=1}^s N(B_j^{*T}) \backslash \{0\}. \tag{12}$$

Thus, the checking operation for the existence of a P satisfying Condition (7) can be simplified to proceed for those worst cases associated with each nested set only.

To obtain a matrix P which meets Condition (11), we can choose a matrix among all the worst cases $B_1^*, \cdots, B_s^*$, say B_1^*, having minimum rank. Let the rank of B_1^* be l. That is,

$$\text{rank}(B_1^*) = \min_{1 \leq j \leq s} \text{rank}(B_j^*) = l. \tag{13}$$

Before proceeding the derivation of checking condition to provide relation (11) or (12), we present the next lemma.

Lemma 1: Suppose $L \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $M \in \mathbb{R}^{n \times m}$ and rank$(M) = l$. Then $y^T Ly < 0$ for all $y \in N(M^T) \backslash \{0\}$ if and only if $(M^\perp)^T LM^\perp$ is a negative definite matrix, where $M^\perp$ is a $n \times (n-l)$ matrix whose columns form an orthonormal basis for $N(M^T)$.

Proof: Note that, $(M^\perp)^T LM^\perp$ is a negative definite matrix if and only if $v^T (M^\perp)^T LM^\perp v < 0$ for every nonzero $v \in \mathbb{R}^{n-l}$. Moreover, the latter condition is equivalent to that $u^T Lu < 0$ for every $u = M^\perp v \in N(M^T) \backslash \{0\}$. The conclusion of the lemma is hence implied. ■

Now, let $L = AW + WA^T$ with $W = P^{-1}$. The next result follows readily from Eq. (10) and Lemma 1.

Theorem 2: Consider the class of systems (A, B_i), $B_i \in \mathcal{B}$. Suppose B_1^* satisfies the relation (13) and $P = W^{-1} > 0$ is the solution of Eq. (10). Then P is a matrix satisfying Condition (11) or (12) if and only if for each $j = 1, \cdots, s$, $(B_j^{*\perp})^T (AW + WA^T) B_j^{*\perp}$ is a negative definite matrix. Here, $B_j^{*\perp}$ denotes a matrix whose columns form an orthonormal basis for $N(B_j^{*T})$.

For the case of which rank$(B_1^*) = 1$ and rank$(B_j^*) = 1$ for some $j \neq 1$, the checking condition (12) corresponding to B_j^* as given in (14) below

$$y^T AWy < 0 \text{ for all } y \in N(B_j^{*T}) \tag{14}$$

can be simplified by verifying the positivity of a scalar instead of checking negative definiteness of the $(n-1) \times (n-1)$ matrix $(B_j^{*\perp})^T (AW + WA^T) B_j^{*\perp}$ as given in Theorem 2 above. Details are discussed as follows.

Suppose A is not a Hurwitz matrix. From Eq. (10) and $W = P^{-1}$ that $AW + WA^T$ has exactly one unstable eigenvalue. The unstable eigenvalue may be zero or a positive real number. If the unstable eigenvalue is zero, then $y^T AWy < 0$ for all $y \notin E_0 = \{z | (AW + WA^T) z = 0\}$. Here, E_0 denotes the eigenspace of $AW + WA^T$ associated with the zero eigenvalue. Thus, Condition (14) hold if and only if $E_0 \not\subset N(B_j^{*T}) = R(B_j^*)^\perp$. On the other hand, if the unstable eigenvalue is a positive real number, an equivalent condition can be constructed. Details are summarized in the next corollary.

Corollary 1: Suppose $\text{rank}(B_1^*) = 1$, $\text{rank}(B_j^*) = 1$ for some $j \neq 1$ and $P = W^{-1} > 0$ denotes the solution of Eq. (10). Let b be a nonzero column of B_j^*. Then the following two statements hold:

(i) If $AW + WA^T$ possesses a zero eigenvalue, then Condition (14) holds if and only if $E_0 \not\subset R(B_j^*)^\perp$. That is, $b^T v \neq 0$ for $v \in E_0 \backslash \{0\}$.

(ii) If $AW + WA^T$ has a positive eigenvalue, then Condition (14) holds if and only if

$$b^T (AW + WA^T)^{-1} b > 0. \tag{15}$$

Proof: Statement (i) has been discussed in the preceding paragraph of Corollary 1. The proof of (ii) is given in Appendix. ■

To summarize the extended reliable design discussed above, a procedure for the construction of common stabilizers for system (1) can be listed as follows.

Procedure for finding common stabilizers:

Step 1: Divide all the stabilizable system pairs into different nested subsets $\mathcal{B}_1, \cdots, \mathcal{B}_s$, and pick up one of the worst cases, say $B_1^* \in \mathcal{B}_1$, among those subsets.

Step 2: Attempt a reliable control design using the method of cited reference [9]. That is, given $H > 0$, solve for P in Eq. (10) and check whether all the matrices $A - B_i^* B_i^{*T} P$, $B_i^* \in \mathcal{B}_i$ for all $i \neq 1$, are Hurwitz. If it fails to provide the desired reliable properties with respective to outages outside the pre-specified set of actuators, then continue to Step 3.

Step 3: Check the sufficient condition (11) or (12) by employing Theorem 2 or Corollary 1, with P being the solution of the Riccati equation used in Step 2. If the condition holds, then a scaling of the feedback gain matrix from Step 2 is guaranteed to work and continue to Step 4. Otherwise, go back to Step 2 with the choice of another worst case.

Step 4: Use the Routh-Hurwitz stability criteria to determine an appropriate scaling α of the control gain from $A - \alpha B_i^* B_i^{*T} P$ being Hurwitz for all $i = 1, \cdots, s$.

Note that, if the above procedure fails to construct a common stabilizer, one might attempt to find a new matrix P by the use of different weighting matrices in the Riccati equation.

IV. Illustrative Examples

This section presents an example to determine the application of the main results as summarize in the procedure above of the paper. In this example, the existence of common stabilizers for all admissible faulty systems can not be provided by using Veillett's design [7] when both weighting matrices Q and R are identity matrices.

Example 2: Consider system (1) with

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0.1 \\ 10 & 0.01 \\ 10 & 0.01 \end{pmatrix}. \tag{16}$$

Let B_1 and B_2 be derived from B which correspond to the failure of the second and first actuators, respectively. That is,

$$B_1 = \begin{pmatrix} 1 & 0 \\ 10 & 0 \\ 10 & 0 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 & 0.1 \\ 0 & 0.01 \\ 0 & 0.01 \end{pmatrix}. \tag{17}$$

It is easy to check that both (A, B_1) and (A, B_2) are stabilizable. This leads to $\mathcal{B} = \{B, B_1, B_2\}$, which is not nested. Clearly, $\mathcal{B}$ contains two nested subsets $\mathcal{B}_1 = \{B_1, B\}$ and $\mathcal{B}_2 = \{B_2, B\}$. The two worst cases associated with $\mathcal{B}_1$ and $\mathcal{B}_2$ are $B_1^* = B_1$ and $B_2^* = B_2$, respectively.

According to Veillette's method [9], the first thing to do is to solve the ARE

$$A^T M_i + M_i A - M_i B_i R^{-1} B_i^T M_i + Q = 0, \quad Q > 0 \tag{18}$$

for $i = 1$. Then, verify if the matrix $A - B_2 R^{-1} B^T M_1$ is stable. If it is not, redo this process for $i = 2$ and check if the matrix $A - B_1 R^{-1} B^T M_2$ is stable. Unfortunately, the method proposed by Veillette does not work in this example for both R and Q being the identity matrix. Indeed, for $i = 1$, the eigenvalues of $A - B_2 R^{-1} B^T M_1$ are $\{0.954, 0.005, -1\}$; and for $i = 2$, the eigenvalues of $A - B_1 R^{-1} B^T M_2$ are $\{-1, 0.438, -3.080 \times 10^4\}$. Although Veillette's method might work for the construction of common stabilizers for this example by a suitable choice of weighting matrices Q and R, however, no guideline of choosing matrices Q and R has been proposed in [9] for reliable design.

To employ the proposed methodology, we first solve the ARE (10) for H being the identity matrix. The unique solution is calculated to be:

$$P = \begin{pmatrix} 4.194 & 4.565 & -4.678 \\ 4.565 & 5.866 & -5.915 \\ -4.678 & -5.915 & 6.109 \end{pmatrix}. \tag{19}$$

Then, by direct calculation, the index as given in (15) is $b^T (AW + WA^T)^{-1} b = 2.464 \times 10^{-4} > 0$, where b is the nonzero column of B_2^* and $W = P^{-1}$. According to Corollary 1, matrix $P = W^{-1}$ as in (19) satisfies Condition (11). The common stabilizer can hence be obtained from Theorem 1 in the form of

$$u = -\alpha \cdot B^T P x \text{ for some } \alpha > 0. \tag{20}$$

By applying Routh-Hurwitz criteria on $A - \alpha \cdot B_2 B^T P$, this matrix is verified to be Hurwitz if $\alpha > 22.869$. By direct calculation, the eigenvalues of $A - \alpha \cdot B B^T P$, $A - \alpha \cdot B_1 B^T P$ and $A - \alpha \cdot B_2 B^T P$ with $\alpha = 25$ are found to be $\{-407.668,\ -1.954,\ -1\}$, $\{-407.516,\ -1.063,\ -1\}$ and $\{-0.022 \pm 0.326j,\ -1\}$, respectively. These verify the reliable stabilization of the system.

V. CONCLUSIONS

This paper has employed the Lyapunov approach to study the existence conditions of common stabilizers for linear control systems. The control systems considered in this paper result from an actual system where some actuators failed. The unique aspect of this study, compared with earlier studies, is that the possible outage of the actuators is not confined within a pre-specified set. We have obtained a sufficient condition for the existence of common stabilizers and provided a procedure to implement such a condition. When the possible outage of actuators are confined within a pre-specified set, the obtained results agree with previous findings.

VI. ACKNOWLEDGMENTS

This research was supported by the National Science Council, Taiwan, R.O.C. under Grants NSC 90-CS-7-009-004, NSC 91-2623-7-009-010 and NSC 91-2212-E-009-037.

REFERENCES

[1] J. Ackermann, "Robustness against sensor failures", *Automatica* 1984, pp. 211-215.

[2] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[3] J.V. Medanic, "Design of reliable controllers using redundant control elements", *Proc. of the 1993 American Control Conference*, San Francisco, CA, June 1993, pp. 3130-3134.

[4] D.C. Liaw, Y.W. Liang and T.C. Lee, "Reliable control of nonlinear systems", *IEEE Transactions on Automatic Control*, 2000, pp. 706-710.

[5] E.S. Pyatnitskii and V.I. Skorodinskii, "A criterion of absolute stability of non-linear sampled-data control systems in the form of numerical procedures", *Automation and Remote Control*, 1986, pp. 1190-1198.

[6] R.J. Veillette, J.V. Medanic and W.R. Perkins, "Design of reliable control systems", *IEEE Transaction on Automatic Control*, 1992, pp. 290-304.

[7] R.J., Veillette, "Reliable linear-quadratic state-feedback control", *Automatica* 1995, pp. 137-143.

[8] M.A. Wicks, P. Peleties and R.A. DeCarlo, "Construction of piecewise Lyapunov functions for stabilizing switched systems", *Proc. 33rd IEEE Conf. on Decision and Control*, 1994, pp. 3492-3497.

[9] G.H. Yang, J. Lam and J. Wang J, "Reliable controller design for nonlinear system", *Proc. 35th IEEE Conf. on Decision and Control*, 1996, pp. 112-117.

[10] G.H. Yang, Y. Liu, J. Wang and C.B. Soh, "Reliable nonlinear control system design using redundant control elements", *Proc. 35th IEEE Conf. on Decision and Control*, 1996, pp. 118-123.

[11] Q. Zhao and J. Jiang, "Reliable state feedback control system design against actuator failures", *Automatica*, 1998, pp. 1267-1272.

VII. APPENDIX

Here we give the Proof for (ii) of Corollary 1). Suppose $AW + WA^T$ is in diagonal form. Let $AW + WA^T$ =diag$(\lambda_1^2, -\lambda_2^2, \cdots, -\lambda_n^2)$. We now show that Condition (15) implies Condition (14). Denote $b = (b_1, \cdots, b_n)^T$ and $y = (y_1, \cdots, y_n)^T \in N(B_j^{*T})\backslash\{0\}$. Condition (15) then becomes

$$\sum_{i=2}^{n} b_i^2/\lambda_i^2 < b_1^2/\lambda_1^2. \tag{A1}$$

This implies that $b_1 \neq 0$. From the structure of $AW+WA^T$, if $y_1 = 0$, we then have $y^T(AW+WA^T)y < 0$. For $y_1 \neq 0$, $b^T y = b_1 y_1 + \sum_{i=2}^n b_i y_i = 0$, which implies that

$$-1 = \sum_{i=2}^{n} \frac{b_i y_i}{b_1 y_1} = \sum_{i=2}^{n} \left(\frac{b_i}{b_1} \frac{\lambda_1}{\lambda_i} \frac{\lambda_i}{\lambda_1} \frac{y_i}{y_1} \right). \tag{A2}$$

By employing Cauchy-Schwartz inequality from (A2) and the inequality from (A1), we have

$$1 \leq \left(\sum_{i=2}^{n} \frac{b_i^2 \lambda_1^2}{b_1^2 \lambda_i^2} \right) \left(\sum_{i=2}^{n} \frac{\lambda_i^2 y_i^2}{\lambda_1^2 y_1^2} \right) < \sum_{i=2}^{n} \frac{\lambda_i^2 y_i^2}{\lambda_1^2 y_1^2}, \tag{A3}$$

which leads to $\sum_{i=2}^n \lambda_i^2 y_i^2 > \lambda_1^2 y_1^2$ and $y^T(AW + WA^T)y = \lambda_1^2 y_1^2 - \sum_{i=2}^n \lambda_i^2 y_i^2 < 0$.

Next, we show that Condition (14) implies (15) by contradiction. Suppose there exists a nonzero vector $y \in N(B_j^{*T})$ such that Condition (15) does not hold. Thus, we have $b_1^2/\lambda_1^2 - \sum_{i=2}^n b_i^2/\lambda_i^2 \leq 0$. This implies that $(b_2, \cdots, b_n)^T$ is a nonzero vector since b is a nonzero vector. Choose $y = (\sum_{i=2}^n b_i^2/\lambda_i^2, -b_1 b_2/\lambda_2^2, \cdots, -b_1 b_n/\lambda_n^2)^T$. It is clear that y is a nonzero vector and $y \in N(B_j^{*T})$. That is, $b^T y = 0$. By direct calculation, we have

$$y^T(AW + WA^T)y = \left(\lambda_1^2 \sum_{i=2}^{n} \frac{b_i^2}{\lambda_i^2} \right) \left(\sum_{i=2}^{n} \frac{b_i^2}{\lambda_i^2} - \frac{b_1^2}{\lambda_1^2} \right) \geq 0.$$

For the case of which $AW + WA^T$ is not a diagonal matrix, a similarity transformation can be pre-applied to fulfill the proof. Since $AW + WA^T$ is a symmetric matrix, there exists an orthogonal matrix U such that $U^T(AW + WA^T)U$ =diag$(\lambda_1^2, -\lambda_2^2, \cdots, -\lambda_n^2)$, where $\lambda_i > 0$ for all $i = 1, \cdots, n$. Let $z = Uy$ and D =diag$(\lambda_1^2, -\lambda_2^2, \cdots, -\lambda_n^2)$. It is clear that $z^T D z < 0$ for all $z \in N((UB_j^*)^T)$ and Condition (15) becomes $(Ub)^T D^{-1}(Ub) > 0$. The rest of the proof is similar to the one given above and is hence omitted. The conclusion of Corollary 1 is hence implied. ■