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H_∞ Tracking Control of a Rigid Spacecraft

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ABSTRACT

The attitude tracking control problem of a rigid spacecraft with external disturbances is addressed using the concept of extended disturbance and the inverse optimal control method. The proposed attitude tracking control law is inverse optimal with respect to a meaningful cost functional and the associated Lyapunov function satisfies a Hamilton-Jacobi-Isaacs partial differential equation. Hence, it is H_∞ optimal with respect to the extended disturbance. The performance limitation of the H_∞ inverse optimal PD control law is also analyzed.

I. INTRODUCTION

In the spacecraft engineering, PID controllers of different forms are widely used for its simplicity and high reliability (see [1]). For the past ten years, many authors have investigated the relationship between nonlinear H_∞ control method [2] and PID control and their applications to the attitude control problem of spacecraft with external disturbances [3][4]. In nonlinear optimal control theory, nonlinear H_∞ control method is robust and should be a potential approach to the attitude control problem. However, the practical applications of the nonlinear H_∞ control method still remain open due to the difficulty in solving the associated Hamilton-Jacobi-Isaacs (HJI) partial differential equation. There have been some attempts to solve the HJI equation. One of the approximation methods is the state-dependent-Riccati equation (SDRE) method [5], while it only guarantees local asymptotic stability. The concept of extended disturbances, including system error dynamics, was developed in robotics by [6] to help to solve the HJI equation for nonlinear H_∞ control.

Parallel to nonlinear H_∞ control, the framework of input-to-state stability (ISS) introduced by Sontag [7] has triggered great efforts to input-to-state stabilizing controllers [8]. The inverse optimal control approach [9] was proposed to achieve optimality for a set of meaningful cost functions without solving the HJI equation. The inverse optimal controller exists if and only if the nonlinear system is ISS with respect to the disturbance.

In this paper, the results of [3][4][8][10] are extended to the attitude tracking control problem of a rigid spacecraft with external disturbances by using the concept of extended disturbance and the inverse optimal control method. H_∞-suboptimal attitude controllers of PD form were designed in [3][4] for the setpoint regulation problem of the rigid spacecraft, in which the $\mathcal{L}_2$-gain γ was limited to $\gamma > 1$. An inverse optimal control was designed in [10] for the setpoint regulation problem of a rigid spacecraft without any disturbances. A H_∞ PID controller was designed in [8] for the trajectory tracking control problem of robotic manipulators. In this paper, a PD state-feedback control law that solves the inverse optimal gain assignment problem is designed for the attitude tracking control problem. The PD control law is also H_∞ optimal with respect to the extended disturbance, allows the L_2-gain γ being any positive magnitudes instead of $\gamma > 1$ and thus achieves disturbance attenuation at any given attenuation level γ with the target signals being included in the extended disturbance.

This paper is organized as follows. In section II, preliminaries on nonlinear H_∞ control and inverse optimal control are briefly reviewed. In Section III the attitude tracking control problem of a rigid spacecraft is formulated using the unit quaternions and the extended disturbance is introduced. In Section IV, a H_∞ inverse optimal PD state-feedback controller is designed using the inverse optimal control method. The performance estimates of the H_∞ controller are analyzed in Section V. A simple example is used in Section VI to demonstrate the performance of the proposed controller, and finally conclusions follow in Section VII.

The following notations are used in the paper. $L_f V$ denotes the Lie derivative of the smooth function $V(x)$ with respect to x, i.e., $L_f V = \frac{\partial V}{\partial x} f$. A function $\alpha : \mathcal{R}_+ \to \mathcal{R}_+$ is of class $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta : \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}_+$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s \geq 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$. We use $||M||$ to denote the induced 2-norm of M and $|v|$ to denote the vector norm.

II. PRELIMINARIES

Consider a nonlinear system of the form

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u, \quad y = h(x), \qquad (1)$$

where x is the state vector, d is the exogenous disturbance to be rejected, u is the control input, and y is the penalized output signal. $f(x)$, $g_1(x)$, $g_2(x)$ and $h(x)$ are assumed to be smooth functions and $x = 0$ is the equilibrium point of the nonlinear system, i.e., $f(0) = h(0) = 0$.

The *nonlinear state-feedback* H_∞ *control problem* is to find a state-feedback control $u = k(x)$ for (1), with $k(0) = 0$, such that the $\mathcal{L}_2$ gain from the disturbance d to a block vector of output y and input u is not larger than γ, i.e., there exists a function $K(x) \geq 0$ with $K(0) = 0$ such that

$$\int_0^\infty \left(|y|^2 + u^T R_2(x) u\right) dt \leq \gamma^2 \int_0^\infty |d(t)|^2 dt + K(x_0)$$

is satisfied for any initial condition $x(0) = x_0$ of (1), where $R_2(x)$ is symmetric and positive definite for all x. The H_∞ *optimal problem* is to find, if exists, the smallest value γ^* of such $\mathcal{L}_2$ gains γ.

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Lemma 1: Consider the nonlinear system (1). Let the constant $\gamma > 0$ and the matrix $R_2(x) = R_2^T(x) > 0$ for all x. Suppose that there exists a smooth solution $V(x) \geq 0$ with $V(x_0) = 0$ to the Hamilton-Jacobi-Issacs partial differential (HJIPD) equation

$$\frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x}\left[\frac{1}{\gamma^2} g_1 g_1^T - g_2 R_2^{-1} g_2^T\right]\frac{\partial^T V}{\partial x} + \frac{1}{4} h^T h = 0, \quad (2)$$

or to the HJIPD inequality

$$\frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x}\left[\frac{1}{\gamma^2} g_1 g_1^T - g_2 R_2^{-1} g_2^T\right]\frac{\partial^T V}{\partial x} + \frac{1}{4} h^T h \leq 0, \quad (3)$$

then the closed-loop system for the feedback

$$u = -2R_2^{-1}(x)(L_{g_2} V)^T \quad (4)$$

has the $\mathcal{L}_2$-gain less than or equal to γ from the disturbance d to the block vector of outputs y and control input u.

It is a direct variation of Theorem 16 in [2]. In general, the HJIPD inequality (3) only guarantees a suboptimal solution. The most challenging task in solving the nonlinear H_∞ control problem is to find a smooth positive function $V(x)$ satisfying the HJIPD equation (2) or the inequality (3). However, it is very difficult to reach this goal.

Compared to the nonlinear H_∞ control, the inverse optimal method [9] solves the nonlinear optimal assignment problem without solving the HJI partial differential equation explicitly. For completeness of this paper, we quote the following theorem of the inverse optimal method.

Theorem 1: [9] *Consider the nonlinear affine system*

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u \quad (5)$$

and its auxiliary system

$$\dot{x} = f(x) + g_1(x)\ell_\rho(2|L_{g_1}V|)\frac{(L_{g_1}V)^T}{|L_{g_1}V|^2} + g_2(x)u, \quad (6)$$

where $V(x)$ is a Lyapunov function candidate; ρ is a class $\mathcal{K}_\infty$ function whose derivative ρ' is also a class $\mathcal{K}_\infty$ function; ℓ_ρ denotes the transform $\ell_\rho(r) = \int_0^r (\rho')^{-1}(s)ds$, where $(\rho')^{-1}(r)$ stands for the inverse function of $\frac{d\rho(r)}{dr}$. Suppose that there exists a matrix-valued function $R_2(x) = R_2^T(x) > 0$ such that the control law

$$u = \alpha(x) = -R_2^T(x)(L_{g_2}V)^T \quad (7)$$

globally asymptotically stabilizes (6) with respect to $V(x)$. Then the control law

$$u = \alpha^*(x) = \beta\alpha(x) = -\beta R_2^T(x)(L_{g_2}V)^T \quad (8)$$

with $\beta \geq 2$ solves the inverse optimal gain assignment problem for the nonlinear system (5) by minimizing the cost functional

$$\begin{aligned} J(u) = &\sup_d \Big\{ \lim_{t\to\infty} \big[2\beta V(x(t)) \\ &+ \int_0^t \big(l(x) + u^T R_2(x) u - \beta\lambda\rho(\tfrac{|d|}{\lambda})\big)d\tau\big]\Big\} \end{aligned}$$

for any $\lambda \in (0, 2]$, where

$$\begin{aligned} l(x) = &-2\beta[L_f V + \ell_\rho(2|L_{g_1}V|) - L_{g_2}VR_2^{-1}(L_{g_2}V)^T] \\ &+\beta(2-\lambda)\ell_\rho(2|L_{g_1}V|) + \beta(\beta-2)L_{g_2}VR_2^{-1}(L_{g_2}V)^T. \end{aligned}$$

III. ATTITUDE TRACKING PROBLEM

The spacecraft is modelled as a rigid body with actuators providing torques about three mutually perpendicular axes that define a body-fixed frame $\mathcal{B}$. Let the unit quaternion $q = [q_v^T, q_4]^T$ satisfying the constraint $q_v^T q_v + q_4^2 = 1$ denote the attitude of the rigid spacecraft in the body-fixed frame with respect to an inertial frame. Let S^3 be the unit sphere in R^4 where the unit quaternion lies, and TS^3 be the tangent bundle of S^3. The attitude kinematics and dynamics of a rigid spacecraft can be represented as (see [12, Chapter 4])

$$\begin{aligned} \dot{q}_v =& \tfrac{1}{2}(q_4 I_3 + S(q_v))\omega, && (9)\\ \dot{q}_4 =& -\tfrac{1}{2}q_v^T\omega, && (10)\\ \dot{\omega} =& -M^{-1}S(\omega)M\omega + M^{-1}u + M^{-1}d, && (11) \end{aligned}$$

where ω is the angular velocity of the spacecraft with respect to the inertial frame and expressed in the body-fixed frame; $M = M^T > 0$ is the inertia matrix of the spacecraft; d and u are the external disturbance and the control torque, respectively; I_3 is the identity matrix of dimensions 3×3;

$$S(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

denotes a skew-symmetric matrix acting on a vector $a = [a_1, a_2, a_3]^T$ and satisfies the following properties:

$$\begin{aligned} &S^T(a) = -S(a),\ S(a)b = -S(b)a,\ S(a)a = 0, \\ &S(a)S(b) = ba^T - a^T bI_3,\ S(S(a)b) = ba^T - ab^T, \end{aligned} \quad (12)$$

which offer very useful insights to designing a robust H_∞ inverse optimal PD control law in next section.

Our objective of attitude control is to track the attitude target trajectory to achieve the attitude maneuver with a satisfactory accuracy. Let the target unit quaternion be given by $q_c = [q_{cv}^T, q_{c4}]^T$. Assume that the target angular velocity $w_c(t)$ and its derivative $\dot{w}_c(t)$ are bounded. From (9) and (10) we have the differential equations:

$$\begin{aligned} \dot{q}_{cv} &= \tfrac{1}{2}(q_{c4}I_3 + S(q_{cv}))\omega_c \\ \dot{q}_{c4} &= -\tfrac{1}{2}q_{cv}^T\omega_c \end{aligned} \quad (13)$$

By the quaternion multiplication [12, Appendix A], the error quaternion $q_e = [q_{ev}^T, q_{e4}]^T$ can be expressed as

$$\begin{aligned} q_{ev} &= q_{c4}q_v - S(q_{cv})q_v - q_4 q_{cv}, \\ q_{e4} &= q_{cv}^T q_v + q_4 q_{c4}. \end{aligned} \quad (14)$$

The rate error w_e is defined by

$$\omega_e = \omega - \omega_c. \quad (15)$$

Applying (13)–(15) to (9)–(11), we obtain the differential error dynamics for the tracking control problem

$$\begin{aligned} \dot{q}_{ev} &= \tfrac{1}{2}[q_{e4}I_3 + S(q_{ev})]\omega_e + S(q_{ev})\omega_c && (16)\\ \dot{q}_{e4} &= -\tfrac{1}{2}q_{ev}^T\omega_e && (17)\\ M\dot{\omega}_e &= -[S(\omega_e)M\omega_e + S(\omega_c)M\omega_e + S(\omega_e)M\omega_c] \\ &\quad + u + [d - M\dot{\omega}_c - S(\omega_c)M\omega_c]. && (18) \end{aligned}$$

For simplicity of notation, let $\epsilon = q_{ev}$, $\eta = q_{e4}$ and define the extended disturbance (which includes information of system error dynamics, see [6]) as

$$\hat{d}(\omega_e, t) = \begin{bmatrix} \omega_c \\ d_c - S(\omega_c)M\omega_e - S(\omega_e)M\omega_c \end{bmatrix}, \quad (19)$$

where $d_c = d - M\dot{\omega}_c - S(\omega_c)M\omega_c$ is a combination of the reference signals $\dot{\omega}_c(t)$, $\omega_c(t)$ and the external disturbance $d(t)$. Hence we can represent (16)–(18) as

$$\begin{bmatrix}\dot{\epsilon}\\ \dot{\eta}\\ \dot{\omega}_e\end{bmatrix} = \begin{bmatrix}\frac{1}{2}[\eta I_3 + S(\epsilon)]\omega_e\\ -\frac{1}{2}\epsilon^T\omega_e\\ -M^{-1}S(\omega_e)M\omega_e\end{bmatrix} + \begin{bmatrix}0_{33}\\ 0_{13}\\ M^{-1}\end{bmatrix}u + \begin{bmatrix}S(\epsilon) & 0_{33}\\ 0_{13} & 0_{13}\\ 0_{33} & M^{-1}\end{bmatrix}\hat{d}(\omega_e, t). \tag{20}$$

Thus, the attitude tracking problem is transformed into the problem of stabilizing the error system (20) with respect to the extended disturbance $\hat{d}(\omega_e, t)$.

Lemma 2: [11] *Two coordinate systems, corresponding to q and q_c respectively, coincide if and only if $\epsilon = 0$.*

Remark 1: Both (ϵ, η) and $(-\epsilon, -\eta)$ represent exactly the same physical attitude orientation of the spacecraft. By Lemma 2 and the fact $\epsilon^T\epsilon + \eta^2 = 1$, we then conclude that the attitude tracking problem is solved if and only if $\epsilon \to 0$ and $\omega_e \to 0$ as $t \to \infty$. $\otimes$

IV. H_∞ PD Controller Design

Now we proceed to design the attitude tracking control law. Define the state

$$x = [\epsilon^T,\ \eta,\ \omega_e^T]^T.$$

Since η is not an independent variable, we also write

$$\tilde{x} = [\epsilon^T,\ \omega_e^T]^T. \tag{21}$$

Let $\bar{\lambda}_m$ denote the maximum eigenvalue of the inertia matrix M, i.e., $\bar{\lambda}_m = \lambda_{\max}(M) = ||M||$. We choose the Lyapunov function candidate V of the form:

$$V(x) = \tfrac{1}{2}\omega_e^T M\omega_e + b\omega_e^T M\epsilon + c(1-\eta)^2 + c\epsilon^T\epsilon, \tag{22}$$

where b should be chosen small enough to ensure that V is positive definite, and $c > 0$. A sufficient condition for V being positive definite about $(\epsilon, \eta, \omega_e) = (0, 1, 0)$ is that

$$Q_V = \begin{bmatrix} 2cI_3 & bM\\ bM & M\end{bmatrix} > 0, \quad i.e., \quad 2cI_3 > b^2M. \tag{23}$$

Along the trajectory of (20) and applying the properties in (12), we have the following Lie derivatives of the Lyapunov function candidate V:

$$\begin{aligned} L_fV &= \tfrac{b}{2}\omega_e^T M[\eta I_3 + S(\epsilon)]\omega_e - b\epsilon^T S(\omega_e)M\omega_e + c\epsilon^T\omega_e\\ &= \tfrac{b}{2}\omega_e^T M[\eta I_3 - S(\epsilon)]\omega_e + c\epsilon^T\omega_e\\ L_{g_1}V &= \begin{bmatrix} b\omega_e^T MS(\epsilon), & \omega_e^T + b\epsilon^T\end{bmatrix}\\ L_{g_2}V &= \omega_e^T + b\epsilon^T. \end{aligned}$$

Prior to an inverse optimal control law in Theorem 3, we first present a PD controller that globally asymptotically stabilizes an auxiliary system of the form (26) of the nonlinear attitude tracking system (20) on $(S^3 \times R^3) \setminus (0, -1, 0)$ by the following theorem.

Theorem 2: The PD control law

$$u = -R_2^{-1}(L_{g_2}V)^T = -(k_1 + \tfrac{1}{\gamma^2}k_2)(\omega_e + b\epsilon) \tag{24}$$

with the matrix $R_2(x)$ being

$$R_2^{-1}(x) = (k_1 + \tfrac{1}{\gamma^2}k_2)I_3 \tag{25}$$

globally asymptotically stabilizes the auxiliary system

$$\begin{bmatrix}\dot{\epsilon}\\ \dot{\eta}\\ \dot{w}_e\end{bmatrix} = \begin{bmatrix}\frac{1}{2}[\eta I_3 + S(\epsilon)]\omega_e\\ -\frac{1}{2}\epsilon^T\omega_e\\ -M^{-1}S(\omega_e)M\omega_e\end{bmatrix} + \begin{bmatrix}0_{33}\\ 0_{13}\\ M^{-1}\end{bmatrix}u + \frac{1}{\gamma^2}\begin{bmatrix}S(\epsilon) & 0_{33}\\ 0_{13} & 0_{13}\\ 0_{33} & M^{-1}\end{bmatrix}(L_{g_1}V)^T \tag{26}$$

about the equilibrium point $(\epsilon, \eta, \omega_e) = (0, 1, 0)$ on $(S^3 \times R^3) \setminus (0, -1, 0)$ if the controller gains in (22) and (24) satisfy the following conditions:

$$\begin{aligned} &c = 2b\big[k_1 + \tfrac{1}{\gamma^2}(k_2 - 1)\big], \quad 1 \le k_2 \le 1 + b^2\bar{\lambda}_m^2,\\ &b > 0, \quad k_1 > \tfrac{b}{2}\bar{\lambda}_m + \tfrac{b^2}{\gamma^2}\bar{\lambda}_m^2 - \tfrac{1}{\gamma^2}(k_2 - 1). \end{aligned} \tag{27}$$

Proof: If we consider a class $\mathcal{K}_\infty$ function $\rho(r) = \gamma^2r^2$, it follows that $\rho'(r) = 2\gamma^2r$, $(\rho')^{-1}(r) = \frac{r}{2\gamma^2}$, $\ell_\rho(r) = \int_0^r(\rho')^{-1}(s)ds = \frac{r^2}{4\gamma^2}$ and $\ell_\rho(2r) = \frac{r^2}{\gamma^2}$. We can then construct an auxiliary system as follows:

$$\dot{x}(t) = f(x) + \tfrac{1}{\gamma^2}g_1(x)(L_{g_1}V)^T + g_2u, \tag{28}$$

which is the state representation of (26).

Applying (12), we have the following derivative of V along the solutions of (26):

$$\begin{aligned} \dot{V} =& L_fV + \tfrac{1}{\gamma^2}(L_{g_1}V)(L_{g_1}V)^T + L_{g_2}Vu\\ =& \tfrac{b}{2}\omega_e^T M[\eta I_3 - S(\epsilon)]\omega_e + c\epsilon^T\omega_e + \tfrac{1}{\gamma^2}\big|\omega_e + b\epsilon\big|^2\\ &+ \tfrac{1}{\gamma^2}\big|b\omega_e^T MS(\epsilon)\big|^2 + (\omega_e^T + b\epsilon^T)u\\ =& \tfrac{b}{2}\omega_e^T M[\eta I_3 - S(\epsilon)]\omega_e + (\omega_e^T + b\epsilon^T)u\\ &+ \tfrac{1}{\gamma^2}(|\omega_e|^2 + b^2|\epsilon|^2) + (c + \tfrac{2}{\gamma^2}b)\epsilon^T\omega_e\\ &- \tfrac{b^2}{\gamma^2}\big[\omega_e^T M\big(\epsilon\epsilon^T - \epsilon^T\epsilon I_3\big)M\omega_e\big]. \end{aligned} \tag{29}$$

Since $\epsilon^T\epsilon + \eta^2 = 1$, it can be shown that $||\eta I_3 - S(\epsilon)|| = 1$ and

$$|\omega_e^T M[\eta I_3 - S(\epsilon)]\omega_e| \le ||M|||\omega_e|^2. \tag{30}$$

By (30) and the inequality $|\epsilon|^2 \le 1$, it follows

$$\begin{aligned} \dot{V} \le& \tfrac{b}{2}\bar{\lambda}_m|\omega_e|^2 + (c + \tfrac{2}{\gamma^2}b)\epsilon^T\omega_e + (\omega_e^T + b\epsilon^T)u\\ &+ \tfrac{1}{\gamma^2}(|\omega_e|^2 + b^2|\epsilon|^2) + \tfrac{b^2}{\gamma^2}\epsilon^T\epsilon(\omega_e^T MM\omega_e)\\ \le& (\tfrac{b}{2}\bar{\lambda}_m + \tfrac{1}{\gamma^2} + \tfrac{b^2}{\gamma^2}\bar{\lambda}_m^2)|\omega_e|^2 + \tfrac{b^2}{\gamma^2}|\epsilon|^2\\ &+ (c + \tfrac{2}{\gamma^2}b)\epsilon^T\omega_e + (\omega_e^T + b\epsilon^T)u. \end{aligned} \tag{31}$$

We design the control law $u(x)$ to be of a PD form, expressed by (24), and select the controller parameters b, c, k_1, k_2 satisfying the constraints (27). Clearly, such parameters guarantee (23) such that the Lyapunov function V in (22) is positive definite because

$$2cI_3 \ge 4bk_1I_3 > 2b^2||M||I_3 > b^2M.$$

Hence,

$$\begin{aligned} \dot{V} \le& \big(\tfrac{b}{2}\bar{\lambda}_m + \tfrac{1}{\gamma^2} + \tfrac{1}{\gamma^2}b^2\bar{\lambda}_m^2\big)|\omega_e|^2 + \tfrac{1}{\gamma^2}b^2|\epsilon|^2\\ &+ (c + \tfrac{2b}{\gamma^2})\epsilon^T\omega_e - (k_1 + \tfrac{1}{\gamma^2}k_2)|\omega_e + b\epsilon|^2\\ \le& -\lambda_b|\tilde{x}|^2, \end{aligned}$$

where $\lambda_b > 0$ is defined by

$$\lambda_b = \min\{k_1 - \tfrac{b}{2}\bar{\lambda}_m - \tfrac{b^2}{\gamma^2}\bar{\lambda}_m^2 + \tfrac{k_2-1}{\gamma^2},\ \tfrac{b^2(k_2-1)}{\gamma^2} + k_1b^2\}.$$

It follows from the Barbalat's theorem [13, pp.192] that $\tilde{x}(t) \to 0$ as $t \to \infty$. By Lemma 2, this corresponds to the zero orientation error.

Since $|\epsilon|^2 + \eta^2 = 1$, the derivative $\dot{V} = 0$ implies two equilibrium points $(\epsilon, \eta, \omega_e) = (0, \pm 1, 0)$ on $S^3 \times R^3$, standing for exactly the same physical attitude orientation. However, it is clear that $(0, -1, 0)$ is an unstable equilibrium point, because it is a local maximum of $V(x)$ on $S^3 \times R^3$ and $\dot{V} < 0$ whenever $|\tilde{x}| \neq 0$. We therefore conclude that the PD control law (24) results in the global asymptotic stability of the auxiliary system (26) on $(S^3 \times R^3) \setminus (0, -1, 0)$. Note also that given any initial attitude orientation, one may always choose $\eta(0) \geq 0$. ■

Note that $l(x)$ in Theorem 1 can be positive semidefinite, which is also an meaningful cost functional. Applying Theorem 1 and Theorem 2, we will design an inverse optimal attitude tracking controller by the following theorem.

Theorem 3: If we let $\beta = \lambda = 2$, then the PD feedback control law

$$u = \beta\alpha(x) = -2(k_1 + \tfrac{1}{\gamma^2}k_2)(\omega_e + b\epsilon) \qquad (32)$$

with the parameters b, c, k_1, k_2 and $R_2(x)$ being given in Theorem 2, solves the inverse optimal gain assignment problem for the attitude tracking control problem on $(S^3 \times R^3) \setminus (0, -1, 0)$ with respect to the extended disturbance $\hat{d}(\omega_e, t)$ by minimizing the cost function

$$\begin{aligned} J(u) = &\sup_{\hat{d}} \Big\{ \lim_{t\to\infty} \big[4V(x(t)) + \\ &+ \int_0^t \big(l(x) + u^T R_2(x) u - \gamma^2 |\hat{d}|^2\big) d\tau\big]\Big\}, \end{aligned} \qquad (33)$$

where $l(x)$ is defined by

$$l(x) = -4L_f V - \tfrac{4}{\gamma^2}|L_{g_1}V|^2 + 4L_{g_2}VR_2^{-1}(L_{g_2}V)^T. \quad (34)$$

Proof: This theorem is a consequence of Theorem 1 and Theorem 2. From the derivations in Theorem 2, we have

$$\begin{aligned} l(x) \geq &4(k_1 - \tfrac{b}{2}\bar{\lambda}_m - \tfrac{b^2}{\gamma^2}\bar{\lambda}_m^2 + \tfrac{k_2 - 1}{\gamma^2})|\omega_e|^2 \\ &+ 4b^2(k_1 + \tfrac{k_2-1}{\gamma^2})|\epsilon|^2, \end{aligned} \qquad (35)$$

which shows that $l(x)$ is positive semidefinite (precisely, $l(x)$ is positive definite in ϵ and ω_e). Therefore, $J(u)$ is a meaningful cost functional for the attitude tracking control problem, penalizing both the tracking errors ϵ, ω_e and the control effort u. Substituting $l(x)$ in (34) into $J(u)$ in (33), we get the optimal cost $J(u) = 4V(x(0))$ and the "worse-case" extended disturbance

$$\hat{d}^*(x) = \lambda(\rho')^{-1}(2|L_{g_1}V|)\frac{(L_{g_1}V)^T}{|L_{g_1}V|} = \frac{2}{\gamma^2}(L_{g_1}V)^T \quad (36)$$

(see [9] for the detailed computations) where ρ is defined in the same way as in the proof of Theorem 2. ■

Remark 2: It can be seen from (35) that the state penalty function $l(x)$ in the performance index (33) can be rewritten as $l(x) = \tilde{x}^T Q(x)\tilde{x}$ where the weighting matrix $Q(x)$ is positive definite and its magnitude depends mainly on the gain k_1 and on γ. In the viewpoint of optimal theory, if we are to reduce the system error, we can enlarge k_1 and decrease γ, producing a larger weight $Q(x)$ and a bigger control effort. Conversely, if k_1 is reduced, then a smaller control effort and a bigger error will result. ⊗

It is interesting that such an inverse optimal PD controller shows H_∞ optimality, which implies that the $\mathcal{L}_2$-gain of the closed-loop system from $\hat{d}(\omega_e, t)$ to $\tilde{x}(t)$ is finite, thus achieving disturbance attenuation.

Theorem 4: The inverse optimal control law designed by (32) in Theorem 3 is also H_∞ optimal for the closed-loop attitude control system with respect to the extended disturbance $\hat{d}(\omega_e, t)$ and the H_∞ performance index (33).

Proof: From the derivations in Theorems 2 and 3, the Lyapunov function candidate $V(x)$ solves the following Hamilton-Jacobi-Isaacs partial differential equation:

$$\frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x}\big[\tfrac{1}{\gamma^2}g_1 g_1^T - g_2 R_2^{-1} g_2^T\big]\frac{\partial^T V}{\partial x} + \tfrac{1}{4}l(x) = 0.$$

Substituting $l(x)$ in (34) into the cost functional $J(u)$ in (33), it follows that $J(u) = 4V(x(0))$, which means

$$\int_0^\infty \big(l(x) + u^T R_2 u\big)dt \leq \int_0^\infty \gamma^2 |\hat{d}|^2 dt + 4V(x(0)) \quad (37)$$

with the "worst-case" extended disturbance $\hat{d}^*(x)$ of (36). Note that $l(x)$ is positive semidefinite and $R_2(x)$ is positive definite. Therefore, the inverse optimal PD controller (32) implies the H_∞ optimality of the attitude tracking control with respect to the extended disturbance $\hat{d}(\omega_e, t)$. ■

In general, we say that the system (1) with control input $u = k(x)$ is integral-input-to-state stable (iISS) with respect to d if for some functions α, $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for all initial states $x(0)$ and all d, the following estimate holds:

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|d(s)|)ds, \quad \forall t \geq 0.$$

For nonlinear control systems with disturbances or in the case of trajectory tracking, the *integral-input-to-state stability* [14] is a useful theoretical tool to analyze the stability of the closed-loop system; see [8][14] for examples. Under the control law (32), the stability of the closed-loop attitude control system is summarized by the following theorem.

Theorem 5: The attitude tracking full state $(\epsilon, \eta, \omega_e)$-system (20) under the inverse optimal PD control law (32) is integral-input-to-state stable on $(S^3 \times R^3) \setminus (0, -1, 0)$ with respect to the extended disturbance $\hat{d}(\omega_e, t)$.

Proof: The derivative of the Lyapunov function candidate $V(x)$ can be expressed as

$$\begin{aligned} \dot{V} = &L_f V(x) + L_{g_1}V\hat{d} - 2L_{g_2}VR_2^{-1}(x)(L_{g_2}V)^T \\ = &-\tfrac{1}{4}l(x) - \tfrac{1}{\gamma^2}|L_{g_1}V(x)|^2 + L_{g_1}V(x)\hat{d}(t) \\ &- L_{g_2}V(x)R_2^{-1}(x)(L_{g_2}V(x))^T. \end{aligned}$$

By Young's inequality, it follows

$$L_{g_1}V\hat{d} \leq \tfrac{\gamma^2}{4}|\hat{d}|^2 + \tfrac{1}{\gamma^2}|L_{g_1}V|^2,$$

where the equal sign is satisfied only when $\hat{d}(\omega_e, t) = \hat{d}^*(x) = \frac{2}{\gamma^2}(L_{g_1}V(x))^T$. Therefore,

$$\dot{V} \leq -\tfrac{1}{4}l(x) - L_{g_2}VR_2^{-1}(x)(L_{g_2}V)^T + \tfrac{\gamma^2}{4}|\hat{d}(t)|^2. \quad (38)$$

Note that $l(x)$ and $R_2(x)$ are positive definite. As analyzed in the proof of Theorem 2, the full-state system is GAS on $(S^3 \times R^3) \setminus (0, -1, 0)$ if the extended disturbance $\hat{d} = 0$. It is also zero-output dissipative because, if we let the output function $h(x) = 0$, $\dot{V} < \frac{\gamma^2}{4}|\hat{d}|^2$. Then, according to [14], the system is iISS with respect to $\hat{d}(\omega_e, t)$. ■

V. Performance Analysis

In the previous section, although the inverse optimal PD control law (32) guarantees the system iISS, it does not provide the global asymptotic stability due to the extended disturbance. In this section, we will introduce the concept of performance limitation and some results in [8] to analyze the performance of the attitude tracking controller.

The extended disturbance $\hat{d}(\omega_e, t)$ expressed by (19) can be represented as:

$$\hat{d}(\omega_e, t) = H(t)\omega_e + h(t), \tag{39}$$

where

$$H(t) = \begin{bmatrix} 0 \\ -S(\omega_c)M + S(M\omega_c) \end{bmatrix}, \quad h(t) = \begin{bmatrix} \omega_c \\ d_c \end{bmatrix}.$$

Under the assumption of the boundedness of the target signals $\omega_c(t)$, $\dot{w}_c(t)$ and the external disturbance $d(t)$ in the attitude dynamics (11), there exist some finite positive time-varying coefficients c_1, c_2, c_3, β_1 and β_2 such that

$$\begin{aligned} |\hat{d}(\omega_e, t)|^2 &= \omega_e^T(H^TH)\omega_e + 2\omega_e^T(H^Th) + (h^Th) \\ &\le c_1|\omega_e|^2 + c_2|\omega_e| + c_3, \\ &\le \beta_1|\omega_e|^2 + \beta_2. \end{aligned} \tag{40}$$

Such time-varying coefficients can be chosen as

$$\begin{aligned} c_1 &= ||H||^2 \le 4\bar{\lambda}_m^2|\omega_c|^2, \\ c_2 &= 2|H^Th| \le 4\bar{\lambda}_m|\omega_c||d_c|, \\ c_3 &= |h^Th| = |\omega_c|^2 + |d_c|^2, \\ \beta_1 &= 6\bar{\lambda}_m^2|\omega_c|^2, \qquad \beta_2 = 3|d_c|^2 + |\omega_c|^2. \end{aligned}$$

If $d(t) = 0$, the GAS holds for the setpoint regulation problem (corresponding to $\dot{w}_c = \omega_c = 0$ and $q_c = constant$) because $c_1 = c_2 = c_3 = 0$ and then $\dot{V} < 0$. However, the static PD controller cannot guarantee the GAS either in the trajectory tracking or in the existence of external disturbances. This fact brings a performance limitation of the inverse optimal PD controller. The control performance is determined by the gain values of the controller. Therefore, it is important to set up the relation between the gain values and the system errors, which is found by examining points that satisfy $\dot{V} = 0$ where the state cannot be further reduced.

Theorem 6: Choose $k_2 = 1$. Suppose that λ_c is the minimum eigenvalue of the matrix

$$Q_c = \begin{bmatrix} (2k_1 + \frac{1}{\gamma^2})b^2I_3 & (k_1 + \frac{1}{\gamma^2})bI_3 \\ (k_1 + \frac{1}{\gamma^2})bI_3 & Q_{c2} \end{bmatrix},$$

with $Q_{c2} = (2k_1 + \frac{1}{\gamma^2} - \frac{b}{2}\bar{\lambda}_m - \frac{b^2}{\gamma^2}\bar{\lambda}_m^2 - \frac{\gamma^2}{4}\bar{c}_1)I_3$. Let the performance limitation $|\tilde{x}|_{PL}$ be defined as the Euclidian norm of $\tilde{x}$ that satisfies $\dot{V} = 0$. If the inverse optimal PD controller law (32) with $k_2 = 1$ is applied to the attitude tracking system (20) and $\lambda_c > 0$ is satisfied, then its performance limitation is upper bounded by

$$|\tilde{x}|_{PL} \le \frac{\gamma^2}{8\lambda_c}\left[\bar{c}_2 + \sqrt{\bar{c}_2^2 + \frac{16}{\gamma^2}\lambda_c\bar{c}_3}\right], \tag{41}$$

where $\bar{c}_1 = \sup\{c_1(t) : \forall t \ge 0\}$, $\bar{c}_2 = \sup\{c_2(t) : \forall t \ge 0\}$ and $\bar{c}_3 = \sup\{c_3(t) : \forall t \ge 0\}$ with $c_1(t)$, $c_2(t)$ and $c_3(t)$ defined by (40).

Proof: Substituting $l(x)$ of (34) with $k_2 = 1$ into the derivative $\dot{V}$ in (38), we have

$$\begin{aligned} \dot{V} \le& -\left(k_1 - \tfrac{b}{2}\bar{\lambda}_m - \tfrac{b^2}{\gamma^2}\bar{\lambda}_m^2\right)|\omega_e|^2 - k_1b^2|\epsilon|^2 \\ & -\left(k_1 + \tfrac{1}{\gamma^2}\right)|\omega_e + b\epsilon|^2 + \tfrac{\gamma^2}{4}|\hat{d}(t)|^2 \\ \le& -\tilde{x}^TQ_c\tilde{x} + \tfrac{\gamma^2}{4}(c_2|\omega_e| + c_3) \\ \le& -\lambda_c|\tilde{x}|^2 + \tfrac{\gamma^2}{4}\bar{c}_2|\tilde{x}| + \tfrac{\gamma^2}{4}\bar{c}_3, \end{aligned} \tag{42}$$

where λ_c is the minimum eigenvalue of the matrix Q_c. The state vector cannot be further reduced at a point satisfying $\dot{V} = 0$. By the definition of the performance limitation, the inequality (42) brings the performance limitation of (41). ■

Remark 3: The inequality (41) can be considered as an upper bound of the system error for all time and then can be used as a formula to predict the performance of the closed-loop system for various values of the controller gains. ⊗

Remark 4: Since the right-hand side of (41) is monotonically decreasing with λ_c, the inequality holds if λ_c is replaced by any smaller positive value. The minimum eigenvalue λ_c of the matrix Q_c in Theorem 6 satisfies

$$\lambda_c \ge \min\left\{b^2k_1, \left(k_1 - \tfrac{b}{2}\bar{\lambda}_m - \tfrac{b^2}{\gamma^2}\bar{\lambda}_m^2 - \tfrac{\gamma^2}{4}c_1\right)\right\}. \tag{43}$$

It is because

$$Q_c = \begin{bmatrix} k_1b^2I_3 & 0 \\ 0 & q_{22}I_3 \end{bmatrix} + (k_1 + \tfrac{1}{\gamma^2})\begin{bmatrix} I_3 & 0 \\ \frac{1}{b}I_3 & I_3 \end{bmatrix}\begin{bmatrix} b^2I_3 & bI_3 \\ 0 & 0 \end{bmatrix}$$

with $q_{22} = k_1 - \frac{b}{2}\bar{\lambda}_m - \frac{b^2}{\gamma^2}\bar{\lambda}_m^2 - \frac{\gamma^2}{4}c_1$, and the minimum eigenvalue of the second part of Q_c is zero. Hence, λ_c satisfies (43). ⊗

Remark 5: Suppose that $|\beta_1(t)| \le \bar{\beta}_1 = \sup_t\{|\beta_1(t)|\}$. Then it follows from (40) and (38) that

$$\dot{V} \le -(k_1 - \tfrac{b\bar{\lambda}_m}{2} - \tfrac{b^2\bar{\lambda}_m^2}{\gamma^2} - \tfrac{\gamma^2\bar{\beta}_1}{4})|\omega_e|^2 - k_1b^2|\epsilon|^2 + \tfrac{\gamma^2\beta_2}{4}.$$

If $(k_1 - \frac{b\bar{\lambda}_m}{2} - \frac{b^2\bar{\lambda}_m^2}{\gamma^2} - \frac{\gamma^2\bar{\beta}_1}{4}) > 0$, it follows that the closed-loop system is β_2-to-(ϵ, ω_e) stable. In addition, if $\omega_c(t)$, $\dot{\omega}_c(t)$, $d(t) \in \mathcal{L}_2[0,\infty)$, then the $\mathcal{L}_2$-gain from β_2 to (ϵ, ω_e) is finite and ϵ, $\omega_e \in \mathcal{L}_2[0,\infty)$. ⊗

VI. Simulation Results

In this section, a rigid-body micro-satellite is simulated to demonstrate the performance of the H_∞ inverse optimal tracking controller. We assume that the inertia matrix of the satellite is diagonal,

$$M = \mathrm{diag}\{10,\ 10,\ 8\}\ \mathrm{kg{\cdot}m^2},$$

and the external disturbance $d(t)$ is given by

$$d(t) = \begin{bmatrix} 0.005 - 0.05\sin(\frac{2\pi t}{400}) + \delta(200, 0.2) + v_1 \\ 0.005 + 0.05\sin(\frac{2\pi t}{400}) + \delta(250, 0.2) + v_2 \\ 0.005 - 0.03\sin(\frac{2\pi t}{400}) + \delta(300, 0.2) + v_3 \end{bmatrix} \mathrm{Nm},$$

where $\delta(t_1, \Delta T)$ denotes an impulsive disturbance with magnitude 1 and width ΔT, starting at the time point t_1. The terms v_1, v_2 and v_3 denote white Gaussian noises with mean values $m_{v_1} = m_{v_2} = m_{v_3} = 0$ and variances $\sigma_{v_1} = \sigma_{v_2} = \sigma_{v_3} = 0.005$.

The desired angular velocity ω_c is given by

$$\omega_c = \begin{bmatrix} 0.05\sin(2\pi t/400) \\ -0.05\sin(2\pi t/400) \\ 0.03\sin(2\pi t/400) \end{bmatrix} \cdot \mathrm{rad/s}$$

and is plotted as in Fig.1 in dotted lines. The target quaternion q_c can then be obtained by integrating (13) with the initial condition $q_c(0) = [0,\ 0,\ 0,\ 1]^T$.

The initial conditions of the quaternion q and the angular velocity w are given by $q(0) = [0.3,\ 0.2,\ 0.3,\ -0.8832]^T$ and $w(0) = [0,\ 0,\ 0]^T$.

We choose a set of gains, $\gamma = 1$, $k_1 = 4.0$, $k_2 = 1$ and $b = 0.18$, to demonstrate the performance of the proposed H_∞ inverse optimal PD tracking controller (32). Fig.1 and Fig.2 depict the time responses of the angular velocities ω, ω_c and the corresponding rate error ω_e, from which we can say that the actual angular velocity ω tracks the target angular velocity ω_c well with a small error. The sharp peaks are due to the large impulsive disturbances. Fig.3 and Fig.4 plot the time behaviors of the actual quaternion q compared to the desired quaternion q_c and the corresponding tracking error ϵ, from which it is seen that the H_∞ inverse optimal controller (32) achieves a good performance on the attitude tracking with a satisfactory orientation error.

VII. Conclusions

With the introduction of extended disturbance, the inverse optimal control method has been applied to the attitude tracking control problem of a rigid spacecraft with external disturbances. The proposed state-feedback control law is inverse optimal with respect to a meaningful cost functional. The associated control Lyapunov function satisfies a Hamilton-Jacobi-Isaacs equation. Thus, nonlinear H_∞ optimality with respect to the extended disturbance is achieved without obtaining a direct solution to the HJI equation, and H_∞ disturbance attenuation is also achieved. Performance estimates have been given in terms of the performance limitation. Such a feedback law is in the form of a PD controller, which is easy to implement in practice.

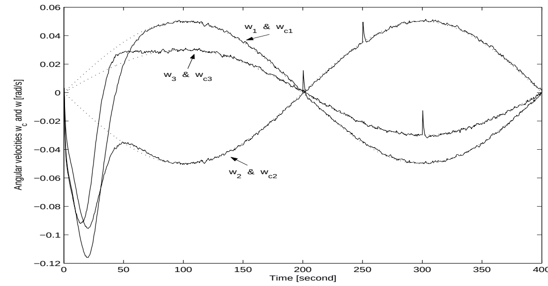

Fig.1: The angular velocities ω and ω_c. The dotted lines stand for ω_c, the solid lines denote ω.

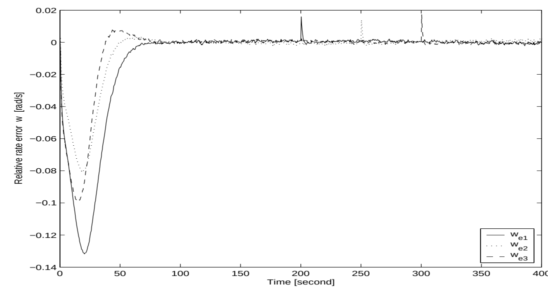

Fig.2: The tracking error $\omega_e = [\omega_{e1}, \omega_{e2}, \omega_{e3}]^T$.

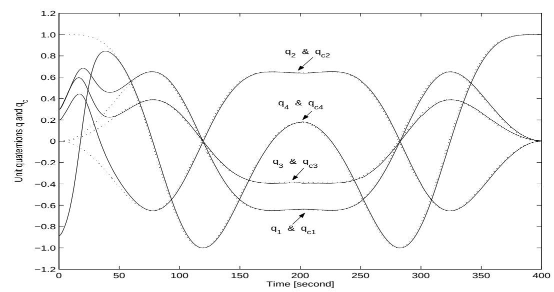

Fig.3: The unit quaternions q and q_c. The dotted lines stand for q_c, the solid lines denote q.

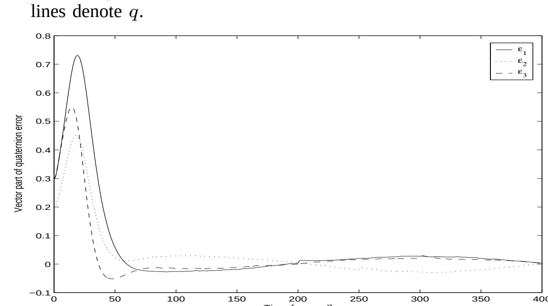

Fig.4: The tracking error $\epsilon = [\epsilon_1, \epsilon_2, \epsilon_3]^T$.

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