Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Probabilistic Controllability Analysis of Sampled-Data/Discrete-Time Piecewise Affine Systems

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***Abstract*— This paper proposes a new approach to solve the controllability (reachability) problem of the sampled-data/discrete-time piecewise affine systems. First, an algebraic characterization for the system to be controllable/reachable is derived. Next, based on this characterization, an approach to determine if the system is controllable/reachable in a probabilistic sense is proposed based on a randomized algorithm. Finally, it is shown by numerical examples that the proposed approach is useful.**

I. INTRODUCTION

In the research field of hybrid systems, the controllability/reachability analysis is one of the important topics. In particular, for the hybrid systems with the autonomous switching, e.g., piecewise affine (PWA) systems and mixed logical dynamical (MLD) systems, only a few analytical results have been obtained so far, and the problem has been negatively solved in the sense that it is undecidable [1]. In addition, it has been shown in [2] that the PWA system is not always controllable even if the subsystem in every mode is controllable in the usual sense. In this way, the controllability analysis for the hybrid systems with the autonomous switching is a very complex issue and one of the challenging research topics.

In spite of these theoretical limitations, Bemporad *et al.* have discussed the controllability problem of the discrete-time PWA/MLD systems by specifying in advance the control time period, where the problem is reduced into the verification problem [3]. However, this approach involves the hardness of the combinatorial problem and the computation on polyhedra. As the control time period is taken larger, the computation amount becomes exponentially larger. In addition, this dose not expose any algebraical structure of the controllability properties.

On the other hand, one of the authors has proposed a new model of continuous-time hybrid systems called the sampled-data PWA systems, where the switching action of the discrete state is determined depending upon if some condition on the continuous and/or discrete state holds or not at each sampling time fixed in the digital device [4]. Furthermore, the authors have provided a (necessary and) sufficient condition for such a system to be controllable [5]. However, for the multi-modal case, the class of the systems to which the sufficient condition can be applied is limited.

This paper proposes a new approach to the controllability/reachability analysis for both the sampled-data PWA systems and the discrete-time PWA systems. First, a necessary and sufficient condition for the system to be controllable/reachable is derived by characterizing the set of the initial/final state from/to which there exists a control input driving to/from the origin with the control time period fixed. This condition provides the geometrical structure of the controllability/reachability spaces, from which it turns out that large computation amount is required to check this condition in a deterministic way. Motivated by this discussion, we next propose a probabilistic method to determine if the system is controllable/reachable with arbitrarily specified accuracy, where a randomized algorithm is not only applied to check if the obtained necessary and sufficient condition for the controllability/reachability holds or not, but also several techniques for determining it in an efficient way are developed. It is stressed here that *no* probabilistic approach to the controllability/reachability problem of hybrid systems has been derived. Finally, it is shown that for some examples for which it may be hopeless to check the controllability in a deterministic way, the proposed method can solve their controllability problems in a probabilistic sense within a practically short time.

In the sequel, we will use the following notation: $\mathcal{R}$, $\mathcal{N}$, $\mathcal{N}_+$, and $\mathcal{PC}$ denote the real number field, the set of nonnegative integers, the set of positive integers, and the set of all piecewise continuous functions, respectively. Let the vector inequality $x_1 \leq (<)\, x_2$ express that each element of the vector $x_1 - x_2$ is nonpositive (negative) and let $x_{(i)}$ denote the i-th element of the vector x. In addition, $0_{n\times m}$ and $\boldsymbol{I}_n$ express the $n \times m$ zero matrix and the $n \times n$ identity matrix, respectively, and, for simplicity of notation, we sometimes use the symbol 0 instead of $0_{n\times m}$, and the symbol $\boldsymbol{I}$ instead of $\boldsymbol{I}_n$. Finally, the set $\mathcal{S}$ given as the form $\mathcal{S} := \{x \in \mathcal{R}^n |\ Ax + b \leq 0, Cx + d < 0\}$ is called here the polyhedron where A, b, C, and d are some matrices.

II. SAMPLED-DATA/DISCRETE-TIME PWA SYSTEMS

The hybrid system in general involves two kinds of discrete events; the discontinuous phenomena of physical systems (*physical discrete event*) such as the collision of a mass to a wall and the logic designed artificially (*logical discrete event*) such as emergency measures. Contrary to the physical discrete events, the logical event is mostly embedded in the digital device, which means that the switching action of the discrete state is determined at each switching time fixed in the digital device. Taking this fact into account, in this paper, we focus on the class of the PWA systems with the logical events as shown in Fig. 1, and discuss the controllability/reachability properties of this

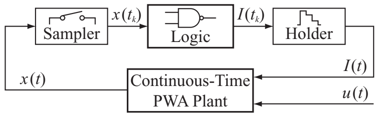


Fig. 1. Hybrid system with logic.

system from the viewpoints of both the continuous-time system (sampled-data PWA system) and the discrete-time system (discrete-time PWA system).

Let us consider the sampled-data PWA system Σ_{sd} described by

$$\begin{aligned} &\dot{x}(t) = A_{I(t)}x(t) + B_{I(t)}u(t) + a_{I(t)}, \\ &I(t) = I(t_k), \qquad \forall t \in [t_k, t_{k+1}), \\ &\begin{cases} x(t_{k+1}) = \phi(h, I(t_k), x(t_k)), \\ I(t_{k+1}) = I_+ \quad \text{if} \quad x(t_{k+1}) \in \mathcal{S}_{I_+} \end{cases} \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$ is the continuous state, $I \in \mathcal{M}(= \{0, 1, \dots, M-1\})$ is the discrete state (or mode), $M \in \mathcal{N}_+$ is the number of discrete state, $u \in \mathcal{R}^m$ is the control input, $h \in \mathcal{R}$ is the switching period, $t_k = kh$ $(k \in \mathcal{N})$ is the switching time, $I_+ \in \mathcal{M}$ is the new value of the discrete state at the switching time, $A_I \in \mathcal{R}^{n\times n}$, $B_I \in \mathcal{R}^{n\times m}$, and $a_I \in \mathcal{R}^n$ are constant matrices for mode I. We call $(I, x) \in \mathcal{M} \times \mathcal{R}^n$ the hybrid state (or simply the state) of Σ_{sd}. In addition, $\phi(h, I(t_k), x(t_k))$ denotes the solution $x(t_k+h)$ at time t_k+h of $\dot{x}(t) = A_{I(t_k)}x(t)+B_{I(t_k)}u(t)+a_{I(t_k)}$ with the initial state $x(t_k)$, and the subregion of the continuous state assigned to I is given by the polyhedron

$$\mathcal{S}_I := \{x \in \mathcal{R}^n |\ C_I x + d_I \le 0,\ \hat{C}_I x + \hat{d}_I < 0\} \tag{2}$$

where $C_I \in \mathcal{R}^{p_I\times n}$, $\hat{C}_I \in \mathcal{R}^{\hat{p}_I\times n}$, $d_I \in \mathcal{R}^{p_I}$, and $\hat{d}_I \in \mathcal{R}^{\hat{p}_I}$. For this subregion, it is assumed that

(A1) $\bigcup_{I\in\mathcal{M}} \mathcal{S}_I = \mathcal{R}^n$ and $\mathcal{S}_I \cap \mathcal{S}_J = \emptyset$ for every $I, J \in \mathcal{M}$ such that $I \neq J$.

This assumption implies that I is uniquely determined for each x, which guarantees that Σ_{sd} is well-posed for all $u \in \mathcal{PC}$. Note that, in this system, the discrete transition (if possible) occurs only at each switching time t_k.

On the other hand, the discrete-time PWA system Σ_d with sampling time h, which is the discrete-time model of the system in Fig. 1, is described by

$$\begin{aligned} &x(t_{k+1}) = A^d_{I(t_k)}x(t_k) + B^d_{I(t_k)}u(t_k) + a^d_{I(t_k)}, \\ &I(t_{k+1}) = I_+, \qquad\qquad \text{if} \quad x(t_{k+1}) \in \mathcal{S}_{I_+} \end{aligned} \tag{3}$$

where the symbols $x \in \mathcal{R}^n$, $I \in \mathcal{M}$, $M \in \mathcal{N}_+$, $u \in \mathcal{R}^m$, $h \in \mathcal{R}$, t_k, $I_+ \in \mathcal{M}$, and $\mathcal{S}_I \subseteq \mathcal{R}^n$ are defined in a similar way to Σ_{sd}, and $A^d_I \in \mathcal{R}^{n\times n}$, $B^d_I \in \mathcal{R}^{n\times m}$, and $a^d_I \in \mathcal{R}^n$ are constant matrices for mode I. For this system, the condition (A1) is also assumed to guarantee that Σ_d is well-posed for all $u(t_k) \in \mathcal{R}^m$. For simplicity of notation, we use hereafter $x(0) = x_0 \in \mathcal{R}^n$ as the initial state instead of the hybrid state $(I(0), x(0)) = (I_0, x_0) \in \mathcal{M} \times \mathcal{S}_{I_0}$, since the value of the initial discrete state $I_0 \in \mathcal{M}$ is uniquely determined by each $x_0 \in \mathcal{R}^n$.

It is remarked that, although at first sight, the system structures of Σ_{sd} and Σ_d seem similar, the controllability/reachability properties of Σ_{sd} and Σ_d are quite different due to the difference of the class of control input signals, i.e., an control input for Σ_{sd} is given by any piecewise continuous functions of time and an control input for Σ_d is given by any piecewise constant functions of time.

III. Controllability/Reachability Analysis

A. Definition of Controllability/Reachability

We define the following notion.

Definition 1: For Σ_{sd} (Σ_d), suppose that the set $\mathcal{X} \subseteq \mathcal{R}^n$ of the continuous state and the final time $T_f \in (0, \infty)$ $(T_f \in \{t_1, t_2, \dots\})$ are given. Let $f \in \mathcal{N}_+$ denote the integer satisfying $t_{f-1} < T_f \le t_f$.
(i) Σ_{sd} (Σ_d) is said to be $\boldsymbol{(\mathcal{X}, T_f)}$**-controllable** if for each $x_0 \in \mathcal{X}$, there exists an input function $u \in \mathcal{PC}$ (an input vector sequence $\{\, u(t_k) \in \mathcal{R}^m \mid k = 0, 1, \dots, f-1\,\}$) satisfying $x(T_f) = 0$ under the initial state $x(0) = x_0$.
(ii) Σ_{sd} (Σ_d) is said to be $\boldsymbol{(\mathcal{X}, T_f)}$**-reachable** if for each $x_f \in \mathcal{X}$, there exists an input function $u \in \mathcal{PC}$ (an input vector sequence $\{\, u(t_k) \in \mathcal{R}^m \mid k = 0, 1, \dots, f-1\,\}$) satisfying $x(T_f) = x_f$ under the initial state $x(0) = 0$. ■

Note here that the set of the initial/final state and the final time are explicitly specified in the above definition. Such a definition is useful in checking the feasibility of the finite-time optimal control (model predictive control) problem with the final state fixed. Note also that the controllability in (i) and the reachability in (ii) are not in general equivalent; we can show that there is a Σ_{sd} (Σ_d) which is $(\mathcal{X}, T_f)$-controllable, not $(\mathcal{X}, T_f)$-reachable, and that there is a Σ_{sd} (Σ_d) which is $(\mathcal{X}, T_f)$-reachable, not $(\mathcal{X}, T_f)$-controllable.

Next, let $I_{or} \in \mathcal{M}$ be the value of the discrete state satisfying $0 \in \mathcal{S}_{I_{or}}$. Then for the above definition, the following fact holds straightforwardly.

Lemma 1: For Σ_{sd} (Σ_d), suppose that $\mathcal{X} \subseteq \mathcal{R}^n$ and $T_f \in (0, \infty)$ are given. Let $\mathcal{U} := \{U \in \mathcal{R}^{mn} | [B_{I_{or}}\ A_{I_{or}}B_{I_{or}} \cdots A^{n-1}_{I_{or}}B_{I_{or}}]U + \int_0^h e^{A_{I_{or}}(h-\tau)}a_{I_{or}}d\tau = 0\}$ $(\mathcal{U} := \{u \in \mathcal{R}^m | B^d_{I_{or}}u + a^d_{I_{or}} = 0\})$. Then if $\mathcal{U} \neq \emptyset$ holds, the following statements hold for all $k \in \mathcal{N}$.
(i) If $T_f \in \{t_1, t_2, \dots\}$ and Σ_{sd} (Σ_d) is $(\mathcal{X}, T_f)$-controllable, then it is $(\mathcal{X}, T_f + kh)$-controllable.
(ii) If Σ_{sd} (Σ_d) is $(\mathcal{X}, T_f)$-reachable, then it is $(\mathcal{X}, T_f + kh)$-reachable. ■

Lemma 1 implies that, under the condition $\mathcal{U} \neq \emptyset$, if Σ_{sd} (Σ_d) is $(\mathcal{X}, T_f)$-controllable/reachable, it is also $(\mathcal{X}, T_f + kh)$-controllable/reachable for any $k \in \mathcal{N}_+$.

B. Controllability/Reachability Criteria

First, let us consider the controllability/reachability criteria of Σ_{sd}. For simplicity of notation, letting $\bar{x} := [x^T\ 1]^T$ and

$$\bar{A}_I := \begin{bmatrix} A_I & a_I \\ 0 & 0 \end{bmatrix}, \qquad \bar{B}_I := \begin{bmatrix} B_I \\ 0 \end{bmatrix} \tag{4}$$

where $\bar{A}_I \in \mathcal{R}^{(n+1)\times(n+1)}$ and $\bar{B}_I \in \mathcal{R}^{(n+1)\times m}$, we rewrite (1) as $\dot{\bar{x}}(t) = \bar{A}_{I(t)}\bar{x}(t) + \bar{B}_{I(t)}u(t)$. Let $\bar{V}_I := [\bar{B}_I \;\; \bar{A}_I\bar{B}_I \;\; \cdots \;\; \bar{A}_I^n\bar{B}_I]$ and $r_I := \mathrm{rank}\,\bar{V}_I$, and let $\bar{V}_I^{\perp} \in \mathcal{R}^{(n+1)\times(n+1-r_I)}$ be a matrix such that $\mathrm{rank}[\bar{V}_I \;\; \bar{V}_I^{\perp}] = n+1$ and $(\bar{V}_I^{\perp})^T\bar{V}_I = 0$. Thus $\mathrm{span}(\bar{V}_I^{\perp})$ expresses the orthogonal complement of $\mathrm{span}(\bar{V}_I)$. Furthermore, we denote the hybrid states $(I_k, x_k) \in \mathcal{M}\times\mathcal{S}_{I_k},\ k = 1,2,\ldots,f-1$, as $\mathcal{I} := [I_1 \;\; I_2 \;\; \cdots \;\; I_{f-1}]^T$ and $\mathsf{x} := [x_1^T \;\; x_2^T \;\; \cdots \;\; x_{f-1}^T]^T$; so $\mathcal{I}\in\mathcal{M}^{f-1}$ and $\mathsf{x}\in\mathcal{S}_{\mathcal{I}}$ for $\mathcal{S}_{\mathcal{I}} := \mathcal{S}_{I_1}\times\mathcal{S}_{I_2}\times\cdots\times\mathcal{S}_{I_{f-1}}$. Then we obtain the following result. This characterizes a condition for the system to have $u\in\mathcal{PC}$ satisfying $x(T_f) = x_f$ for the initial state $(I(0), x(0)) = (I_0, x_0)$, in terms of the intermediate hybrid states $(\mathcal{I}, \mathsf{x})$.

Lemma 2: For Σ_{sd}, suppose that the hybrid state $(I_k, x_k)\in\mathcal{M}\times\mathcal{S}_{I_k}$ $(k = 0,1,\ldots,f-1)$, the final state $x_f\in\mathcal{R}^n$, and the final time $T_f\in(0,\infty)$ are given. Then the following statements are equivalent.

(i) For the initial state $x(0) = x_0$ (that is, $(I(0), x(0)) = (I_0, x_0)$), there exists a $u\in\mathcal{PC}$ satisfying $(I(t_k), x(t_k)) = (I_k, x_k)$ $(k = 1,2,\ldots,f-1)$ and $x(T_f) = x_f$.

(ii) The following relation holds:

$$\mathcal{E}^a_{I_0\mathcal{I}}\,\mathsf{x} + \mathcal{E}^{b_1}_{I_0\mathcal{I}} + \mathcal{E}^{b_2}_{I_0\mathcal{I}}\,x_0 + \mathcal{E}^{b_3}_{\mathcal{I}}x_f = 0 \tag{5}$$

where $\mathcal{E}^a_{I_0\mathcal{I}}$, $\mathcal{E}^{b_1}_{I_0\mathcal{I}}$, $\mathcal{E}^{b_2}_{I_0\mathcal{I}}$, and $\mathcal{E}^{b_3}_{\mathcal{I}}$ are given by

$$\mathcal{E}^a_{I_0\mathcal{I}} := \begin{bmatrix} E^2_{I_0} & 0 & 0 & \cdots & 0 \\ E^1_{I_1} & E^2_{I_1} & 0 & \cdots & 0 \\ 0 & E^1_{I_2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & E^1_{I_{f-2}} & E^2_{I_{f-2}} \\ 0 & \cdots & \cdots & 0 & E^1_{I_{f-1}} \end{bmatrix},$$

$$\mathcal{E}^{b_1}_{I_0\mathcal{I}} := \begin{bmatrix} E^3_{I_0} \\ E^3_{I_1} \\ \vdots \\ E^3_{I_{f-1}} \end{bmatrix},\quad \mathcal{E}^{b_2}_{I_0\mathcal{I}} := \begin{bmatrix} E^1_{I_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix},\quad \mathcal{E}^{b_3}_{\mathcal{I}} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E^2_{I_{f-1}} \end{bmatrix},$$

$$E^1_{I_k} := (\bar{V}^{\perp}_{I_k})^T\begin{bmatrix} -e^{A_{I_k}h_k} \\ 0 \end{bmatrix},\quad E^2_{I_k} := (\bar{V}^{\perp}_{I_k})^T\begin{bmatrix} \boldsymbol{I}_n \\ 0 \end{bmatrix},$$

$$E^3_{I_k} := (\bar{V}^{\perp}_{I_k})^T\begin{bmatrix} -\int_0^{h_k} e^{A_{I_k}(h_k-\tau)}a_{I_k}\,d\tau \\ 0 \end{bmatrix},$$

$$h_k := \begin{cases} h & \text{if} \quad k\le f-2, \\ T_f - t_{f-1} & \text{if} \quad k = f-1. \end{cases}$$

Proof: Since the discrete transition in Σ_{sd} does not occur at $t\in(t_k, t_{k+1})$, $I(t) = I_k$ holds for all $t\in[t_k, t_{k+1})$. Then letting $\bar{x}_k := [x_k^T \;\; 1]^T$, it follows that for the state $(I(t_k), x(t_k)) = (I_k, x_k)$ of Σ_{sd}, there exists a $u\in\mathcal{PC}$ such that $x(t_k + h_k) = x_{k+1}$ if and only if $\bar{x}_{k+1} - e^{\bar{A}_{I_k}h_k}\bar{x}_k \in \mathrm{span}(\bar{V}_{I_k})$, namely, $(\bar{V}^{\perp}_{I_k})^T(\bar{x}_{k+1} - e^{\bar{A}_{I_k}h_k}\bar{x}_k) = 0$. This can be expressed as $E^1_{I_k}x_k + E^2_{I_k}x_{k+1} + E^3_{I_k} = 0$. Thus (5) is obtained by putting together the above relations for all $k\in\{0,1,\ldots,f-1\}$. This proves the equivalence between (i) and (ii). ■

From Lemma 2, it follows that for given $(I_0, x_0)\in\mathcal{M}\times\mathcal{S}_{I_0}$, $\mathcal{I}\in\mathcal{M}^{f-1}$, $x_f\in\mathcal{R}^n$, and $T_f\in(0,\infty)$, there exists a $u\in\mathcal{PC}$ satisfying $[I(t_1) \; I(t_2) \; \cdots \; I(t_{f-1})]^T = \mathcal{I}$ and $x(T_f) = x_f$ under the initial state $(I(0), x(0)) = (I_0, x_0)$ if and only if there exists an $\mathsf{x}\in\mathcal{S}_{\mathcal{I}}$ satisfying (5), i.e., $\{\mathsf{x}\in\mathcal{S}_{\mathcal{I}} | \mathcal{E}^a_{I_0\mathcal{I}}\mathsf{x} + \mathcal{E}^{b_1}_{I_0\mathcal{I}} + \mathcal{E}^{b_2}_{I_0\mathcal{I}}x_0 + \mathcal{E}^{b_3}_{\mathcal{I}}x_f = 0\} \neq \emptyset$ holds. Thus if I_0 and $\mathcal{I}$ are fixed, the set of $x_0\in\mathcal{S}_{I_0}\cap\mathcal{X}$ such that there exists a $u\in\mathcal{PC}$ satisfying $[I(t_1) \; I(t_2) \; \cdots \; I(t_{f-1})]^T = \mathcal{I}$ and $x(T_f) = 0$ under the initial state $(I(0), x(0)) = (I_0, x_0)$ can be expressed as $\{x_0\in\mathcal{S}_{I_0}\cap\mathcal{X} \,|\, \{\mathsf{x}\in\mathcal{S}_{\mathcal{I}} | \mathcal{E}^a_{I_0\mathcal{I}}\mathsf{x} + \mathcal{E}^{b_1}_{I_0\mathcal{I}} + \mathcal{E}^{b_2}_{I_0\mathcal{I}}x_0 = 0\} \neq \emptyset\}$. Then, since the relation $\mathcal{X} = \bigcup_{I_0\in\mathcal{M}}\mathcal{S}_{I_0}\cap\mathcal{X}$ holds under (A1), let us define the set

$$\begin{aligned}&\mathcal{X}_0^{\mathcal{I}}(\mathcal{X}, T_f) := \\ &\bigcup_{I_0\in\mathcal{M}}\{x_0\in\mathcal{S}_{I_0}\cap\mathcal{X} \,| \\ &\qquad \{\mathsf{x}\in\mathcal{S}_{\mathcal{I}} | \mathcal{E}^a_{I_0\mathcal{I}}\mathsf{x} + \mathcal{E}^{b_1}_{I_0\mathcal{I}} + \mathcal{E}^{b_2}_{I_0\mathcal{I}}x_0 = 0\} \neq \emptyset\ \}.\end{aligned} \tag{6}$$

Finally, we can express the set of $x_0\in\mathcal{X}$ such that there exists a $u\in\mathcal{PC}$ satisfying $x(T_f) = 0$ under the initial state $x(0) = x_0$ as follows:

$$\mathcal{X}_0(\mathcal{X}, T_f) := \bigcup_{\mathcal{I}\in\mathcal{M}^{f-1}} \mathcal{X}_0^{\mathcal{I}}(\mathcal{X}, T_f). \tag{7}$$

In a similar way, the set of $x_f\in\mathcal{X}$ such that there exists a $u\in\mathcal{PC}$ satisfying $x(T_f) = x_f$ under the initial state $x(0) = 0$ (that is, $(I(0), x(0)) = (I_{or}, 0)$) is given by

$$\mathcal{X}_f(\mathcal{X}, T_f) := \bigcup_{\mathcal{I}\in\mathcal{M}^{f-1}} \mathcal{X}_f^{\mathcal{I}}(\mathcal{X}, T_f) \tag{8}$$

where $\mathcal{X}_f^{\mathcal{I}}(\mathcal{X}, T_f) := \{x_f\in\mathcal{X} \big| \{\mathsf{x}\in\mathcal{S}_{\mathcal{I}} | \mathcal{E}^a_{I_{or}\mathcal{I}}\mathsf{x} + \mathcal{E}^{b_1}_{I_{or}\mathcal{I}} + \mathcal{E}^{b_3}_{\mathcal{I}}x_f = 0\} \neq \emptyset\}$. Thus the following result is obtained.

Theorem 1: For Σ_{sd}, suppose that $\mathcal{X}\subseteq\mathcal{R}^n$ and $T_f\in(0,\infty)$ are given. Then the following statements hold.

(i) Σ_{sd} is $(\mathcal{X}, T_f)$-controllable if and only if the relation $\mathcal{X}_0(\mathcal{X}, T_f) = \mathcal{X}$ holds.

(ii) Σ_{sd} is $(\mathcal{X}, T_f)$-reachable if and only if the relation $\mathcal{X}_f(\mathcal{X}, T_f) = \mathcal{X}$ holds. ■

The above theorem allows us to analyze these two properties in a unified way.

Next, let us consider the controllability/reachability criteria of Σ_d. This is discussed in a similar way to that of Σ_{sd}. Let

$$\bar{A}_I^d := \begin{bmatrix} A_I^d & a_I^d \\ 0 & 1 \end{bmatrix}, \qquad \bar{B}_I^d := \begin{bmatrix} B_I^d \\ 0 \end{bmatrix}$$

where $\bar{A}_I^d \in \mathcal{R}^{(n+1)\times(n+1)}$ and $\bar{B}_I^d \in \mathcal{R}^{(n+1)\times m}$. In addition, let $\bar{V}_I^d := \bar{B}_I^d$ and $r_I^d := \mathrm{rank}\,\bar{V}_I^d$, and let $\bar{V}_I^{d\perp} \in \mathcal{R}^{(n+1)\times(n+1-r_I^d)}$ be the matrix satisfying $\mathrm{rank}[\bar{V}_I^d \;\; \bar{V}_I^{d\perp}] = n+1$ and $(\bar{V}_I^{d\perp})^T\bar{V}_I^d = 0$. Then by defining $E^1_{I_k} := (\bar{V}^{d\perp}_{I_k})^T[(-A^d_{I_k})^T \;\; 0^T_{1\times n}]^T$, $E^2_{I_k} := (\bar{V}^{d\perp}_{I_k})^T[\boldsymbol{I}_n^T \;\; 0^T_{1\times n}]^T$, $E^3_{I_k} := (\bar{V}^{d\perp}_{I_k})^T[(-a^d_{I_k})^T \;\; 0]^T$ instead of $E^1_{I_k}$, $E^2_{I_k}$, $E^3_{I_k}$ in Lemma 2 (ii) and by defining $\mathcal{X}_0$ and $\mathcal{X}_f$ in a similar way to (7) and (8), the same result as Theorem 1 is obtained for Σ_d.

In this way, since the controllability/reachability conditions of Σ_{sd} and Σ_d are characterized in a similar form, so we mainly discuss the $(\mathcal{X}, T_f)$-controllability of Σ_{sd} hereafter. Furthermore, for simplicity of notation, we sometimes use the symbol $\mathcal{X}_0^{\mathcal{I}}$ instead of $\mathcal{X}_0^{\mathcal{I}}(\mathcal{X}, T_f)$, and the symbol $\mathcal{X}_0$ instead of $\mathcal{X}_0(\mathcal{X}, T_f)$.

C. Deterministic Controllability Analysis and Its Problems

Based on Theorem 1, let us discuss how to check the $(\mathcal{X}, T_f)$-controllability of Σ_{sd}. For this purpose, we prepare the following lemma.

Lemma 3: For Σ_{sd}, suppose that $\mathcal{X} \subseteq \mathcal{R}^n$ and $T_f \in (0, \infty)$ are given. Then if $\mathcal{X}$ is the bounded polyhedron, $\mathcal{X}_0$ can be expressed as

$$\mathcal{X}_0 = \bigcup_{\mathcal{I} \in \mathcal{M}^{f-1}} \bigcup_{I_0 \in \mathcal{M}} \mathcal{X}_0^{I_0 \mathcal{I}} \tag{9}$$

by using the set $\mathcal{X}_0^{I_0\mathcal{I}} := \bigcup_{i \in \mathcal{N}_+} \mathcal{O}_{I_0\mathcal{I}}^i$ where $\mathcal{O}_{I_0\mathcal{I}}^i$ is some polyhedron.

Proof: Since $\mathcal{X}_0^{I_0\mathcal{I}}$ is considered as $\{x_0 \in \mathcal{S}_{I_0} \cap \mathcal{X} | \{\mathsf{x} \in \mathcal{S}_{\mathcal{I}} | \mathcal{E}_{I_0\mathcal{I}}^a \mathsf{x} + \mathcal{E}_{I_0\mathcal{I}}^{b_1} + \mathcal{E}_{I_0\mathcal{I}}^{b_2} x_0 = 0\} \neq \emptyset\}$ from (6) and (7), we prove that it is characterized by the union set of some polyhedra. For this purpose, we first define $\bar{\mathcal{S}}_{I_0} \cap \bar{\mathcal{X}}$ as the closure of $\mathcal{S}_{I_0} \cap \mathcal{X}$. For given $(I_0, x_0) \in \mathcal{M} \times \mathcal{S}_{I_0}$ and $\mathcal{I} \in \mathcal{M}^{f-1}$, let us consider the linear programming (LP) problem LP$(\mathcal{I}, I_0, x_0)$:

$$\begin{aligned} &\min_{\mathsf{x}, w} \quad w \\ &\text{subject to} \begin{cases} \mathcal{C}_{\mathcal{I}} \mathsf{x} + \mathcal{D}_{\mathcal{I}} \leq 0, \;\; \hat{\mathcal{C}}_{\mathcal{I}} \mathsf{x} + \hat{\mathcal{D}}_{\mathcal{I}} \leq w\mathbf{1}, \;\; -1 \leq w, \\ \mathcal{E}_{I_0\mathcal{I}}^a \mathsf{x} + \mathcal{E}_{I_0\mathcal{I}}^{b_1} + \mathcal{E}_{I_0\mathcal{I}}^{b_2} x_0 = 0 \end{cases} \end{aligned}$$

where w is the scalar variable, $\mathcal{C}_{\mathcal{I}}$, $\mathcal{D}_{\mathcal{I}}$, $\hat{\mathcal{C}}_{\mathcal{I}}$, and $\hat{\mathcal{D}}_{\mathcal{I}}$ are the matrices satisfying $\mathcal{S}_{\mathcal{I}} = \{\mathsf{x} \in \mathcal{R}^{(f-1)n} \mid \mathcal{C}_{\mathcal{I}} \mathsf{x} + \mathcal{D}_{\mathcal{I}} \leq 0,\ \hat{\mathcal{C}}_{\mathcal{I}} \mathsf{x} + \hat{\mathcal{D}}_{\mathcal{I}} < 0\}$, and $\mathbf{1} := [1\ 1\ \cdots\ 1]^T$. Then, by defining the optimal value of LP$(\mathcal{I}, I_0, x_0)$ as $w^*(\mathcal{I}, I_0, x_0)$, it turns out that $\{\mathsf{x} \in \mathcal{S}_{\mathcal{I}} | \mathcal{E}_{I_0\mathcal{I}}^a \mathsf{x} + \mathcal{E}_{I_0\mathcal{I}}^{b_1} + \mathcal{E}_{I_0\mathcal{I}}^{b_2} x_0 = 0\} \neq \emptyset$ is satisfied if and only if LP$(\mathcal{I}, I_0, x_0)$ is feasible and the relation $w^*(\mathcal{I}, I_0, x_0) < 0$ holds. Thus, for given $(I_0, \mathcal{I}) \in \mathcal{M}^f$, if LP$(\mathcal{I}, I_0, x_0)$ is not feasible for all $x_0 \in \bar{\mathcal{S}}_{I_0} \cap \bar{\mathcal{X}}$, then the relation $\{x_0 \in \mathcal{S}_{I_0} \cap \mathcal{X} | \{\mathsf{x} \in \mathcal{S}_{\mathcal{I}} | \mathcal{E}_{I_0\mathcal{I}}^a \mathsf{x} + \mathcal{E}_{I_0\mathcal{I}}^{b_1} + \mathcal{E}_{I_0\mathcal{I}}^{b_2} x_0 = 0\} \neq \emptyset\} = \emptyset$ holds. On the other hand, for given $(I_0, \mathcal{I}) \in \mathcal{M}^f$, if LP$(\mathcal{I}, I_0, x_0)$ is feasible for some $x_0 \in \bar{\mathcal{S}}_{I_0} \cap \bar{\mathcal{X}}$, then by considering LP$(\mathcal{I}, I_0, x_0)$ as the multiparametric LP (mp-LP) problem with the parameter $x_0 \in \bar{\mathcal{S}}_{I_0} \cap \bar{\mathcal{X}}$, we can obtain

$$w^*(\mathcal{I}, I_0, x_0) = G_{I_0\mathcal{I}}^i\ x_0 + g_{I_0\mathcal{I}}^i, \qquad \text{if} \quad x_0 \in \mathcal{G}_{I_0\mathcal{I}}^i \tag{10}$$

where $i \in \mathcal{N}_+$, $G_{I_0\mathcal{I}}^i$ and $g_{I_0\mathcal{I}}^i$ are some vectors, and $\mathcal{G}_{I_0\mathcal{I}}^i$ $(\subseteq \bar{\mathcal{S}}_{I_0} \cap \bar{\mathcal{X}})$ is some polyhedron [6]. Note that LP$(\mathcal{I}, I_0, x_0)$ for given $(I_0, \mathcal{I}) \in \mathcal{M}^f$ and $x_0 \in \bar{\mathcal{S}}_{I_0} \cap \bar{\mathcal{X}}$ is feasible if and only if $x_0 \in \bigcup_{i \in \mathcal{N}_+} \mathcal{G}_{I_0\mathcal{I}}^i$. Thus, since $\mathcal{S}_{I_0} \cap \mathcal{X} \subseteq \bar{\mathcal{S}}_{I_0} \cap \bar{\mathcal{X}}$ holds, the relation $\{x_0 \in \mathcal{S}_{I_0} \cap \mathcal{X} | \{\mathsf{x} \in \mathcal{S}_{\mathcal{I}} | \mathcal{E}_{I_0\mathcal{I}}^a \mathsf{x} + \mathcal{E}_{I_0\mathcal{I}}^{b_1} + \mathcal{E}_{I_0\mathcal{I}}^{b_2} x_0 = 0\} \neq \emptyset\} = \bigcup_{i \in \mathcal{N}_+} \{x_0 \in \mathcal{G}_{I_0\mathcal{I}}^i | G_{I_0\mathcal{I}}^i\ x_0 + g_{I_0\mathcal{I}}^i < 0\} \cap (\mathcal{S}_{I_0} \cap \mathcal{X})$ is obtained, which completes the proof. ■

Lemma 3 implies that, for the bounded polyhedron $\mathcal{X}$, $\mathcal{X}_0$ is characterized by the union set of some polyhedra obtained by solving the mp-LP problems LP$(\mathcal{I}, I_0, x_0)$. Hence, calculating $\mathcal{X}_0^{I_0\mathcal{I}}$ and then $\bigcup_{\mathcal{I} \in \mathcal{M}^{f-1}} \bigcup_{I_0 \in \mathcal{M}} \mathcal{X}_0^{I_0\mathcal{I}}$ is required for checking the condition $\mathcal{X}_0 = \mathcal{X}$. However, such a deterministic way is not always practical; as T_f is taken larger, the number of the mp-LP problems to obtain $\mathcal{X}_0^{I_0\mathcal{I}}$ for all $(I_0, \mathcal{I}) \in \mathcal{M}^f$ becomes exponentially large in the worst case. In addition, in the mp-LP problem, even when the dimension of the parameter x_0 is fixed, the computation amount grows exponentially with the dimension of the variable and the number of the constraints in general. In fact, the minimum dimension of the variable and the minimum number of the constraints are at least $n(f-1)+1$ and $\min_{I \in \mathcal{M}}(p_I + \hat{p}_I)\cdot(f-1)+1$, respectively, where $p_I + \hat{p}_I$ is the number of the inequalities characterizing $\mathcal{S}_I$ in (2). Hence, for example, if $n = 3$, $M = 5$, $\min_{I \in \mathcal{M}}(p_I + \hat{p}_I) = 2$, and $f = 10$, then 9,765,625 mp-LP problems with 3-dimensional parameter and at least 28-dimensional variable and 19 constraints have to be solved.

IV. Probabilistic Controllability Analysis

In Section III-C, we have discussed the hardness of the deterministic way based on Theorem 1. Hence, as a practical method, we consider here a randomized algorithm to solve the condition in Theorem 1 in a probabilistic sense. For simplicity of discussion, we suppose $\mathcal{X}$ is the bounded and measurable set whose measure is not zero.

A. Principle of Probabilistic Controllability Analysis

Let us define the measures of $\mathcal{X}$ and $\mathcal{X}_0$ as $\mathrm{vol}(\mathcal{X}) := \int_{\mathcal{X}} dx_0$ and $\mathrm{vol}(\mathcal{X}_0) := \int_{\mathcal{X}_0} dx_0$, respectively. In addition, letting x_0 be a random vector with a uniform probability density function ϕ_{x_0} on $\mathcal{X}$, we formally define

$$\mathrm{Prob}\{x_0 \in \mathcal{X}_0\} := \int_{\mathcal{X}_0} \phi_{x_0}\ dx_0. \tag{11}$$

Note that $\mathrm{Prob}\{x_0 \in \mathcal{X}_0\} = \mathrm{vol}(\mathcal{X}_0)/\mathrm{vol}(\mathcal{X})$ holds. Then the following result is straightforwardly obtained from the result in [7].

Lemma 4: For Σ_{sd}, suppose that $\mathcal{X} \subseteq \mathcal{R}^n$ and $T_f \in (0, \infty)$ are given. For given $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, let N_s be an integer satisfying

$$N_s \geq \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\varepsilon}}. \tag{12}$$

Then if for all i.i.d. random vectors $x_0^i \in \mathcal{X}$ $(i = 1, 2, \ldots, N_s)$, $x_0^i \in \mathcal{X}_0$ holds, the relation

$$\mathrm{Prob}\{\mathrm{Prob}\{x_0 \in \mathcal{X} - \mathcal{X}_0\} \leq \varepsilon\} \geq 1 - \delta \tag{13}$$

holds. ■

Lemma 4 implies that, when $x_0^i \in \mathcal{X}_0$ holds for every i.i.d. random vectors $x_0^i \in \mathcal{X}$ $(i = 1, 2, \ldots, N_s)$, the fact that the volume of $\mathcal{X} - \mathcal{X}_0$ $(= \mathrm{vol}(\mathcal{X} - \mathcal{X}_0)/\mathrm{vol}(\mathcal{X}))$ is less than ε holds with the probability more than $1 - \delta$. Thus if ε and δ are sufficiently small, the condition $\mathcal{X}_0 = \mathcal{X}$ is approximately satisfied, in other words, for almost all $x_0 \in \mathcal{X}$, there exists a $u \in \mathcal{PC}$ in Σ_{sd} satisfying $x(T_f) = 0$ under the initial state $x(0) = x_0$. We may feel here that something is missing for such a analysis. However, as mentioned in section III-C, we have to recall that we face on the hardness of the computation on the analysis of the complex systems such as hybrid systems. Thus for hybrid systems that can *not* be analyzed in a deterministic way, the probabilistic method will be the alternative.

Now, by letting $\varphi(x_0)$ be the function to check whether $x_0 \in \mathcal{X}_0$ holds or not, given by

$$\varphi(x_0) := \begin{cases} 0 & \text{if} \quad x_0 \in \mathcal{X}_0, \\ 1 & \text{if} \quad x_0 \notin \mathcal{X}_0, \end{cases} \tag{14}$$

and $F(\mathcal{I}, I_0, x_0)$ be the propositional function given by

$$F(\mathcal{I}, I_0, x_0) := \begin{cases} 0 & \text{if} \quad \text{LP}(\mathcal{I}, I_0, x_0) \text{ is feasible and} \\ & \qquad w^*(\mathcal{I}, I_0, x_0) < 0, \\ 1 & \text{otherwise}, \end{cases} \tag{15}$$

the probabilistic controllability analysis of Σ_{sd} is executed by the following randomized algorithm.

[Algorithm 1: Probabilistic Controllability Analysis]

0: Given $\varepsilon \in (0,1)$ and $\delta \in (0,1)$;
1: Let N_s be an integer s.t. (12);
2: Generate the i.i.d. random vectors $x_0^1, x_0^2, \dots, x_0^{N_s} \in \mathcal{X}$;
3: i := 1;
4: If $\varphi(x_0^i)$ == 1
  then Halt: return "Not $(\mathcal{X}, T_f)$-controllable";
5: If ($\varphi(x_0^i)$ == 0) and (i == N_s)
  then Halt: return "$(\mathcal{X}, T_f)$-controllable in the sense of (13)";
6: i := $i+1$;   goto Line 4;

[Algorithm 2: Computation of $\varphi(x_0)$]

0: Given $x_0 \in \mathcal{X}$;
1: Stack := $\mathcal{M}^{f-1}$;
2: Let I_0 be the value of the discrete state s.t. $x_0 \in \mathcal{S}_{I_0}$;
3: $\mathcal{I}$:= Pop(Stack);
4: Determine $F(\mathcal{I}, I_0, x_0)$ by solving LP$(\mathcal{I}, I_0, x_0)$;
5: If $F(\mathcal{I}, I_0, x_0)$ == 0
  then Halt: return $\varphi(x_0) = 0$;
6: If Stack == empty
  then Halt: return $\varphi(x_0) = 1$; else goto Line 3;

Algorithm 1 shows the procedure of the probabilistic controllability analysis: for given $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, N_s is determined by (12) (line 1), and N_s i.i.d. random vectors $x_0^1, x_0^2, \dots, x_0^{N_s} \in \mathcal{X}$ are generated (line 2). Then if $x_0^i \notin \mathcal{X}_0$ for some i, the output is given by "Not $(\mathcal{X}, T_f)$-controllable" (line 4). Note that this is the deterministic result since $\mathcal{X}_0 \neq \mathcal{X}$ holds. Otherwise, that is, $x_0^i \in \mathcal{X}_0$ for all $i \in \{1, 2, \dots, N_s\}$, the output is given by "$(\mathcal{X}, T_f)$-controllable in the sense of (13)" (line 5).

On the other hand, Algorithm 2 shows the procedure to compute $\varphi(x_0)$ in Algorithm 1: for $x_0 \in \mathcal{X}$ and $\mathcal{I} \in \mathcal{M}^{f-1}$, since the condition $x_0 \in \mathcal{X}_0^{\mathcal{I}}$ holds if and only if $\{\mathsf{x} \in \mathcal{S}_{\mathcal{I}} | \mathcal{E}^a_{I_0\mathcal{I}}\mathsf{x} + \mathcal{E}^{b_1}_{I_0\mathcal{I}} + \mathcal{E}^{b_2}_{I_0\mathcal{I}}x_0 = 0\} \neq \emptyset$ holds for $I_0 \in \mathcal{M}$ satisfying $x_0 \in \mathcal{S}_{I_0}$, we can determine if the condition $x_0 \in \mathcal{X}_0^{\mathcal{I}}$ holds or not by solving LP$(\mathcal{I}, I_0, x_0)$ for $I_0 \in \mathcal{M}$ satisfying $x_0 \in \mathcal{S}_{I_0}$ and verifying $F(\mathcal{I}, I_0, x_0) = 0$. Therefore, if $F(\mathcal{I}, I_0, x_0) = 0$ for some element $\mathcal{I}$ in Stack$(= \mathcal{M}^{f-1})$, then the output is $\varphi(x_0) = 0$ (line 5), and if there exists no $\mathcal{I} \in \mathcal{M}^{f-1}$ such that $F(\mathcal{I}, I_0, x_0) = 0$, then the output is given by $\varphi(x_0) = 1$ (line 6).

The number $N_{LP} \in \mathcal{N}$ of LP problems solved in the proposed algorithm is estimated by

$$\min\{N_s, \ M^{f-1}\} \leq N_{LP} \leq N_s M^{f-1}. \tag{16}$$

In fact, Algorithm 2 needs to solve at most M^{f-1} times LP problems for step i in Algorithm 1, and if Σ_{sd} is $(\mathcal{X}, T_f)$-controllable, the condition $\varphi(x_0^i) = 0$ is checked for all $x_0^1, x_0^2, \dots, x_0^{N_s} \in \mathcal{X}$, and if Σ_{sd} is not $(\mathcal{X}, T_f)$-controllable, for some $x_0^i \in \mathcal{X}$, the condition $x_0^i \notin \mathcal{X}_0^{\mathcal{I}}$ is checked for every $\mathcal{I} \in \mathcal{M}^{f-1}$. Hence we obtain (16).

The probabilistic controllability analysis has some good properties. First, the large memory in the computer is not needed. Although the mode sequence set $\mathcal{M}^{f-1}$ is stored in Stack at a time in Algorithm 2 for simplicity of discussion, we do not have to do such a way: if we use ordered sequences, the large memory is not required. Second, even when we check if $x_0^i \in \mathcal{X}_0$ holds for some $N_s'(< N_s)$ sampled data, we can estimate ε and δ for N_s' samples, based on Lemma 4. Thus it is possible to estimate the controllability of Σ_{sd} for a fixed computation time. For any deterministic methods, such advantages will not be satisfied.

B. Techniques for Efficient Probabilistic Controllability Analysis

In this subsection, we present several techniques to more efficiently execute the above randomized algorithm. First, the following result is obtained from (6) and (7).

Lemma 5: For Σ_{sd}, suppose that $\mathcal{X} \subseteq \mathcal{R}^n$ and $T_f \in (0, \infty)$ are given. Let $\mathcal{M}_C := \{I_0 \in \mathcal{M} | r_{I_0} = n\}$ and

$$\mathcal{X}_C := \begin{cases} \emptyset & \text{if} \quad \mathcal{M}_C = \emptyset, \\ \bigcup_{I_0 \in \mathcal{M}_C} \mathcal{S}_{I_0} \cap \mathcal{X} & \text{if} \quad \mathcal{M}_C \neq \emptyset. \end{cases} \tag{17}$$

Then, the relation $\mathcal{X}_C \subseteq \mathcal{X}_0$ holds. ■

Lemma 5 implies that Σ_{sd} is $(\mathcal{X}, T_f)$-controllable if and only if it is $(\mathcal{X} - \mathcal{X}_C, T_f)$-controllable. Thus since the set $\mathcal{X} - \mathcal{X}_C$ is smaller than $\mathcal{X}$, we can check the controllability by smaller size of samples than N_s defined by (12) as follows.

Lemma 6: Suppose that $\mathcal{X} \subseteq \mathcal{R}^n$, $T_f \in (0, \infty)$, $\varepsilon \in (0,1)$, and $\delta \in (0,1)$ are given and that $\mathrm{vol}(\mathcal{X} - \mathcal{X}_C) \neq 0$ holds. Let $\varepsilon' := \varepsilon(\mathrm{vol}(\mathcal{X}) / \mathrm{vol}(\mathcal{X} - \mathcal{X}_C))$ and let $\tilde{N}_s$ be the integer satisfying

$$\tilde{N}_s \geq \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\varepsilon'}}. \tag{18}$$

Then if for all i.i.d. random vectors $x_0^i \in \mathcal{X} - \mathcal{X}_C$ $(i = 1, 2, \dots, \tilde{N}_s)$, $x_0^i \in \mathcal{X}_0$ holds, the relation (13) holds. ■

Thus we obtain an efficient algorithm, where the statements in lines 1 and 2 of Algorithm 1 are replaced by

```
1: Let Ñ_s be the integer s.t. (18);
2: Generate the i.i.d. random vectors
                         x_0^1, x_0^2, ..., x_0^{Ñ_s} ∈ X − X_C;
```

and N_s in line 5 is replaced by $\tilde{N}_s$.

Next, the following result obtained from Lemma 2, the proof of Lemma 3, and (15), plays a central role to compute $\varphi(x_0)$ in an efficient way.

Lemma 7: For Σ_{sd}, suppose that $\mathcal{X} \subseteq \mathcal{R}^n$ and $T_f \in (0, \infty)$ are given. Then the following statements hold.
(i) For given $(I_0, x_0) \in \mathcal{M} \times \mathcal{S}_{I_0}$ and $\mathcal{I} \in \mathcal{M}^{f-1}$, if $\mathrm{rank}\, \mathcal{E}^a_{I_0\mathcal{I}} \neq \mathrm{rank}[\mathcal{E}^a_{I_0\mathcal{I}} \;\; \mathcal{E}^{b_1}_{I_0\mathcal{I}} + \mathcal{E}^{b_2}_{I_0\mathcal{I}} x_0]$, then $F(\mathcal{I}, I_0, x_0) = 1$ holds.
(ii) Suppose that $(I_0, x_0) \in \mathcal{M} \times \mathcal{S}_{I_0}$, $j \in \{1, 2, \ldots, f-1\}$, and $\mathcal{I}_j \in \mathcal{M}^j$ are given. Then for the initial state $x(0) = x_0$ if there exists no $u \in \mathcal{PC}$ satisfying $[I(t_1) \;\; I(t_2) \;\; \cdots \;\; I(t_j)]^T = \mathcal{I}_j$, $F(\mathcal{I}, I_0, x_0) = 1$ for all $\mathcal{I} \in \mathcal{M}^{f-1}$ such that $[\boldsymbol{I}_j \;\; 0_{j \times (f-1-j)}]\mathcal{I} = \mathcal{I}_j$. ■

Lemma 7 (i) implies that, without solving LP($\mathcal{I}, I_0, x_0$), we can often check if $F(\mathcal{I}, I_0, x_0) = 1$ holds; thus the computation amount N_{LP} can be decreased. Lemma 7 (ii) is also very useful. For example, if there exists no $u \in \mathcal{PC}$ satisfying $I(t_1) = \mathcal{I}_1$ for the initial state $x(0) = x_0$ and $\mathcal{I}_1 = 0$ (this can be checked by solving some LP problem), it turns out that $F(\mathcal{I}, I_0, x_0) = 1$ for all $\mathcal{I} \in \mathcal{M}^{f-1}$ whose first element is "0". Thus the corresponding elements in `Stack` can be removed. It is stressed that, by integrating the above techniques into Algorithms 1 and 2, we can expect that the computation amount N_{LP} is made smaller than the lower bound of (16) derived for Algorithms 1 and 2.

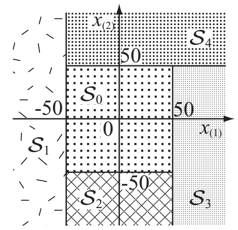
V. Example

Let us consider the following Σ_{sd} with $n = 2$, $M = 5$, and $h = 1$, which include a parameter $\zeta \in \{0, 1\}$:

$$A_0 := \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad B_0 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad a_0 := \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$A_1 := \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad B_1 := \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad a_1 := \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_2 := \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_2 := \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_3 := \begin{bmatrix} -\zeta & 0 \\ 0 & -3 \end{bmatrix}, \quad B_3 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad a_3 := \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$A_4 := \begin{bmatrix} 0 & 0 \\ 0 & -9 \end{bmatrix}, \quad B_4 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad a_4 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The subregions of the continuous state assigned to each value of the discrete state are shown in Fig. 2. Note that, the subsystems in only mode 0 and mode 1 are controllable for any $\zeta \in \{0, 1\}$, so $\mathcal{M}_C = \{0, 1\}$ and $\mathcal{X}_C = (\mathcal{S}_0 \cup \mathcal{S}_1) \cap \mathcal{X}$ in Lemma 5.

Let us apply the proposed algorithm to Σ_{sd} for $T_f = 10$, $\mathcal{X} = [-100, 100]^2$, in order to determine if there is a



$$\mathcal{S}_0 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} -50 \\ -50 \\ -50 \\ -50 \end{bmatrix} \leq 0 \right\},$$
$$\mathcal{S}_1 := \left\{ x \in \mathcal{R}^2 \,\middle|\, [\, 1 \;\; 0 \,]\, x + 50 < 0 \right\},$$
$$\mathcal{S}_2 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{array}{l} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} -50 \\ -50 \end{bmatrix} \leq 0 \\ [\, 0 \;\; 1 \,]\, x + 50 < 0 \end{array} \right\},$$
$$\mathcal{S}_3 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{array}{l} [\, 0 \;\; 1 \,]\, x - 50 \leq 0 \\ [\, -1 \;\; 0 \,]\, x + 50 < 0 \end{array} \right\},$$
$$\mathcal{S}_4 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{array}{l} [\, -1 \;\; 0 \,]\, x - 50 \leq 0 \\ [\, 0 \;\; -1 \,]\, x + 50 < 0 \end{array} \right\}.$$

Fig. 2. Subregions of the continuous state assigned to each value of the discrete state.

TABLE I
RESULT OF PROBABILISTIC CONTROLLABILITY ANALYSIS.

Value of ζ	0	1
Result	Not $(\mathcal{X}, T_f)$-controllable	$(\mathcal{X}, T_f)$-controllable
Computation time [sec] (min. / mean / max.)	804 / 959 / 1,107	202 / 233 / 260
N_{LP} [times] (min. / mean / max.)	27,063 / 32,929 / 38,778	5,398 / 6,325 / 7,550

probability more than 99.9 [%] that $\mathrm{vol}(\mathcal{X} - \mathcal{X}_0) / \mathrm{vol}(\mathcal{X}) \leq 0.1$ [%]. So we set $\varepsilon = 0.001$, $\delta = 0.001$, and $\tilde{N}_s = 3,500$ by (18). We used MATLAB on the computer with the Intel Pentium 4 2.20GHz processor and the 768MB memory and the techniques of Lemmas 5–7 are used. Table I shows the numerical results based on ten trials. For each case, the algorithm answered the same result in every trial.

From (16), the computation amount is given by $7,000 \leq N_{LP} \leq 13,672,187,500$. These values are the case where we do not use any techniques of Lemmas 5–7. However, if these lemmas are applied, we can see that the actual number N_{LP} in Table I for each case is around the lower bound of the estimated times. In contrast, note that, if we apply the deterministic method in section III-C to these examples, in worst case, we will have to solve 9,765,625 mp-LP problems with 2-dimensional parameter and at least 19 variables and 10 constraints. Thus it will be hopeless to get any solutions.

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