Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Positive μ-modification for Stable Adaptation in the Presence of Input Constraints

Eugene Lavretsky and Naira Hovakimyan

***Abstract*— For a class of linear dynamical systems with unknown parameters a direct model reference adaptive control framework is developed that provides stable adaptation in the presence of input constraints. The proposed design methodology, termed "positive μ-modification", protects the control law from actuator position saturation. Moreover, the design is Lyapunov based and ensures global asymptotic tracking for open-loop stable systems. For unstable systems an estimate for the domain of attraction is derived based on the input saturation magnitude and system parameters. Simulation of a benchmark example verifies the theoretical statements.**

I. INTRODUCTION

During the past decade control design in the presence of input saturation has attracted a vast amount of research effort (for chronological bibliography see [1]). This issue is especially challenging in adaptive systems, because continued adaptation during input saturation may easily lead to instability. In order to overcome the effects of control saturation during adaptation, a modification to both the tracking error and the reference model dynamics was proposed by Monopoli in [2] but without any formal proof of stability. In [3], a rigorous proof of asymptotic stability has been laid out for model reference adaptive control framework, considering an adaptive modification of the reference model dynamics with a gain proportional to control deficiency. In the PCH method of Johnson and Calise a fixed gain adjustment (proportional to control deficiency) to the reference model was introduced [4]. Adaptive control with amplitude saturation was also addressed for linear systems in [5]–[7] and for affine-in-control nonlinear systems in [8].

In this paper, we propose a direct model reference adaptive control framework (MRAC) that yields stable adaptation in the presence of input constraints. The nouvelle design approach is termed "Positive μ-modification", or simply "μ-mod", and can be viewed as an extension of the results in [2], [3]. Compared to [2] and [3], the approach guarantees that the control *will never* incur saturation. Similar to [3], we show that for open-loop stable (unstable) systems global (local) stability results are attained. We also derive an upper bound for the corresponding closed-loop system stability domain and prove that it depends upon the system parameters and control saturation level.

Research of the second author is supported in part by AFOSR under Grant F49620-03-1-0443, and MURI subcontract F49620-03-1-0401.

Eugene Lavretsky is an Associate Technical Fellow of The Boeing Company - Phantom Works, Huntington Beach, CA 92647-2099, e-mail: eugene.lavretsky@boeing.com, Associate Fellow, AIAA

Naira Hovakimyan is an Associate Professor of Aerospace & Ocean Engineering, Virginia Polytechnic Institute & State University, Blacksburg, VA 24061-0203, e-mail: nhovakim@vt.edu, Senior Member IEEE, Corresponding author

This paper is organized as follows. In Section II we formulate the design problem for linear in parameters adaptive control with input saturation. Section III defines the proposed μ-modification to the adaptive signal and discusses some of its properties. In Section IV, reference model dynamics and the classical matching conditions are formulated. Stability properties of the μ-mod based adaptive control are analyzed in Section V. As an application example, in Section VI we discuss the results for uncertain linear scalar systems and give specific interpretations of the conditions of Theorem 5.1. In Section VII, a simulation of a benchmark example is presented that verifies the theoretical statements and the benefits of the "μ-mod" based adaptive control design process. Conclusions, recommendations, and future research directions are given at the end of the paper.

II. PROBLEM FORMULATION

Let the system dynamics propagate according to the following differential equation:

$$\dot{x}(t) = Ax(t) + b\lambda u(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, A is an unknown matrix, b is a known constant vector, λ is an unknown constant of known sign. Without loss of generality let $\lambda > 0$. The control input $u \in \mathbb{R}$ is amplitude limited and is calculated using the following static actuator model:

$$\begin{aligned} u(t) &= u_{\max}\,\text{sat}\left(\frac{u_c(t)}{u_{\max}}\right) \\ &= \begin{cases} u_c(t), & |u_c(t)| \le u_{\max} \\ u_{\max}\,\text{sgn}\left(u_c(t)\right), & |u_c(t)| > u_{\max} \end{cases} \end{aligned} \tag{2}$$

Here $u_c(t)$ is the commanded control input, while $u_{\max} > 0$ defines amplitude saturation level of the actuator. Rewrite the system dynamics in (1) in the form:

$$\dot{x}(t) = Ax(t) + b\lambda u_c(t) + b\lambda\Delta u(t) \tag{3}$$

where $\Delta u(t) = u(t) - u_c(t)$ denotes the *control deficiency* due to the actuator amplitude saturation constraint. Consider the following reference model dynamics, driven by a uniformly bounded reference input $\{r \in \mathbb{R} : |r(t)| \le r_{\max}\}$:

$$\dot{x}_m^*(t) = A_m x_m^*(t) + b_m r(t)\,, \tag{4}$$

In (4), $x_m^* \in \mathbb{R}^n$ is the state of the reference model, A_m is a Hurwitz matrix. The control design problem, addressed in this paper, can be stated as follows:

Given reference model (4), define an adaptive control signal $u_c(t)$ and, if necessary, augment the input $r(t)$ to the reference model, so that the state $x(t)$ of the system (1) in the presence of input constraint (2) tracks the state $x_m(t)$ of the augmented reference model asymptotically, while all the signals in both systems remain bounded.

III. Positive μ-modification and Closed Loop System Dynamics

The main challenge in designing an adaptive controller for the system in (1), (2) is associated with the control deficiency $\Delta u(t) = u(t) - u_c(t)$ that appears in (3). Using this signal, in [3] a modification to the reference model dynamics was suggested and the corresponding direct adaptive laws were formulated. Motivated by [3], we propose yet another control design modification that protects the adaptive input signal from position saturation. To this end, choose a constant $0 < \delta < u_{\max}$ and define $u^\delta_{\max} = u_{\max} - \delta$. Then the control deficiency can be represented as:

$$\Delta u(t) = \Delta u_c(t) + \Delta_{\mathrm{sat}}(t) \tag{5}$$

where

$$\Delta u_c(t) = u^\delta_{\max}\mathrm{sat}\left(\frac{u_c(t)}{u^\delta_{\max}}\right) - u_c(t) \tag{6}$$

$$\Delta_{\mathrm{sat}}(t) = u_{\max}\mathrm{sat}\left(\frac{u_c(t)}{u_{\max}}\right) - u^\delta_{\max}\mathrm{sat}\left(\frac{u_c(t)}{u^\delta_{\max}}\right) \tag{7}$$

Direct adaptive model reference control with μ-modification is defined as:

$$\begin{aligned} u_c(t) &= u_{lin}(t) + \mu\Delta u_c(t) \\ u_{lin}(t) &= k_x^T(t)x(t) + k_r(t)r(t) \end{aligned} \tag{8}$$

In (8), $u_{lin}(t)$ denotes the conventional linear in parameters adaptive control, $k_x(t) \in \mathbb{R}^n,\ k_r(t) \in \mathbb{R}$ are adaptive gains, and μ is the design constant. Note that the relation (8) defines the commanded control input $u_c(t)$ implicitly. Next we show that explicit solution of the latter can be found.

Lemma 3.1: If $\mu \geq 0$, then the solution to (8) is given by a *convex combination* of $u_{lin}(t)$ and $u^\delta_{\max}\ \mathrm{sat}\left(\frac{u_{lin}(t)}{u^\delta_{\max}}\right)\ \ \forall t > 0$:

$$\begin{aligned} u_c(t) &= \frac{1}{1+\mu}\left(u_{lin}(t) + \mu u^\delta_{\max}\ \mathrm{sat}\left(\frac{u_{lin}(t)}{u^\delta_{\max}}\right)\right) \\ &= \begin{cases} u_{lin}(t), & |u_{lin}(t)| \leq u^\delta_{\max} \\ \frac{1}{1+\mu}\left(u_{lin}(t) + \mu u^\delta_{\max}\right), & u_{lin}(t) > u^\delta_{\max} \\ \frac{1}{1+\mu}\left(u_{lin}(t) - \mu u^\delta_{\max}\right), & u_{lin}(t) < -u^\delta_{\max} \end{cases} \end{aligned} \tag{9}$$

Proof. If $|u_c(t)| \leq u^\delta_{\max}$, then $\Delta u_c(t) = 0$, and the first relationship in (9) immediately takes place. If $|u_c(t)| > u^\delta_{\max}$, then using (2) and (8), we get

$$u_c(t) = u_{lin}(t) + \mu\left(u^\delta_{\max}\ \mathrm{sgn}\left(u_c(t)\right) - u_c(t)\right) \tag{10}$$

or equivalently

$$\begin{aligned} u_c(t) &= \frac{1}{1+\mu}\left(u_{lin}(t) + \mu u^\delta_{\max}\ \mathrm{sgn}\left(u_c(t)\right)\right) \\ &= \begin{cases} \frac{1}{1+\mu}\left(u_{lin}(t) + \mu u^\delta_{\max}\right), & u_c > u^\delta_{\max} \\ \frac{1}{1+\mu}\left(u_{lin}(t) - \mu u^\delta_{\max}\right), & u_c < -u^\delta_{\max} \end{cases} \end{aligned} \tag{11}$$

It is easy to see that since $\mu \geq 0$, then the second and the third lines in the above relationship are equivalent to the corresponding ones in (9). The proof is complete.

Remark 3.1: The solution given by (9) is valid also for any $\mu \neq -1$, but in that case the *convexity* condition is violated. The significance of the latter will be apparent during the stability proof. Moreover, for $\delta = 0$ setting $\mu = 0$ recovers the adaptive architecture of [3]. On the other hand, if $u_{lin}(t)$ is uniformly bounded, then as μ tends to infinity, $\lim u_c(t) = u^\delta_{\max}\mathrm{sat}\left(\frac{u_{lin}(t)}{u^\delta_{\max}}\right)$. Consequently, setting $\delta = 0$ and $\mu = \infty$ results in $u_c(t) = u_{\max}\mathrm{sat}\left(\frac{u_{lin}(t)}{u_{\max}}\right)$. The latter yields exactly the same closed-loop dynamics as does the linear in parameters adaptive signal $u_{lin}(t)$ in [3].

Remark 3.2: From (9), it immediately follows that the commanded control signal is continuous in time, but not continuously differentiable. The use of the $\mathrm{sat}(\cdot)$ function in (9) implies that

$$\begin{aligned} \lim_{u_{lin}(t)\to u^\delta_{\max}-} \dot{u}_c(t) &= \dot{u}_{lin}(t)\,, \\ \lim_{u_{lin}(t)\to u^\delta_{\max}+} \dot{u}_c(t) &= \frac{1}{1+\mu}\dot{u}_{lin}(t) \end{aligned}$$

Another point of $\dot{u}_c(t)$ discontinuity exists at $u_c(t) = -u^\delta_{\max}$.

Remark 3.3: Solving (8) for $\Delta u_c(t)$ and substituting $u_c(t)$ from (9), one obtains:

$$\Delta u_c(t) = \frac{1}{1+\mu}\Delta u^\delta_{lin}(t) \tag{12}$$

where $\Delta u^\delta_{lin}(t) \triangleq u^\delta_{\max}\ \mathrm{sat}\left(\frac{u_{lin}(t)}{u^\delta_{\max}}\right) - u_{lin}(t)$. Consequently, if $u^\delta_{lin}(t)$ is bounded, then the control deficiency $\Delta u_c(t)$ is inversely proportional to μ: $\Delta u_c(t) = O(1/\mu)$.

Lemma 3.2: The following inequality is true for all $t > 0$:

$$u_c(t)\Delta u_c(t) \leq 0 \tag{13}$$

Proof. If $|u_c(t)| \leq u_{\max}$, then $\Delta u_c(t) = 0$, and (13) holds with the equality sign. If $|u_c(t)| > u_{\max}$, then using (2) and the definition for $\Delta u_c(t)$, we get

$$\begin{cases} u_c(t) > u^\delta_{\max} \ \Leftrightarrow\ \Delta u_c(t) = u^\delta_{\max} - u_c(t) < 0 \\ u_c(t) < -u^\delta_{\max} \ \Leftrightarrow\ \Delta u_c(t) = -u^\delta_{\max} - u_c(t) > 0 \end{cases} \tag{14}$$

which implies (13). The proof is complete.

Substituting (5) and (8) into (3), yields the following closed-loop system dynamics:

$$\dot{x}(t) = (A + b\lambda k_x^T(t))x(t) + b\lambda k_r(t)r(t) + b\lambda\Delta u_{lin}(t) \tag{15}$$

where

$$\Delta u_{lin}(t) \triangleq \Delta u^{\delta}_{lin}(t) + \Delta_{\rm sat}(t)$$
$$= u_{\max}{\rm sat}\left(\frac{u_c(t)}{u_{\max}}\right) - u_{lin}(t) \qquad (16)$$

defines the deficiency of the linear in parameters adaptive signal $u_{lin}(t)$.

IV. Adaptive Reference Model

The system dynamics in (15) leads to consideration of the following *adaptive* reference model dynamics:

$$\dot{x}_m(t) = A_m x_m(t) + b_m(r(t) + k_u(t)\Delta u_{lin}(t)) \qquad (17)$$

where $x_m \in \mathbb{R}^n$ is the state of the reference model, A_m is Hurwitz, $k_u(t)$ is an adaptive gain to be determined through stability proof. Comparing (17) with the system dynamics in (15), assumptions are formulated that guarantee existence of the adaptive control signal with μ-modification in (8).

Assumption 4.1: (Reference model matching conditions)

$$\exists\ k_x^*, k_r^*, k_u^*,\ b\lambda(k_x^*)^T = A_m - A, \qquad (18)$$
$$b\lambda k_r^* = b_m,\ b_m k_u^* = b\lambda$$

Remark 4.1: The true knowledge of the gains k_x^*, k_r^*, k_u^* is not required, only their existence is assumed. The second and the third matching conditions in (18) imply that $k_r^* k_u^* = 1$.

V. Error Dynamics and Stability Analysis

Let $e(t) = x(t) - x_m(t)$ be the tracking error signal. Then the tracking error dynamics can be written:

$$\begin{aligned}\dot{e}(t) &= A_m e(t) + b\lambda\left(\Delta k_x^T(t)x(t) + \Delta k_r(t)r(t)\right)\\ &- b_m\Delta k_u(t)\Delta u_{lin}(t) \qquad (19)\end{aligned}$$

where $\Delta k_x(t) = k_x(t) - k_x^*$, $\Delta k_r(t) = k_r(t) - k_r^*$, $\Delta k_u(t) = k_u(t) - k_u^*$ denote parameter errors. Consider the following adaptation laws:

$$\begin{aligned}\dot{k}_x(t) &= -\Gamma_x x(t)e^T(t)Pb\\ \dot{k}_r(t) &= -\gamma_r r(t)e^T(t)Pb\\ \dot{k}_u(t) &= \gamma_u\Delta u_{lin}(t)e^T(t)Pb_m \qquad (20)\end{aligned}$$

where $\Gamma_x = \Gamma_x^T > 0$, $\gamma_r > 0, \gamma_u > 0$ are corresponding rates of adaptation. In order to assess the closed-loop system stability, define the following Lyapunov function candidate:

$$V = e^T(t)Pe(t) + \lambda\Big(\Delta k_x^T(t)\Gamma_x^{-1}\Delta k_x(t) \qquad (21)$$
$$+\gamma_r^{-1}(\Delta k_r(t))^2\Big) + \gamma_u^{-1}(\Delta k_u(t))^2$$

where $P = P^T > 0$ solves the algebraic Lyapunov equation

$$A_m^T P + PA_m = -Q \qquad (22)$$

for arbitrary $Q > 0$. The time derivative of the candidate Lyapunov function in (21) along the system trajectories (19), (20) is:

$$\dot{V}(t) = -e^T(t)Qe(t) \le 0 \qquad (23)$$

Hence the equilibrium of (19), (20) is Lyapunov stable, i.e. the signals $e(t)$, $\Delta k_x(t)$, $\Delta k_r(t)$, $\Delta k_u(t)$ are bounded. Consequently, there exist $\Delta k_x^{\max}$, $\Delta k_r^{\max}$, such that $\|\Delta k_x(t)\| < \Delta k_x^{\max}$, $|\Delta k_r(t)| < \Delta k_r^{\max} = \alpha\Delta k_x^{\max}$, $\forall t > 0$, where $\alpha = \sqrt{\gamma_r/\lambda_{\min}(\Gamma_x)}$.

For the statement of our main result introduce the following notations: $\rho = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$, $\kappa = \Big|\lambda_{\min}(Q) - 2\lambda\|Pb\|\|k_x^*\|\Big|$, $\varrho = \frac{r_{\max}}{u_{\max}}$.

Theorem 5.1: For A and b in (1), $u_{\max}$ in (2), k_x^*, k_r^* in (18) and P and Q in (22), assume that the maximum amplitude of the reference signal $r_{\max}$ is chosen such that:

$$r_{\max} < \frac{\lambda_{\min}(Q)}{|k_r^*|\kappa\rho}u_{\max}\,, \qquad (24)$$

and for arbitrary $\delta > 0$ the design parameter μ is selected to satisfy the lower bound:

$$\begin{aligned}\mu &> \frac{(\kappa + 2\lambda\|Pb\|(\Delta k_x^{\max} + ||k_x^*||))\,u_{\max}}{\kappa\delta}\\ &+ \frac{(\Delta k_r^{\max} + |k_r^*|)\kappa\ r_{\max}}{\kappa\delta} - 2 \qquad (25)\end{aligned}$$

If the system initial condition and the initial value of the candidate Lyapunov function in (21) satisfy:

$$x^T(0)Px(0) < \lambda_{\min}(P)\left[\frac{2\lambda\|Pb\|}{\kappa}u_{\max}\right]^2 \qquad (26)$$

$$\sqrt{V(0)} < \sqrt{\frac{\lambda}{\lambda_{\max}(\Gamma_x)}}\left(\frac{\lambda_{\min}(Q) - |k_r^*|\ \kappa\rho\varrho}{2\lambda\|Pb\| + \alpha\ \kappa\rho\varrho}\right) \qquad (27)$$

then

- the adaptive system in (19), (20) has bounded solutions $\forall r(t),\ |r(t)| \le r_{\max}$,
- the tracking error $e(t)$ goes to zero asymptotically, while
$$x^T(t)Px(t) < \lambda_{\min}(P)\left[\frac{2\lambda\|Pb\|}{\kappa}u_{\max}\right]^2,\ \forall t > 0\,,$$
- $|u_c(t)| \le u_{\max}$, i.e. the position saturation of the commanded control signal $u_c(t)$ is overly prevented for all $t > 0$.

Proof. If $\Delta u(t) = 0$, then the adaptive reference model dynamics in (17) reduces to the one in (4), leading to the following form of the error dynamics in (19):

$$\dot{e}(t) = A_m e(t) + b\lambda\left(\Delta k_x^T(t)x(t) + \Delta k_r(t)r(t)\right) \qquad (28)$$

Since (4) defines a stable reference model, then $x_m^*(t)$ is bounded, which together with (23), and using Barbalat's lemma, leads to asymptotic convergence of the tracking error $e(t)$ to zero.

If $\Delta u(t) \ne 0$, then in order to prove asymptotic convergence of the tracking error to zero, one needs to show additionally boundedness of at least one of the two states: either $x_m(t)$ or $x(t)$. To this end, suppose that A is Hurwitz matrix and consider $W(x) = x^T(t)P_A x(t)$ as a candidate Lyapunov function for the system dynamics, where $P_A =$

$P_A^T > 0$ solves the algebraic Lyapunov equation (22) for some positive definite $Q > 0$. Since $\Delta u(t) \neq 0$, then $|u_c(t)| > u_{\max}$, $u(t) = u_{\max}\text{sgn}(u_c(t))$, and the system dynamics in (1) becomes:

$$\dot{x}(t) = Ax(t) + b\lambda u_{\max}\text{sgn}(u_c(t)) \quad (29)$$

Consequently

$$\begin{aligned}\dot{W} &= -x^T(t)Qx(t) + 2x^T(t)P_A b\lambda u_{\max}\text{sgn}(u_c(t)) \\ &\leq -\lambda_{\min}(Q)\|x(t)\|^2 + 2u_{\max}\lambda\|x(t)\|\|P_A b\| \quad (30)\end{aligned}$$

where $\lambda_{\min}(Q)$ is the minimum eigenvalue of Q. For asymptotically stable systems it immediately follows that $\dot{W} < 0$ if $\|x\| > 2\lambda u_{\max}\|P_A b\|/\lambda_{\min}(Q)$. Therefore the system states remain bounded, and the adaptive laws in (20) ensure global asymptotic stability of the error dynamics in (19).

For unstable systems, i.e. when A is not Hurwitz, add and subtract $b\lambda(k_x^*)^T x(t)$, to write the system dynamics in the following form:

$$\dot{x}(t) = A_m x(t) - b\lambda(k_x^*)^T x(t) + b\lambda u_{\max}\text{sgn}(u_c(t)) \quad (31)$$

and consider the following Lyapunov function candidate $W(x(t)) = x^T(t)Px(t)$, where $P = P^T > 0$ solves the algebraic Lyapunov equation (22) for some positive definite $Q > 0$. Then $\dot{W}(x(t))$ can be presented as:

$$\begin{aligned}\dot{W}(x(t)) &= -x^T(t)Qx(t) - 2x^T(t)Pb\lambda(k_x^*)^T x(t) \\ &+2u_{\max}\lambda|x^T(t)Pb|\text{sgn}(u_c(t))\text{sgn}(x^T(t)Pb) \quad (32)\end{aligned}$$

Consider two possibilities:

1) $\text{sgn}(u_c(t)) = -\text{sgn}(x^T(t)Pb)$.
2) $\text{sgn}(u_c(t)) = \text{sgn}(x^T(t)Pb)$.

If $\text{sgn}(u_c(t)) = -\text{sgn}(x^T(t)Pb)$, then it follows from (32) that

$$\begin{aligned}\dot{W} &\leq -\lambda_{\min}(Q)\|x(t)\|^2 \\ &+ 2\lambda|x^T(t)Pb|\left(\|k_x^*\|\|x(t)\| - u_{\max}\right) \quad (33)\end{aligned}$$

If $\|k_x^*\|\|x(t)\| - u_{\max} < 0$, which is equivalent to $\|x(t)\| < \frac{u_{\max}}{\|k_x^*\|}$, we have $\dot{W} < 0$. If $\|k_x^*\|\|x(t)\| - u_{\max} > 0$, which is equivalent to $\|x(t)\| > \frac{u_{\max}}{\|k_x^*\|}$, the expression in (33) implies $\dot{W} \leq \kappa\|x(t)\|^2 - 2\lambda u_{\max}\|x(t)\|\|Pb\|$, where κ has been defined before the statement of the theorem. Therefore $\dot{W}(x(t)) < 0$, if $x \in \Omega_1 \triangleq \left\{x \mid \|x\| < \frac{2u_{\max}\lambda\|Pb\|}{\kappa}\right\}$. Consider the largest set $\mathcal{B}_1$, enclosed in Ω_1, whose boundary forms a level set of the function $W(x(t))$:

$$\mathcal{B}_1 = \left\{x \mid W(x) \leq \lambda_{\min}(P)\left[\frac{2u_{\max}\lambda\|Pb\|}{\kappa}\right]^2\right\} \quad (34)$$

It is obvious that for all initial conditions of $x(t)$ from the set $\mathcal{B}_1$ we have $\dot{W}(x(t)) < 0$, implying that the system states remain bounded.

In the second case, i.e. when $\text{sgn}(u_c(t)) = \text{sgn}(x^T(t)Pb)$, it follows from (32) that

$$\begin{aligned}\dot{W} &\leq -x^T(t)Qx(t) - 2x^T(t)Pb\lambda(k_x^*)^T x(t) \\ &+ 2\lambda|u_c(t)||x^T(t)Pb| = -x^T(t)Qx(t) \\ &- 2x^T(t)Pb\lambda(k_x^*)^T x(t) + 2u_c(t)x^T(t)Pb\lambda\end{aligned}$$

Recalling that $k_x(t) = k_x^* + \Delta k_x(t)$, $k_r(t) = k_r^* + \Delta k_r(t)$, $\text{sgn}(u_c(t)) = -\text{sgn}(\Delta u_c(t))$, substituting $u_c(t)$ from (8), and following the requirement on the sign of $\mu > 0$, stated in Lemma 3.1, we arrive at:

$$\begin{aligned}\dot{W} &\leq -\left(\lambda_{\min}(Q) - 2\lambda\Delta k_x^{\max}\|Pb\|\right)\|x(t)\|^2 \\ &+2\lambda\|Pb\|\|x(t)\|(|k_r^*| + \Delta k_r^{\max})r_{\max} \quad (35)\end{aligned}$$

Notice that since $V(e, \Delta k_x, \Delta k_r, \Delta k_u)$ is radially unbounded, and its derivative is negative $\dot{V}(t) < 0$, then the maximal values of all errors, including $\Delta k_x^{\max}, \Delta k_r^{\max}$, do not exceed the level set value of the Lyapunov function $V = V_0 = V(0)$. Therefore the assumed inequality (27) implies that

$$\Delta k_x^{\max} < \frac{\lambda_{\min}(Q) - |k_r^*|\kappa\rho\varrho}{2\lambda\|Pb\| + \alpha\ \kappa\rho\varrho} \quad (36)$$

This in turn guarantees that $\lambda_{\min}(Q) - 2\lambda\Delta k_x^{\max}\|Pb\| > 0$. Consequently, it follows from (35) that $\dot{W}(x(t)) < 0$, if

$$\|x\| > \frac{2\lambda\|Pb\|(|k_r^*| + \Delta k_r^{\max})r_{\max}}{\lambda_{\min}(Q) - 2\lambda\Delta k_x^{\max}\|Pb\|}$$

Define the ball $\Omega_2 = \left\{x \mid \|x\| = \frac{2\lambda\|Pb\|(|k_r^*| + \Delta k_r^{\max})r_{\max}}{\lambda_{\min}(Q) - 2\lambda\Delta k_x^{\max}\|Pb\|}\right\}$, and the smallest set $\mathcal{B}_2$ that encloses Ω_2, the boundary of which is a level set of the Lyapunov function $W(x(t))$:

$$\mathcal{B}_2 = \left\{x | W \leq \lambda_{\max}(P)\left[\frac{2\lambda\|Pb\|(|k_r^*| + \Delta k_r^{\max})r_{\max}}{\lambda_{\min}(Q) - 2\lambda\Delta k_x^{\max}\|Pb\|}\right]^2\right\}$$

By rearranging the terms in (36), it follows that

$$\sqrt{\lambda_{\max}(P)}\ \frac{(|k_r^*| + \Delta k_r^{\max})r_{\max}}{\lambda_{\min}(Q) - 2\lambda\Delta k_x^{\max}\|Pb\|} < \sqrt{\lambda_{\min}(P)}\frac{u_{\max}}{\kappa}$$

and consequently $\mathcal{B}_2 \subset \mathcal{B}_1$, implying that the annulus region $\mathcal{B}_1 \setminus \mathcal{B}_2 \neq \emptyset$. Thus, our analysis of the closed-loop system dynamics reveals that when $\Delta u(t) \neq 0$, there always exists a *non-empty* annulus region

$$\frac{2\lambda\|Pb\|(|k_r^*| + \Delta k_r^{\max})r_{\max}}{\lambda_{\min}(Q) - 2\lambda\Delta k_x^{\max}\|Pb\|} < \|x\| < \frac{2u_{\max}\lambda\|Pb\|}{\kappa} \quad (37)$$

such that $\dot{W}(x(t)) < 0$ holds $\forall x$ satisfying (37). In other words, asymptotic convergence of the tracking error to zero and boundedness of all the signals are guaranteed as long as the system initial conditions satisfy (26) and the initial parameter errors comply with (27).

It remains only to show that the control signal will never incur saturation. To this end, notice that from (12) it follows that $\Delta u_c(t)$ can be upper bounded

$$\begin{aligned}|\Delta u_c(t)| &\leq \frac{|\Delta u_{lin}^\delta(t)|}{1+\mu} \\ &\leq \frac{u_{\max}^\delta + (\Delta k_x^{\max} + ||k_x^*||)||x|| + (\Delta k_r^{\max} + |k_r^*|)r_{\max}}{1+\mu}\end{aligned}$$

By definition $\Delta u_c(t) = u_{\max}^{\delta}\text{sat}\left(\frac{u_c(t)}{u_{\max}^{\delta}}\right) - u_c(t)$. Hence $|\Delta u_c(t)| \geq |u_c(t)| - \left|u_{\max}^{\delta}\text{sat}\left(\frac{u_c(t)}{u_{\max}^{\delta}}\right)\right|$, and consequently $|u_c(t)| \leq u_{\max}^{\delta} + \frac{C}{1+\mu}$, where $C = u_{\max}^{\delta} + \frac{2\lambda\|Pb\|}{\kappa}(\Delta k_x^{\max} + ||k_x^*||)u_{\max} + (\Delta k_r^{\max} + |k_r^*|)r_{\max}$. For a given $\delta > 0$, if one chooses $\mu > 0$ to satisfy $C/(1+\mu) < \delta$, then $|u_c(t)| < u_{\max}$. Recalling that $u_{\max}^{\delta} = u_{\max} - \delta$, one arrives at (25).

Remark 5.1: Using explicit definition of $u_c(t)$ in (9), one can compute its time derivative for $|u_c(t)| > u_{\max}^{\delta}$:

$$\dot{u}_c(t)\Big|_{|u_c(t)|>u_{\max}^{\delta}} = \frac{\dot{u}_{lin}(t)}{1+\mu} \tag{38}$$

Since $\dot{u}_c(t)$ is of order $1/(1+\mu)$, then the actuator limits can be enforced through the choice of μ, by controlling the time derivative $\dot{u}_c(t)$.

Remark 5.2: Notice that if one chooses $\delta \leq \frac{u_{\max}}{2}$, then from (2) one can deduce a less conservative lower bound for μ than the one in (25). Indeed, it is easy to verify that if $\mu \geq \frac{2\lambda\|Pb\|(\Delta k_x^{\max}+||k_x^*||)u_{\max}+(\Delta k_r^{\max}+|k_r^*|)\kappa\ r_{\max}-\kappa\ u_{\max}}{\kappa\delta}$, then $0 < |u_c(t)| \leq u_{\max}$ for all $t > 0$.

Remark 5.3: Inequality in (24) ensures that the numerator in (27) is positive.

Remark 5.4: Theorem 5.1 implies that if the initial conditions of the state and parameter errors lie within certain bounds, then the adaptive system will have bounded solutions. The local nature of the result for *unstable* systems is due to the static actuator model constraints (2) imposed on the control input. For open-loop stable systems the results are global.

Remark 5.5: The condition in (27) can be viewed as an upper bound for α, which limits the choice of the adaptation gains Γ_x and γ_r.

Remark 5.6: Remark 3.3 allows to estimate the change between the adaptive reference model (17) and the ideal one (4). Let $\Delta e(t) = x(t) - x_m^*(t)$ define the difference between the system state and the ideal reference model in (4), and let $e^*(t) = x_m^*(t) - x_m(t)$. Then $\dot{e}^*(t) = A_m e^*(t) - b_m k_u(t)\Delta u_{lin}(t)$. Notice that $\Delta e(t)$ can be presented as: $\Delta e(t) = e(t) - e^*(t)$. Consequently

$$\begin{aligned}\Delta\dot{e}(t) &= A_m e(t) + b\lambda\left(\Delta k_x^T(t)x(t) + \Delta k_r(t)r(t)\right)\\ &-b_m\Delta k_u(t)\Delta u_{lin}(t) - A_m e^*(t) + b_m k_u(t)\Delta u_{lin}(t)\end{aligned}$$

From Theorem 6.1 it follows that the tracking error $e(t)$ goes to zero asymptotically. From Barbalat's lemma it follows that $\dot{e}(t) \to 0$ as $t \to \infty$. This consequently implies that $b\lambda\left(\Delta k_x^T(t)x(t) + \Delta k_r(t)r(t)\right) - b_m\Delta k_u(t)\Delta u_{lin}(t) \to 0$ as $t \to \infty$. Therefore

$$\Delta\dot{e}(t) = A_m\Delta e(t) + \varepsilon(t) + b_m k_u(t)\Delta u_{lin}(t) \tag{39}$$

In (39), $\varepsilon(t) = \text{o}(1)$ as $t \to \infty$. Hence, there exists $\varepsilon^{\max} \geq 0$, such that $\max_{0\leq\tau\leq t}\|\varepsilon(\tau)\| \leq \varepsilon^{\max}$, $\forall t \geq t_0$. The solution of (39) is given by:

$$\begin{aligned}\Delta e(t) &= \exp(A_m t)\Delta e(0)\\ &+ \int_0^t \exp(A_m(t-\tau))\Big(\varepsilon(\tau) + b_m k_u(t)\Delta u_{lin}(t)\Big)d\tau\end{aligned}$$

From (16), notice that

$$\begin{aligned}|\Delta u_{lin}(t)| &= |\Delta u_{lin}^{\delta}(t) + \Delta_{\text{sat}}(t)|\\ &= \Big|\frac{1}{1+\mu}\Big(\underbrace{u_{\max}^{\delta}\text{sat}\left(\frac{u_c(t)}{u_{\max}^{\delta}}\right) - u_{\max}\text{sat}\left(\frac{u_c(t)}{u_{\max}}\right)}_{-\Delta_{\text{sat}}(t)}\\ &+ \underbrace{u_{\max}\text{sat}\left(\frac{u_c(t)}{u_{\max}}\right) - u_c(t)}_{0}\Big) + \Delta_{\text{sat}}(t)\Big|\\ &= |-\mu\Delta_{\text{sat}}(t)| \leq \mu\delta\end{aligned} \tag{40}$$

Let ω denote any negative number which is larger than the maximum of the real parts of the eigenvalues of the Hurwitz matrix A_m. Then the following upper bound can be derived:

$$\begin{aligned}\|\Delta e(t)\| &\leq \|\Delta e(0)\|\exp(-|\omega|t) + [1-\exp(-|\omega|t)]\\ &\times\tfrac{\|b_m\|(|k_u^*|+\Delta k_u^{\max})\mu\delta+\varepsilon^{\max}}{|\omega|} \leq \exp(-|\omega|t)\Big[\|\Delta e(0)\|\\ &-\tfrac{\|b_m\|(|k_u^*|+\Delta k_u^{\max})\mu\delta+\varepsilon^{\max}}{|\omega|}\Big] + \tfrac{\|b_m\|(|k_u^*|+\Delta k_u^{\max})\mu\delta+\varepsilon^{\max}}{|\omega|}\end{aligned}$$

As $t \to \infty$, the first term in this upper bound diminishes, and therefore $\|\Delta e(t)\| \leq \text{o}(1) + \frac{\|b_m\|(|k_u^*|+\Delta k_u^{\max})\mu\delta+\varepsilon^{\max}}{|\omega|}$.

VI. μ-MOD BASED ADAPTATION FOR SCALAR SYSTEMS

In order to get further insight into the proposed μ-mod based adaptive control design and the Theorem 5.1 conditions, we consider scalar linear systems of the form:

$$\dot{x}(t) = ax(t) + bu(t) \tag{41}$$

where $x \in \mathbb{R}$ is the state of the system, a and b are unknown constants, $\text{sgn}(b)$ is known, and $u \in \mathbb{R}$ is the control input, subject to the static actuator constraint (2). The μ-mod based adaptive signal is defined similar to (8). At the same time, the adaptive reference model dynamics is formed: $\dot{x}_m(t) = a_m x_m(t) + b_m(r(t) + k_u(t)\Delta u_{lin}(t)),\ a_m < 0$. Similar to (20), the following adaptation laws are chosen: $\dot{k}_x(t) = -\gamma_x x(t)e(t)\text{sgn}(b)$, $\dot{k}_r(t) = -\gamma_r r(t)e(t)\text{sgn}(b)$, $\dot{k}_u(t) = \gamma_u\Delta u_{lin}(t)e(t)b_m$. Define the error signal $e(t) = x(t) - x_m(t)$ and consider the following Lyapunov function:

$$\begin{aligned}&V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta k_u(t))\\ &= e^2(t) + \left(\gamma_x^{-1}\Delta k_x^2(t) + \gamma_r^{-1}\Delta k_r^2(t)\right)|b| + \gamma_u^{-1}\Delta k_u^2(t)\end{aligned} \tag{42}$$

The sufficient conditions (24), (26), and (27) in Theorem 5.1 can now be formulated for scalar linear systems.

Theorem 6.1: For a and b in (41), and $u_{\max}$ in (2), let $r_{\max}$ be chosen such that $r_{\max} < \dfrac{|a_m|}{|b_m|}\dfrac{|b|}{|a|}u_{\max}$ and the design parameter μ be selected to satisfy the lower bound: $\mu > \frac{\left(1+(\Delta k_x^{\max}+||k_x^*||)\frac{|b|}{|a|}\right)u_{\max}+(\Delta k_r^{\max}+|k_r^*|)\ r_{\max}}{\delta} - 2$. If the system initial condition and Lyapunov function in (42) satisfy:

$$|x(0)| < \frac{|b|}{|a|}u_{\max},\ \sqrt{V(0)} < \sqrt{\frac{|b|}{\gamma_x}}\left(\frac{|a_m| - |k_r^*||a|\frac{r_{\max}}{u_{\max}}}{\alpha|a|\frac{r_{\max}}{u_{\max}} + |b|}\right)$$

where $\alpha = \gamma_r/\gamma_x$, then the adaptive system has bounded solutions $\forall r,\ |r(t)| \leq r_{\max}$, and the tracking error $e(t)$ goes to zero asymptotically.

For the scalar systems, the annulus region (37) becomes:

$$\left(\frac{(\Delta k_r^{\max} + |k_r^*|)\, r_{\max}}{|a_m| - |b|\Delta k_x^{\max}}\right)|b| \leq |x| \leq \frac{u_{\max}|b|}{|a|}$$

VII. Simulations

In this section, the proposed μ-mod based design methodology is demonstrated using the first order system $\dot{x} = 0.5x + 2u$, subject to the following actuator constraint $u_{\max} = 0.47$. A positive constant δ is set to 20% of the actuator position limit, that is: $\delta = 0.2u_{\max}$. The reference model without μ-modification is given as $\dot{x}_m = -6x_m + 6r$ along with the reference input $r = 0.7(\sin(2t) + \sin(0.4t))$. Both, the system and the reference model are initialized at zero. The adaptation rates are selected as $\gamma_x = 1, \gamma_r = 1, \gamma_u = 1$. Figs. 1(a)-1(d) demonstrate the closed-loop tracking performance for various values of μ. As expected, the plots indicate that large values of μ result in large changes to the reference model dynamics. At the same time, Figs. 2(a)-2(d) demonstrate that by choosing μ large enough the control deficiency is reduced, thus completely avoiding control saturation phenomenon. Consequently, the design constant μ can be viewed as a tuning "knob" that allows for a trade-off between the adaptive changes to the adaptive reference model and a protection against saturating actuator position which is required for tracking the model.

VIII. Conclusions

A direct adaptive model reference control design methodology is developed for uncertain linear systems in the presence of input constraints and matched uncertainties. The novel term used in the direct adaptive control architecture is termed μ-modification. It ensures asymptotic convergence of the tracking error to zero. For stable systems the obtained results are global, while for unstable systems an estimate of the domain of attraction is derived. The output feedback results can be obtained following the approach of [3]. Extension to nonlinear systems can be done following the lines of [8] for formulating the error dynamics. Current efforts are directed towards extension of the results to multivariable systems with magnitude and rate constrained control input.

References

[1] D. Bernstein and A. A. Michel. Chronological bibliography on saturating actuators. *International Journal of robust and nonlinear control*, 5:375–380, 1995.

[2] R. Monopoli. Adaptive control for systems with hard saturation. *Conference on Decision and Control*, 1975.

[3] S.P. Karason and A.M. Annaswamy. Adaptive control in the presence of input constraints. *IEEE Transactions on Automatic Control*, 39:2325–2330, 1994.

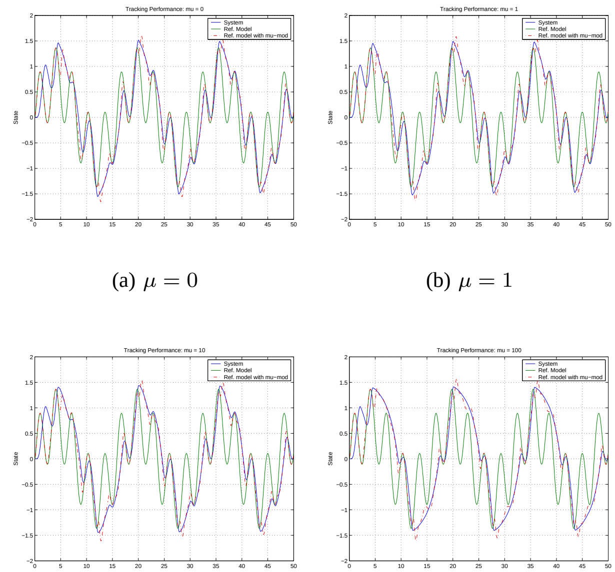
[4] E. N. Johnson and A. J. Calise. Limited authority adaptive flight control for reusable lunch vehicles. *Journal of Guidance, Control and Dynamics*, 26:906–913, 2003.

[5] D.Y. Abramovitch, R.L. Kosut, and G.F. Franklin. Adaptive control with saturating inputs. *Proc. of IEEE Conf. on Decision Control*, pages 848–852, 1986.

[6] A.N. Payne. Adaptive one-step-ahead control subject to input-amplitude constraint. *Int. Journal of Control*, 43:1257–1269, 1986.

[7] C. Zhang and R.J. Evans. Amplitude constrained adaptive control. *Int. Journal of Control*, 46:53–64, 1987.

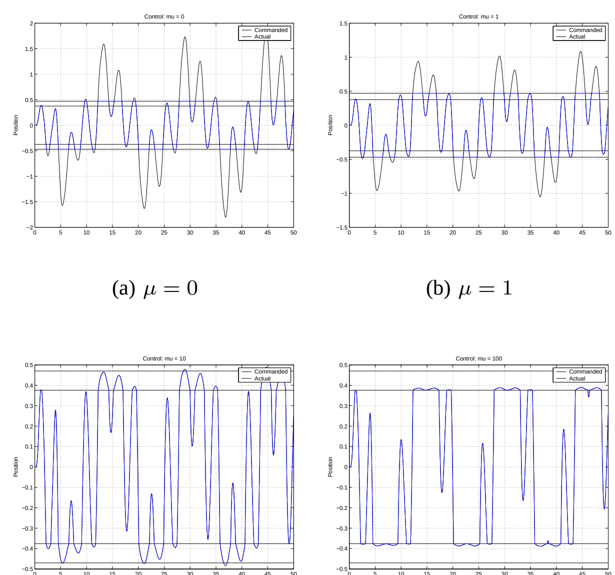
[8] A. Leonessa, W. Haddad, and T. Hayakawa. Adaptive tracking for nonlinear systems with input constraints. *American Control Conference*, 2001.



(a) $\mu = 0$ (b) $\mu = 1$

(c) $\mu = 10$ (d) $\mu = 100$

Fig. 1. Tracking performance for various values of μ.



(a) $\mu = 0$ (b) $\mu = 1$

(c) $\mu = 10$ (d) $\mu = 100$

Fig. 2. Comparison of the control deficiency for various values of μ.