Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Uncertainty Analysis of Complex Dynamical Systems

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Abstract— **A unified framework for studying parametric and initial condition uncertainty in dynamical systems is developed. The notion of input measure of an observable is defined and its propagation to output measure of the observable is studied by means of transfer operators. Uncertainty of these measures is defined in terms of their cumulative probability distributions. The developed formalism is illustrated through an analysis of the effect of pitchfork bifurcation on uncertainty. General results on uncertainty for dynamical systems on an infinite time horizon are derived.**

I. Introduction

Uncertainty analysis is a topic of research that has received much attention in recent years. Indeed, the increased use of physics based models in the study of the dynamical behavior of systems in a wide range of applications calls for the analysis and quantification of model predictions in terms of uncertainties in model descriptions and model operating environments [7]. In this paper we consider systems that can be modeled by discrete maps and adopt the view that uncertainty in the model description can be represented as a random uncertainty in model parameters and initial conditions. We introduce a new framework for uncertainty analysis that is rooted on ideas from Random Dynamical Systems (RDS) [1]. Random Dynamical Systems is an area of research in the intersection of dynamical systems and probability theory and provides the ideal framework for treating uncertainty in dynamical systems that can exhibit complex dynamical behavior.

The analysis of uncertainty sources and classification of uncertainty types in mathematical models has received much attention [12]. Frequently uncertainty is classified into two types: reducible, or *epistemic*, and irreducible, or *aleatory*. An example of epistemic uncertainty is uncertainty initial conditions that can possibly be reduced by improved measurements. Aleatory uncertainty is an uncertainty in the system parameters that are the result of the intrinsic stochasticity of the system. We make a further distinction that both of these can be a-priori and a-posteriori. In particular, a-priori uncertainty is any uncertainty (epistemic or aleatory) that can be captured in an input description of the system and a-posteriori uncertainty is an uncertainty that is inherent to the process dynamics and observations.

The most common approach for propagating uncertainty in mathematical and computational models is to use Monte Carlo type methods [5] [6][7]. Due to the fact that Monte Carlo methods are basically ”wrapper” methods they have the advantage that they apply to a large class of problems but suffer from slow convergence rate and in many problems the computational burden may be prohibitive. An alternative approach for uncertainty propagation are Polynomial Chaos methods (also called Stochastic Finite Elements) [3][4]. Polynomial Chaos is an analytical approach based on expansions of the uncertain quantities of terms of prescribed random basis functions. It has been demonstrated that for certain classes of problems Polynomial Chaos can be considerably (up to several orders of magnitude) faster than Monte Carlo methods. Furthermore, the analytical representation in Polynomial Chaos can be of great benefit in analysis. However, the computational speed of Polynomial Chaos depends very much on the character of the probability distribution of the input parameters as well as characteristics of the system equations and the computational advantage in certain problems may be minimal [15].

Monte Carlo and Polynomial Chaos are methods for uncertainty propagation and have to be combined with other analysis methods for analyzing the effect of input uncertainty on critical system outputs or measureables. Sensitivity Analysis is probably the best known approach for evaluating the effect of variations in system inputs and parameters on system outputs. Sensitivity Analysis of dynamical systems involves evaluating partial derivatives of system outputs with respect to the uncertain quantities and works well for problems that are nearly linear [13].

In this paper we are interested in defining and calculating the effect of probabilistic uncertainty in input parameters and initial conditions on the output of a dynamical system that may exhibit complex dynamic behavior. In particular, we are interested in uncertainty analysis of systems that are far from linearity, may have multiple steady states and exhibit purely nonlinear behavior such as bifurcations that depend on the uncertain parameters of the system. We develop an approach that involves defining uncertainty propagation in the system through the invariant measures of the system (in the sense of Random Dynamical Systems) and defining uncertainty as a worst case distance from a certain system in the space of output measures. Related, but nevertheless quite different concepts have been considered in the reliability literature [2], where uncertainty is defined in terms of a distance between cumulative distribution functions, and in the atmospheric sciences [8], where uncertainty (called there predictability) is defined in terms of the relative entropy between uncertain and certain measures.

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The paper is organized as follows: In Section 2 we formulate our framework for uncertainty analysis in terms of Random Dynamical Systems. In Section 3 the effect of uncertainty in initial conditions is analyzed and used to motivate the definition of an uncertainty in a dynamical system. Uncertainty of an observable is rigorously defined and characterized in Section 4. In Section 5 the uncertainty of a dynamical system on an infinite time horizon is analyzed in detail. Finally conclusions are formulated in Section 6. All proofs are omitted due to space restrictions and can be obtained from the authors upon request.

II. Dynamical evolution of uncertainty

We consider a dynamical system whose evolution is determined by the Discrete Random Dynamical System (DRDS)

$$x_{i+1} = T(x_i, \xi), \qquad y_i = f(x_i) \tag{1}$$

where $x_i \in M$ is the system state, $\xi \in N$ a random parameter, y_i is a scalar valued observable and $i \in \mathbb{Z}$. We assume that M and N are compact Riemannian manifolds, N is endowed with a probability measure p that is absolutely continuous with respect to the Lebesgue measure on N. We assume that $T(x,\xi)$ is $C^r, r \geq 1$ in x for every $\xi \in N$ and consider observables $f : M \to \mathbb{R}$ that satisfy $f \in L^1(M)$. Denote $T^i_\xi(x) = T_\xi \circ \ldots \circ T_\xi$ where $T_\xi(x) = T(x,\xi)$. With some abuse of notation, we will call the above DRDS T. The product space $P = M \times N$ is endowed with the product σ-algebra $\mathcal{P}$ in the usual way.

Definition 1: A probabilistic measure η on P is called an input measure.

Example 2: Assume M is a Riemannian manifold, $N = [a,b] \subset \mathbb{R}$, $\eta = \nu \times \delta(\xi^*)$, where $\xi^* = c$ is a constant. In this case there is no ”parametric uncertainty” in parameter c. If ν is not a Dirac delta measure, it represents uncertainty in initial conditions of the process.

Example 3: Consider the evolution given by

$$x_{i+1} = x_i + \xi x_i - x_i^3, \tag{2}$$

where the input measure η is given by the product measure $\nu \times \rho$, $\nu(E) = l(E \cap [-a,a])/2a$, l the Lebesgue measure on the real line, $a > 0$. This measure is given by the density function $f_\nu(x) = 1/2a$ for $x \in [-a,a]$. The function $f_\nu(x)$ is the density for the uncertainty in initial conditions. The measure ρ is a measure on $\mathbb{R}$ defined by $\nu(E) = l(E \cap [-b,b])/2b$. This measure is given by the density function $f_\rho(\xi) = 1/2b$ for $\xi \in [-b,b]$. The function $f_\rho(\xi)$ is the density for the ”parameter input measure”. In figure 1 the input measure is superposed on the bifurcation diagram of the equation (2) ($b < 1$).

We are interested in the question of how does the uncertainty in the ”output” of the process depend on the input measure. For an observable $f : M \to \mathbb{R}$, the ”initial” uncertainty is described by a probabilistic measure ϖ_i on $\mathbb{R}$ (endowed with the Borel σ-algebra $\mathcal{B}$) defined by

$$\varpi_i(E) = \eta((f)^{-1}(E)),$$

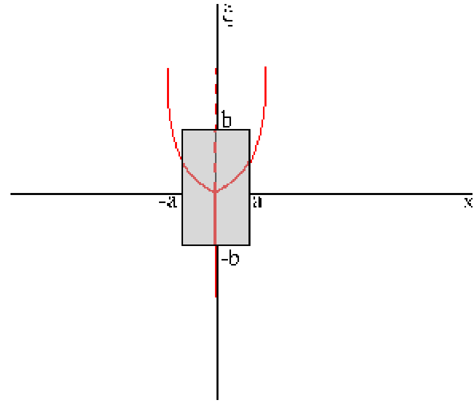

Fig. 1. The bifurcation diagram and uncertainty measure density for equation (2).

where $E \in \mathcal{B}$. This measure evolves in time, becoming

$$\begin{aligned} \varpi^n(E) &= \eta((f \circ T^n_\xi)^{-1}(E)) = \eta((T^n_\xi)^{-1} f^{-1}(E)) \\ &= P^n_\xi \eta(f^{-1}(E)) \end{aligned}$$

where P_ξ is the Perron-Frobenius operator. We call ϖ^n an *output measure*. It describes the uncertainty of observable f at the $n-$th step of the process given the input measure η. The cumulative distribution function $P_\varpi : \mathbb{R} \to \mathbb{R}$ of ϖ is defined in the obvious way by

$$F^n_\varpi(z) = \varpi^n((-\infty, z]).$$

Let us relate these notions with uncertainty semantics in the literature (see e.g. [12]). Frequently uncertainty is classified into two basic types: reducible, or *epistemic*, and irreducible, or *aleatory*. An example of epistemic uncertainty is uncertainty initial conditions that can possibly be reduced by improved measurements. An example of aleatory uncertainty is pressure difference on an airplane wing at fixed speed in clear air turbulence. This is known only as a measure and comes from intrinsic stochasticity of the flow turbulence. We make a further distinction that both of these can be a-priori and a-posteriori. In the example of airplane wing, the aleatory pressure difference uncertainty might be a parameter for a calculation and thus is a-priori. The same is true for initial conditions uncertainty. In this paper we do not make distinction between aleatory and epistemic a-priori uncertainties: they are both described by the input measure η. A-posteriori aleatory uncertainty is the one that is inherent to the process. This uncertainty comes from a possibly chaotic nature of the dynamics of T. In geophysical literature this would be called ”climatological uncertainty” (see [8]). A posteriori epistemic uncertainty is induced by, for example, measurement and simulation error.

III. Uncertainty in initial conditions

Assume now that there is no uncertainty in the parameter values in (1), i.e. the distribution for ξ is concentrated at some point $c \in N$. The uncertainty in initial conditions is assumed to be described by a measure η on M which is absolutely continuous with respect to the Lebegue measure ϱ on M. Systems that evolve from any initial condition (except perhaps for measure zero sets, cf [14]) to a fixed point $\hat{x}$ have no a-posteriori aleatory uncertainty. Thus, if there is no a-posteriori epistemic uncertainty, as $n \to \infty, F_{\varpi}^n$ will approach the step function

$$F_{\varpi}(z) = 0 \text{ for } z \leq f(\hat{x}),$$
$$F_{\varpi}(z) = 1 \text{ for } z > f(\hat{x}).$$

This perfectly (in the probabilistic sense, as measure zero sets are discarded) certain cumulative distribution function motivates developments in the next subsection. In the above described situation, the system is said to have a physical measure [16] that is a Dirac delta measure supported at the fixed point $\hat{x}$. Such systems have the property of *shrinking uncertainty in initial conditions* when the initial measure η is absolutely continuous with respect to ϱ.

In some sense the opposite situation occurs for an expanding map T on a manifold M [10]. A map is called expanding if there is $n > 1$ such that

$$\|DT|_x^n \varsigma\| \geq \|\varsigma\|,$$

for all ς, x, where $\varsigma \in T|_x M$, the tangent space of M at $x \in M$, and DT_x is the derivative of T at x. Such maps posses a unique invariant measure μ_e, absolutely continuous with respect to ϱ. The a-posteriori aleatory uncertainty of an observable f is uncertainty with respect to measure ϱ. A measure-preserving expanding map T on a closed manifold M evolves any initial condition uncertainty of f to uncertainty with respect to Lebesgue measure since, if μ is an invariant measure for $T, \mu = \varrho$ by uniqueness of absolutely continuous invariant measures for such maps. An example is the map $x' = 2x (\bmod 1), x \in S^1$.

The discussion above naturally leads us to try to precisely define a-posteriori aleatory uncertainty within the context of dynamical systems models, paying attention to the fact that we are interested in evolution of a single observable. We consider (1) and assume that for a fixed ξ the system has a physical measure μ, i.e. that for any $g : M \to \mathbb{R}, g \in L^1(\varrho)$, for almost every $x \in M$ with respect to ϱ, the time average of g along trajectory starting at x,

$$g^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T_\xi^i(x)) = \int_M g d\mu.$$

We now define the aleatory uncertainty measure on $\mathbb{R}$ by

$$\varpi_A(E) = (\chi_E \circ f)^* = \int_M \chi_E \circ f d\mu,$$

where χ_E is the characteristic function of the measurable set $E \subset \mathbb{R}$. The intuition about ϖ_A is that it represents the proportion of time that the observable f spends in the set E during the evolution of the system, i.e. ϖ_A is the occupation measure of the observable. Note that ϖ_A can also represented as $\varpi_A(E) = \mu\left(f^{-1}(E)\right)$, i.e. the physical measure μ plays the same role in the definition of the a-posteriori aleatory uncertainty measure as the initial measure η does in the definition of the a-priori output measure. The measure ϖ_A corresponds to the so-called "climatological uncertainty" in geophysics [8]. It is natural to consider uncertain only the part of the evolution of f that does not have any cyclic motion associated with it i.e. that the motion on the attractor that is support for μ does not have periodic factors (see the discussion in [11]). A test for this is provided by the requirement that

$$g_\omega^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \exp(2\pi j i \omega) g(T_\xi^i(x)) = 0, \qquad (3)$$

for every $\omega \neq 0$. In this case ϖ_A is absolutely continuous (has a density) with respect to the Lebesgue measure.

Even though we illustrated above the concept of an certain system in terms of system that has no parametric uncertainty this is not necessary, i.e. a system may have an a-priori uncertainty in parameters and still converge to an certain system. This is best illustrated by an simple example.

Example 4: Consider the linear system

$$x_{i+1} = A(\xi)\, x_i$$
$$y_i = C(\xi)\, x_i$$

where $A(\xi)$ is an $n \times n$ matrix and $C(\xi)$ is $1 \times n$. Note that since $C(\xi)$ depends on ξ this system is more general than the one in (1). Assume the initial condition x_0 is uncertain with distribution ν and the parameter ξ is uncertain with distribution ρ. The solution of the above equation is

$$y_i = C(\xi)\, A(\xi)^i\, x_0$$

If $\sup_{\xi \in support(\rho)} \|A(\xi)\| < 1$ and $\|C(\xi)\| < \infty$ then $y_i \to 0$ as $i \to \infty$ for any finite value of x_0. Therefore, in this case the distribution converges to a certain distribution at 0. In particular, since the solution converges to a fixed point that is independent of ξ for all x_0 there is no a-posteriori uncertainty.

IV. Uncertainty of an Observable

In the previous section we argued that if a system possesses a globally attracting fixed point, initial uncertainty described by an absolutely continuous measure on initial conditions will shrink to a perfectly certain Dirac delta measure concentrated at the fixed point. In this section we wish to provide and analyze quantitative measures of uncertainty. It is clear that a measure of uncertainty of a measure ϖ on $\mathbb{R}$ needs to be invariant with respect to shifts of the measure: the uncertainty of measure defined by $\psi(A) = \varpi(A + c), c \in \mathbb{R}, A \in \mathcal{B}$, should be the same as uncertainty of ϖ. These considerations motivate the following definition:

Definition 5: Let $|\cdot|$ denote a metric (or a pseudo-metric) on the space $\mathcal{M}$ of probabilistic measures on $\mathbb{R}$. For any measure $\varpi \in \mathcal{M}$ the uncertainty of ϖ induced by $|\cdot|$ is defined by

$$u(\varpi) = \min_{z\in\mathbb{R}} |\varpi - \delta_z|,$$

where δ_z is the Dirac delta distribution at z.

A possible pseudo-metric is defined through use of cumulative distribution functions, defining

$$|\varpi_1 - \varpi_2| = \|F_{\varpi_1} - F_{\varpi_2}\|_1 ,$$

where $\|\cdot\|_1$ is the L^1 norm. This is well defined for measures with compactly supported P_ϖ and also for those that satisfy

$$|F_\varpi(z) - 1| \leq \frac{c}{z^2}, z > c > 0,$$
$$|F_\varpi(z)| \leq \frac{c}{z^2}, z < -c < 0.$$

The uncertainty using this norm is denoted by $u_1(\varpi)$. Equivalently, L^p norms can be used in which case the resulting uncertainty is denoted by $u_p(\varpi)$. In general we have the following result for the characterization of $u_p(\varpi)$.

Proposition 6: Assume that $\|F_\varpi - F_{\delta_z}\|_p < \infty$ for at least one $z = \tilde{z} \in \mathbb{R}$. Then $u_p(\varpi) = \min_{z\in\mathbb{R}} \|F_\varpi - F_{\delta_z}\|_p$ is attained at the median value z^* for which

$$\begin{aligned} &\lim_{z\downarrow z^*} F_\varpi(z^*) = F_\varpi(z^*) \geq 1/2, \\ &\lim_{z\uparrow z^*} F_\varpi(z^*) \leq 1/2. \end{aligned} \tag{4}$$

Furthermore,

$$u_p(\varpi) = \left(\int_{-\infty}^{z^*} |F_\varpi(x)|^p \, dx + \int_{z^*}^{\infty} |1 - F_\varpi(x)|^p \, dx \right)^{\frac{1}{p}} . \tag{5}$$

Note that if $z^*$is unique then either $F_\varpi(z^*) = 1/2$ (in that case z^* is a point of continuity of F_ϖ) or there is no z such that $F_\varpi(z) = 1/2$ (in that case z^* is a point of discontinuity of F_ϖ).

The above proposition allows us to characterize maximal uncertainty for output measures with bounded support:

Corollary 7: Consider the set $\mathcal{M}_a^b$ of probability measures on $\mathbb{R}$ with bounded support, $\mathcal{M}_a^b = \{ \varpi \in \mathcal{M} | F_\varpi(z) = 0$ if $z < a, F_\varpi(z) = 1$ if $z \geq b, a < b \in \mathbb{R}\}$. Then

$$\max_{\varpi\in\mathcal{M}_a^b} u_p(\varpi) = \frac{1}{2}(b-a)^{\frac{1}{p}},$$

The maximum is realized for $F_\varpi(z) = 1/2$, for $a \leq z < b$ (i.e. $\varpi = \Delta_{a,1/2}^{b,1/2}$ is the Dirac delta measure concentrated at a and b with strengths $1/2$).

Thus, the most uncertain - in the sense of u_p - output measure with bounded support is the Dirac delta measure concentrated at the support boundaries. It is useful to compare $u_1(\Delta_{a,1/2}^{b,1/2})$ with $u_1(\varpi_u)$, where ϖ_u is the uniform distribution on $[a, b)$. We have $u_1(\varpi_u) = (b-a)/4 = \frac{1}{2}u_1(\Delta_{a,1/2}^{b,1/2})$. In terms of already introduced examples of dynamical systems, we have the following example of a map that leads to maximum uncertainty from specific initial measure η.

Example 8: Consider a map T on a closed manifold M endowed with a Lebesgue measure ϱ. Assume that there is no uncertainty in parameter values. The uncertainty in initial conditions is assumed to be described by a measure η on M which is absolutely continuous with respect to ϱ. Let f be a continuous function on M and assume that $f(x_1^*) = \max_M f$, $f(x_2^*) = \min_M f$, where x_1^*, x_2^* are attracting fixed points of T. Additionally, let $\eta(U_1) = \eta(U_2) = 1/2$ where U_1 and U_2 are the domains of attraction of x_1^*, x_2^*, respectively and $M = U_1 \cup U_2$.Then the asymptotic distribution of uncertainty for f is maximal. Note however that the uncertainty in question is epistemic (reducible) since it is in initial conditions. Changing η such that $\eta(U_1) = 1$ leads to completely certain outcome.

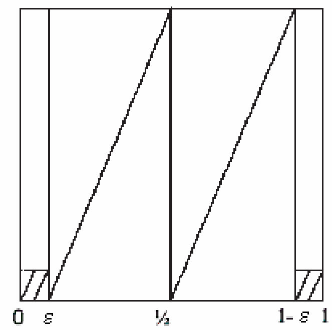

Fig. 2. Graphical representaion of the map $T(x)$ in Example 9

Example 9: Consider the transformation by which $x \in [0, 1]$ is mapped into $T(x) \in [0, 1]$ as shown in figure 2 where all the lines drawn are straight. We take $f(x) = x$ for the observable.The transformation is expanding with $n = 1$. The ergodic invariant measure is uniform on intervals $[0, \varepsilon)$ and $(1-\varepsilon, 1]$, with magnitude $1/2\varepsilon$ on each of these intervals. Any absolutely continuous initial condition measure η converges to the ergodic invariant measure as time goes to infinity. In fact, the system is mixing and f passes the test (3). Thus, uncertainty starting from *any* absolutely continuous initial condition measure can be made arbitrarily close to maximum uncertainty possible, by choosing ε sufficiently small.

It is also useful to compare another possible measure of uncertainty, entropy [9], defined as $H(\varpi) = -\int_a^b f_\varpi(x) \log f_\varpi(x) dx$, where f_ϖ is the probability density function, $f_\varpi = dF_\varpi/dx$, where it has been assumed all the quantities involved exist. We have $H(\varpi_u) = \log(b-a)$. Entropy increases as a $\log$ of the size of the support interval, while uncertainty u_p increases as a power law. Entropy, however, has one property that makes it less appealing for the measure of uncertainty in our context: consider for example the Dirac delta measure $\Delta_{a,1/2}^{b,1/2}$ that maximizes u_p on $\mathcal{M}_a^b$. Since this is a ”discrete” measure, its entropy equals $\log(2)$ and does not depend on the interval bounds a and b!

In the literature, uncertainty of the output is often calculated in terms of the variance of its probability distribution,

$$V(\varpi) = \int_{-\infty}^{\infty} (z - m(\varpi))^2 dF_\varpi(z),$$

where $m(\varpi)$ is the mean given by

$$m(\varpi) = \int_{-\infty}^{\infty} z dF_\varpi(z)$$

However, if one accepts the intuitive notion of the certain distribution as the one having a single value with probability one, the following result shows that measuring uncertainty with variance might be misleading.

Proposition 10: Let $v, u > 0$ be arbitrary constants. Then there is an ϖ such that $V(\varpi) > v, u_1(\varpi) < u$.

We can construct ϖ that has the stated properties as follows. Let the probability density of ϖ be constant and equal to ϵ on the interval $[-l, 0), l > 0$, and a Dirac delta with magnitude m, where $0 \leq m \leq 1$ at $r \in \mathbb{R}, r \geq 0$. Then using the fact that the probability density integrates to 1 gives $\epsilon = \frac{1-m}{l}$. We require that the mean of the distribution, $m(\varpi)$ be at 0. This leads to $r = \frac{\epsilon l^2}{2m} = \frac{(1-m)l}{2m}$. The variance of ϖ is given by

$$V(\varpi) = mr^2 + \epsilon\frac{l^3}{3} = \frac{(1-m)^2l^2}{4m} + \frac{(1-m)l^2}{3}, \quad (6)$$

whilst its uncertainty, $u_1(\varpi)$ for the case $m \geq 1/2$ can be calculated using $z^* = r$ in formula (5), to obtain

$$u_1(\varpi) = \frac{(1-m)l}{2}(1 + \frac{1-m}{m}).$$

Note that by choosing m, l we determine the measure ϖ. Now consider a sequence $\{m_i\}, \{l_i\}, i \in \mathbb{Z}^+, m_i \to 1$ as $i \to \infty$, determining measures ϖ_i and let $(1 - m_i)l_i = k < u$. Note that $l_i \to \infty$ as $i \to \infty$. Thus, $(1 - m_i)l_i^2 = kl_i \to \infty$, and since the first term in (6) is positive, by choosing i big enough, we get $V(\varpi_i) > v, u_1(\varpi_i) < u$.

Reflecting upon the nature of construction of the desired distribution in the proof of the above proposition, it becomes clear that variance measures distance from a certain distribution at 0. However, the closest certain distribution to ϖ_i tends to Dirac delta at $r = k/2$. This failure of variance to measure uncertainty is related to the fact that it pre-sets the "closest certain value" to be at the mean, which is not necessarily true. In fact, as is clear from Proposition 6, if we use u_p as the measure of uncertainty, the "closest certain value" is at the *median* of the distribution.

V. Time-averaged uncertainty

When studying the dynamic behavior of complex systems we are frequently interested in asymptotic properties of solutions. In this context uncertainty in the system can be formulated in terms of uncertainty in invariant measures and, adopting the point of view of [11], this question is transferred to questions of uncertainty in time-averages of observables:

$$\Pi_\xi f(x) = f^*(x,\xi) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T(x_i,\xi_i)) \quad (7)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} U_\xi^i f(x), \quad (8)$$

where Π_ξ is the projection operator on the space of time-averages and $U_\xi^i f = f \circ T_{\xi i}$ is the Koopman operator. Note that the Koopman operator reduces to its usual version when the parameter ξ does not change during the evolution, while Π_ξ becomes the projection operator on the space of constant functions. The *time-averaged uncertainty* is uncertainty of the

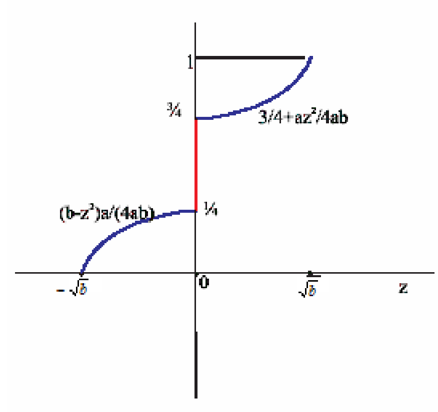

Fig. 3. Cumulative probability distribution for the output measure.

probabilistic measure ϖ_a (*asymptotic output measure*) on $\mathbb{R}$ defined by

$$\varpi_a(E) = \eta((f^*)^{-1}(E)). \quad (9)$$

Example 11: For the problem defined in Example 3, the cumulative probability distribution of the asymptotic output measure corresponding to the function $f(x) = x$ is represented in Figure 3. Applying Proposition 6 gives $u_1(\varpi_a) = \sqrt{b}/3$. Thus the pitchfork bifurcation exhibits, for small b a substantial (of order $\sqrt{b}$ for initial uncertainty of order b) increase of uncertainty.

The system in Example 3 is not ergodic for every value of the parameter ξ. In fact, for the case of a single parameter value ξ^* (i.e. in the situation described in Example 2), ergodic systems will have zero time-averaged uncertainty for any $f \in L^1$.

Proposition 12: Assume $T(x, \xi) : M \to M$ is ergodic for $\xi = \xi^*$ with an invariant ergodic measure μ and the input measure is given by $\eta = \nu \times \delta(\xi^*)$, where η is absolutely continuous with respect to μ. Let $f : M \to \mathbb{R}$ and let ϖ_a be the output measure defined by (9). Then $u_1(\varpi_a) = 0$.

The above result leads us to consider more closely the relationship between the concept of *ergodic partition* and *uncertainty* in initial conditions and the definition of time-average uncertainty in terms of asymptotic properties of systems. We discuss next how ergodic partition [11] is related to uncertainty in initial conditions. For this purpose, we consider the system

$$x_{i+1} = T(x_i, c). \quad (10)$$

where c now represents a (certain) parameter, and as before $x \in M$, a compact Riemannian manifold. The input measure (on M) is denoted by ν.

Definition 13: Let μ be an *a-priori* measure on the Borel σ-algebra on M. System (10) is called *B-regular* (for Birkhoff) with respect to μ (or *B_μ-regular*) if it possesses a finite set of ergodic invariant measures $\mu_i, i = 1, ..., n$ such that for almost every $x \in M$ with respect to μ there is a $j \in \{1, ..., n\}$ such that the time-average defined in (7) satisfies

$$f^*(x) = \int_M f d\mu_j$$

for every $f \in C(M)$ and in addition there is a family of disjoint sets $D_i, i = 1, ..., n$, such that

$$\mu(D_i) > 0, \mu(M - \cup_i D_i) = 0,$$

where

$$D_i = \{x \in M | \int_M f d\mu_i = f^*(x), \text{ for every } f \in C(M)\}.$$

Example 14: The system described in Example 3 is a *B-regular* with respect to l for any value of ξ.

For B_μ*-regular* systems with uncertainty in initial conditions and input measures that are absolutely continuous with respect to μ the distribution functions are simple.

Proposition 15: Let T be a B_μ*-regular* system and ν an input measure absolutely continuous with respect to the a-priori measure μ. Then, for any $f \in L^1_\mu(M), F_{\varpi_a}$ is a step function with a finite number of non-zero steps.

The uncertainty $u_1(\varpi_a)$ is readily calculated for step distribution functions. Assume that $F_{\varpi_a}(z)$ has steps of sizes f_i at z_i, $i = 1, \ldots, n$, i.e.

$$F_{\varpi_a}(z) = \sum_{i=1}^{n} f_i u_{z_i}(z) \tag{11}$$

where u_z is the unit step function at z.

Corollary 16: For the step distribution function F_{ϖ_a} in (11) define $e_j = \sum_{i=1}^{j} f_i$ and let $k = \max\left\{j : e_j \leq \frac{1}{2}\right\}$. Then

$$u_1(\varpi_a) = \sum_{i=1}^{k} e_i (z_{i+1} - z_i) + \sum_{i=k+1}^{n-1} (1 - e_i)(z_{i+1} - z_i) \tag{12}$$

We conclude this section with an extension of the above results to the case when ξ is random but does not change during the evolution, i.e. we consider system (1).

Definition 17: Let μ be an *a-priori* measure on the Borel σ-algebra on $M \times N$. System (1) is called parametric B_μ*-regular* if there exists a partition of $D_i, i = 1, ..., n$ of $M \times N$ such that $\mu(D_i) > 0$, $\mu(M \times N - \cup_i D_i) = 0$, $D_i \cap D_j = \varnothing, i \neq j$ and measures $\mu_i^\xi, i = 1, ..., n$ on M such that for almost every $(x, \xi) \in M \times N$ with respect to μ there is a $j \in \{1, ..., n\}$ such that

$$f^*(x, \xi) = \int_M f d\mu_j^\xi, \quad (x, \xi) \in D_j \tag{13}$$

where $f^*(x, \xi)$ is the time average (7).

Proposition 18: Assume that (1) is parametric B_μ*-regular* and the family of measures $\mu_i^\xi, i = 1, ..., n$ has the property that each $\mu_i^\xi(B)$ is continuous as a function of ξ for any $B \in \mathcal{B}(M)$. Assume that the initial measure η is absolutely continuous with respect to the a-priori measure μ. Then F_{ϖ_a} is piecewise continuous with a finite number of steps.

Example 19: The system in Example 11 satisfies the conditions in Proposition 18 with sets

$$\begin{aligned} D_1 &= \{\xi < 0, -\infty < x < \infty\}, \\ D_2 &= \{\xi \geq 0, -\infty < x < 0\}, \\ D_3 &= \{\xi \geq 0, 0 < x < \infty\} \end{aligned}$$

and $\mu_1^\xi = \delta_{\{x=0\}}$, $\mu_2^\xi = \delta_{\{x=-\sqrt{\xi}\}}$, $\mu_3^\xi = \delta_{\{x=\sqrt{\xi}\}}$.

VI. Conclusions

In this paper we studied,within a unified framework based on measure theoretic concepts from the theory of Random Dynamical Systems, parametric and initial condition uncertainty for discrete-time systems that can exhibit complex time-dependent behavior. The notion of input measure of an observable was defined and its propagation to output measure of the observable studied by means of transfer operators. Uncertainty of these measures was defined in terms of their cumulative probability distributions. Effect of pitchfork bifurcation (and thus nonlinearity, for simple time dependence) on uncertainty was studied within this formalism. General results on uncertainty for dynamical systems of a Sinai-Bowen-Ruelle type were derived thus setting the stage for analysis of nonlinear systems with complex time-dependence.

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