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On a Class of Controlled Invariant Sets

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***Abstract*— In this paper we introduce a new class of controlled invariant sets, called controllable invariant sets. Intuitively, a controllable invariant set has the property that from any "large enough" connected region of the set it is possible to reach any such other region of the set, regardless of disturbances. Disturbances are assumed to be bounded. The range of the control inputs is assumed to be given and is allowed to be bounded. The main result of the paper is a nonrecursive approach for the computation of controllable invariants. The other results of the paper deal with properties of the proposed method and of controllable invariance. The results of the paper assume hybrid system modes with linear discrete-time dynamics.**

I. INTRODUCTION

Controlled invariant sets have been used in the hybrid systems literature for the solution to the safety problem (e.g. [7]). This paper introduces a new class of controlled invariant sets, called controllable invariant sets. The context is that of mode dynamics of hybrid systems with control inputs and bounded disturbances. The controllable invariant sets are defined as follows. Given a closed neighborhood of the origin Ω, let Ω_x denote the neighborhood Ω around x (i.e. $\Omega_x = \{y : y - x \in \Omega\}$). Then J is a controllable invariant set if Ω_x is a controlled invariant at all points x of J, and for all points x_1 and x_2 of J it is possible to reach Ω_{x_2} from any point of Ω_{x_1}, regardless of disturbances. Thus, this definition implies a certain reversibility, meaning that as long as we keep the state within J, it is always possible to return it to the initial condition (i.e., initial neighborhood). Such a reversibility fits most engineering systems. Note that in general the controllable invariant sets are proper subsets of a maximum controlled invariant set. In practice, one factor that may cause a controlled invariant set to be not controllable is the bounded range of the control inputs.

In this paper we approach the computation of the controllable invariant sets for linear discrete-time dynamics and rectangular neighborhoods Ω. The computational approach is nearly optimal, in the sense that the controllable invariant set J that is obtained is an open set whose closure $\overline{J}$ contains all controllable invariant sets with the same neighborhood type Ω as J. The computation involves linear programming and projections (also known as Fourier-Motzkin eliminations). Related approaches have been used in [5], [10] for predecessor operator computations, in [3], [11] for the computation of the maximal controlled invariant set, and also in other contexts, e.g. [4], [2]. Note that some model uncertainties could be incorporated in this framework [6]. To our knowledge, the entire material of this paper is new.

The paper is organized as follows. After presenting our notation and definitions in section II, a motivation is presented in section III. The motivation shows the relevance of the controllable invariant sets to a hybrid system abstraction problem. Then, the computation is approached in section IV. The approach is formally proved in the same section. Section IV includes also an investigation of the properties of the computational approach and of the controllable invariant sets, in general.

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II. DEFINITIONS

This is our notation. Given a hybrid system of set of modes Q, we denote by $Inv(q)$ the invariant set of the mode $q \in Q$. Also, let X denote the domain of the continuous state variable x. In this paper we assume that the dynamics of each mode q can be described by

$$x(t+1) = A(q)x(t) + B(q)u(t) + E(q)d(t) \qquad (1)$$

where u is the control input and d is the disturbance, which will be assumed bounded. For each mode q, we define the following

- The operator *Pre* represents the *predecessor operator*. That is, $Pre(M)$ is the set of continuous states from which M can be robustly reached. In other words, $\forall x_0 \in Pre(M)$ there is a control policy (which may depend on x_0) which, no matter of disturbances, leads the continuous state x from x_0 to some $x_f \in M$.
- $I \subseteq Inv(q)$ is a **controlled invariant set** if for all $x \in I$ there is an admissible control law such that for all subsequent times t: $x(t) \in I$, regardless of the disturbance input.
- Let $Reach : X \to \mathcal{P}(\mathcal{P}(X))$, where for $M \subset X$ we have $M \in Reach(x)$ if it is possible to robustly reach M starting from x (i.e. no matter of disturbances, it is possible to reach M from x.)[1] In other words $Reach(x)$ is the collection of sets M with the property that it is possible to robustly reach M from x.

In this paper, we introduce the following class of controlled invariant sets, that we call controllable invariant sets. Let Ω^o denote the interior of Ω.

Definition 2.1 *Given a set $\Omega \subset \mathbb{R}^n$, let $\Omega_x = \{z \in \mathbb{R}^n : \exists y \in \Omega,\, z = y + x\}$. For some $q \in Q$ we say that $I \subseteq Inv(q)$ is a* **controllable invariant set** *if a connected compact set $\Omega \subset \mathbb{R}^n$ exists such that $0 \in \Omega^o$ and*

[1] $\mathcal{P}(Y) = \{E : E \subseteq Y\}$ denotes the collection of all subsets of Y.

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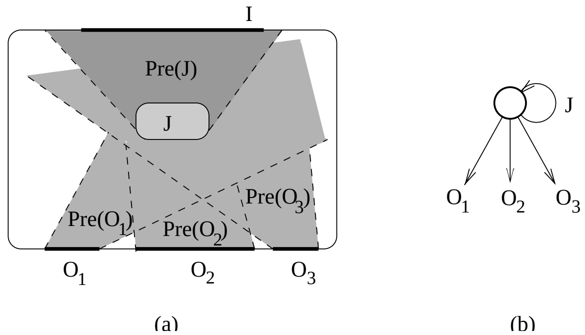

Fig. 1. Illustration of a desirable situation in the controlled behavior of a hybrid system. (a) A hybrid system mode with input set I and output sets O_1, O_2 and O_3 corresponding to the thick lines, controlled invariant set J and $Pre(O_1)$, $Pre(O_2)$, $Pre(O_3)$ and $Pre(J)$ represented through the shaded areas. (b) Equivalent DE abstraction of the mode, where the selfloop corresponds to J and the other transitions to the transitions exiting O_1, O_2 and O_3.

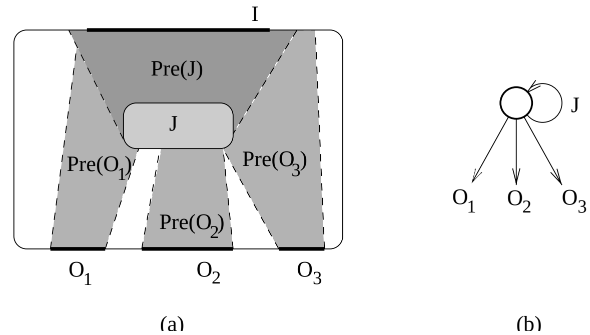

Fig. 2. Illustration of another desirable situation in the controlled behavior of a hybrid system. (a) A hybrid system mode with input set I and output sets O_1, O_2 and O_3 corresponding to the thick lines, controllable invariant set J and $Pre(O_1)$, $Pre(O_2)$, $Pre(O_3)$ and $Pre(J)$ represented through the shaded areas. (b) Equivalent DE abstraction of the mode, where the selfloop corresponds to J and the other transitions to the transitions exiting O_1, O_2 and O_3.

1) $\forall x \in I$: Ω_x *is a controlled invariant set*
2) $\bigcup_{x \in I} \Omega_x \subseteq Inv(q)$
3) $\forall x_1, x_2 \in I$, $\forall x \in \Omega_{x_1}$: $\Omega_{x_2} \in Reach(x)$.

III. Motivation

The ability to move between any desirable setpoints is clearly an interesting benefit of the controllable invariant sets. In this section we show that the controllable invariant sets can be useful also in the discrete-event (DE) abstraction of hybrid systems.

Given (Q, Edg), the state machine of a hybrid automaton with time-invariant mode-dynamics, consider the abstractions (Q', Edg') with the following property. There are maps $\nu : Q' \to Q$ and $\chi : Q' \to X$, such that if $(q_1', q_2') \in Edg'$ is a controllable transition, then $\forall x_1 \in \chi(q_1')$, there is a control law yielding a trajectory from $(q_1, x_1, 0)$ to (q_2, x_2, t_2) for some $x_2 \in \chi(q_2')$ and time $t_2 \geq 0$, regardless of disturbances, and at all intermediary states (q, x, t), $0 \leq t < t_2$, it is true that $q = q_1$ and $x \in \chi(q_1')$. Note that q_1 and q_2 denote $\nu(q_1')$ and $\nu(q_2')$, respectively.

A process by which such abstractions could be found is not presented in this paper. However, note two favorable situations an abstraction process should take advantage of. First, we define for every mode $q \in Q$ the following sets:

(i) $J_q \subseteq Inv(q) \cap Safe(q)$, where $Safe(q)$ is the set specifying the safety specification for the mode q (that is, $Inv(q) \setminus Safe(q)$ is the forbidden state set of the mode q.)
(ii) For every $(q, q') \in Edg$, let $O_{q \to q'} \subseteq Inv(q) \cap Safe(q)$ denote the continuous states for which there is an input leading the system from q to q', no matter of disturbances.
(iii) Let I_q be the set of continuous states in which the mode q may be entered from the modes q_c such that $(q_c, q) \in Edg$.
(iv) Let's write $q' \in q \to$ if q and q' satisfy $(q, q') \in Edg$.

Note that the set I_q could be reduced by an appropriate control law. An ideal situation for the DE abstraction is when for all $q \in Q$ there is J_q such that:

(a) J_q is a controlled invariant set.
(b) $I_q \subseteq Pre(J_q)$.
(c) $J_q \subseteq \bigcap_{q' \in q \to} Pre(O_{q \to q'})$.

This situation is illustrated in Figure 1, together with the DE abstraction of the mode. Thus, once we have the sets I_q and $O_{q \to q'}$, we are interested to compute the maximal controlled invariant set J_q satisfying (i) and (c). Indeed, if the maximal controlled invariant set does not satisfy (b), there is no controlled invariant set J_q satisfying (a-c). However, even when (b) is not satisfied, we may still be able to reduce the set I_q (through a control law) such that (b) is satisfied. An interesting variant of the requirements (a-c) is given below:

(a′) J_q is a controllable invariant set with a set Ω such that $\bigcup_{x \in J_q} \Omega_x \subseteq Inv(q) \cap Safe(q)$.
(b′) $I_q \subseteq Pre(J_q)$.
(c′) $\forall q' \in q \to \exists x \in J_q$: $\Omega_x \subseteq Pre(O_{q \to q'})$.

This situation is illustrated in Figure 2, together with the DE abstraction of the mode. Again, once we have the sets I_q and $O_{q \to q'}$, we are interested to compute a controllable invariant set J_q satisfying (a′) and (c′). This can be achieved by computing a maximal controllable invariant set satisfying (a′). Then, if (c′) is not satisfied and J_q is maximal no solution to (a′-c′) exists, but if (b′) is not satisfied, we may still be able to reduce the set I_q.

Note that the conditions (a′-c′) are a variant of (a-c). Indeed, by the definition of the controllable invariant set, (c′) implies $J_q \subseteq Pre(O_{q \to q'})$. Thus $J_q \subseteq \bigcap_{q' \in q \to} Pre(O_{q \to q'})$. Further, every controllable invariant set is a controlled invariant set. (However, the converse is not true.) The (a′-c′) variant may be computationally advantageous when it is not easy to compute $Pre(O_{q \to q'})$; then we do not need to compute the whole sets $Pre(O_{q \to q'})$, but only to show that they intersect J_q as shown at (c′). This quality may be of interest especially in the discrete-time case, in which the computation of the predecessor operator is iterative and may not terminate. Note also that here the controllable invariant set is computed first, and then the predecessor sets. On the other hand, in the previous situation the maximal controlled invariant set was computed only after the computation of the predecessor sets.

IV. Computation

A. The Intuition

Considering a system of dynamics

$$x(t+1) = Ax(t) + Bu(t) + Ed(t) \tag{2}$$

if the system is stabilizable, there is a state feedback controller $u = Kx$ such that the system is stable. Furthermore, for each $u = Kx + r$, where r is a constant, there is a point x^* to which (in the absence of disturbances) the state converges. Intuitively it is clear that there is a region of attraction around x^*, such that no matter of the disturbances (which are assumed to be bounded), that region is invariant for the given r. Furthermore, if each such point x^* has a region of attraction, the state x can be moved from one region to another. Indeed, if x is in the region of (x_1^*, r_1), by applying the control $u = Kx + r_2$ we can move it to the region of (x_2^*, r_2). Also, in order to keep the control $u = Kx + r$ within its admissible domain $\mathcal{U}$, we can "slowly" change r from r_1 to r_2. Therefore, the controllable invariant set would correspond to the points x^*. While linear state feedback was used in this illustration, we are not going to refer to it in what follows. We consider a more general state feedback solution.

B. The Computation

We consider the dynamics of equation (2) and sets Ω (see Definition 2.1) of the form $\Omega = \{x : |x| \leq b\}$ where $b \in \mathbb{R}^n$ and $b > 0$. Recall, given x^*, $\Omega_{x^*} = \{x : |x - x^*| \leq b\}$. Let $\mathcal{U}$ denote the domain of the control input and $\mathcal{D}$ the (bounded) domain of the disturbance.

Given x_1^*, the set of points x_2^* satisfying that $\exists u(t) \in \mathcal{U}$ $\forall x(t) \in \Omega_{x_1^*}$: $x(t+1) \in \Omega_{x_2^*}$ can be expressed as

$$\exists u \in \mathcal{U}, \forall x \in \Omega_{x_1^*} : \left\{ \begin{array}{lcl} Ax + Bu + d^+ & \leq & x_2^* + b \\ Ax + Bu - d^- & \geq & x_2^* - b \end{array} \right. \tag{3}$$

where $x = x(t)$, $u = u(t)$, $d^+ = \max_{d \in \mathcal{D}} Ed$, and $d^- = -\min_{d \in \mathcal{D}} Ed$ and the maximum/minimum is taken separately on each row of Ed. Note that the requirement that $\Omega_{x_1^*}$ be invariant corresponds to (3) when $x_2^* = x_1^*$.

Assuming a convex domain $\mathcal{U} = \{u : L_u u \leq b_u\}$, the input u can be eliminated from (3) using the Fourier-Motzkin elimination [2] (FME) [8], [9]. The result is of the form:

$$\forall x \in \Omega_{x_1^*} : Gx + Hb + Mx_2^* \leq g \tag{4}$$

[2]Given a system of inequalities $\Delta x \leq \gamma$ and a variable x_j to be eliminated, the FME generates a new system $\Delta' x \leq \gamma'$ in which x_j does not appear. The new system contains the inequalities of $\Delta x \leq \gamma$ that do not involve x_j, and the inequalities obtained from all pairs of inequalities i and k with $\Delta_{ij} > 0$ and $\Delta_{kj} < 0$, by a weighted sum with appropriate positive weights. Geometrically, $\Delta' x \leq \gamma'$ describes the projection of the polyhedron $\Delta x \leq \gamma$ on the hyperplanes $x_j = a$, $a \in \mathbb{R}$.

or

$$\forall \alpha \in \{x : |x| \leq b\} : (G+M)x_1^* + G\alpha + Hb + M\beta \leq g \tag{5}$$

where $\alpha = x - x_1^*$ and $\beta = x_2^* - x_1^*$. Note that α can too be eliminated, as $\max_{\alpha \in \{x:|x| \leq b\}} G\alpha = |G|b$, where the maximum is taken separately on each row of $G\alpha$, and $|G| = [|G_{ij}|]$ denotes the absolute value of G. We obtain:

$$(G+M)x_1^* + (|G| + H)b + M\beta \leq g \tag{6}$$

To satisfy (6) for all $\beta \in [-\delta, \delta]$, where $\delta \in \mathbb{R}^n$, $\delta \geq 0$, is given, the following constraint is obtained:

$$(G+M)x_1^* + (|G| + H)b + |M|\delta \leq g \tag{7}$$

Note that (7) describes the set of points x_1^* such that $\Omega_{x_1^*}$ is a controlled invariant and from all points $x \in \Omega_{x_1^*}$ it is possible to reach any $\Omega_{x_2^*}$ with $|x_2^* - x_1^*| \leq \delta$ in one time step. Obviously, we would like this set of points x_1^* to be as large as possible. At the same time, we are also interested in having the sets Ω_{x^*} as small as possible (i.e., b as small as possible). In view of (3) the minimum value of b is:

$$b \geq \frac{d^+ + d^-}{2} \tag{8}$$

On the other hand, the minimum value of δ is 0. From (7) with $\delta = 0$ we derive the controllable invariant set

$$(G+M)x + (|G| + H)b < g \tag{9}$$

Note that $<$ denotes strict inequality on all elements, that is, $y < z \Rightarrow y_i < z_i$ for all indices i.

Example 4.1 Assume a system described by the dynamics

$$x(t+1) = ax(t) + u(t) + d(t) \tag{10}$$

where $a \in \mathbb{R}$. Assume $d^+ = d^- = d_0$ and the control input domain $-u_0 \leq u \leq u_0$. The relation (3) can be written as

$$\exists u \in \mathcal{U}, \forall x \in \Omega_{x_1^*} : \left\{ \begin{array}{rcr} ax + u + d_0 & \leq & x_2^* + b \\ -ax - u + d_0 & \leq & -x_2^* + b \end{array} \right. \tag{11}$$

Then (4) becomes

$$\forall x \in \Omega_{x_1^*} : \left\{ \begin{array}{rcr} d_0 & \leq & b \\ ax + d_0 & \leq & x_2^* + u_0 + b \\ -ax + d_0 & \leq & -x_2^* + u_0 + b \end{array} \right. \tag{12}$$

while (7) is

$$\left\{ \begin{array}{rcr} d_0 & \leq & b \\ (a-1)x_1^* + (|a|-1)b + \delta + d_0 & \leq & u_0 \\ (-a+1)x_1^* + (|a|-1)b + \delta + d_0 & \leq & u_0 \end{array} \right. \tag{13}$$

We see that there is no solution unless $|a|d_0 < u_0$ or $|a| < 1$. Once these conditions are satisfied, the controllable invariant is given by:

$$\left\{ \begin{array}{rcl} (a-1)x + (|a|-1)b + d_0 & < & u_0 \\ (-a+1)x + (|a|-1)b + d_0 & < & u_0 \end{array} \right. \tag{14}$$

for a b such that $b \geq d_0$ and $(|a|-1)b + d_0 < u_0$. □

The following results establish properties of the controllable invariant sets computed this way. Notably, we prove that (9) describes a controllable invariant set and that, with the possible exception of (some of) its boundary, it coincides with the maximal controllable invariant set with $\Omega = \{x : |x| \leq b\}$. For the moment, we assume that in Definition 2.1 $Inv(q) = \mathbb{R}^n$.

Let $J_\delta = \{x : (G+M)x + (|G|+H)b + |M|\delta \leq g\}$, where the notation of (7) is used.

Proposition 4.1 *The set J_δ is a controllable invariant of set $\Omega = \{x : |x| \leq b\}$.*

Proof: The proof is divided in three parts. Part (a) shows that $\forall x_1^* \in J_\delta\ \forall x_2^* \in [x_1^* - \delta, x_1^* + \delta]\ \forall x(t) \in \Omega_{x_1^*}$ $\exists u \in \mathcal{U}\ \forall d \in \mathcal{D}$: $x(t+1) \in \Omega_{x_2^*}$. Part (b) shows that Ω_{x^*} is a controlled invariant for all $x^* \in J_\delta$. Part (c) shows that $\forall x_1^*, x_2^* \in J_\delta\ \forall x \in \Omega_{x_1^*}$: $\Omega_{x_2^*} \in Reach(x)$.

(a) Let $\alpha = x - x_1^*$ and $\beta = x_2^* - x_1^*$. From $\beta \leq \delta$ we get that $M\beta \leq |M|\delta$. Since x_1^* satisfies (7), it follows that (6) is also satisfied. Similarly, we derive $(G+M)x_1^* + G\alpha + Hb + M\beta \leq g$, and so $Gx + Hb + Mx_2^* \leq g$. However, this is the projection of

$$\left\{ \begin{array}{lll} Ax + Bu + d^+ & \leq & x_2^* + b \\ Ax + Bu - d^- & \geq & x_2^* - b \end{array} \right. \tag{15}$$

that removes the variable $u \in \mathcal{U}$. Therefore, there is $u \in \mathcal{U}$ such that (15) is satisfied for the given x and x_2^* $\forall d \in \mathcal{D}$. However, (15) is precisely the condition that some $x(t+1) \in \Omega_{x_2^*}$ is reached from $x(t) = x$ by applying the input u.

(b) This results from (a) for $x_1^* = x_2^* = x^*$.

(c) Let $x_1^*, x_2^* \in J_\delta$ be chosen arbitrarily. Let $n > 0$ be an integer such that $|x_2^* - x_1^*| \leq n\delta$. Let z_0^*, z_1^*, $\ldots$ z_n^* be such that $z_k^* = \frac{k}{n}x_1^* + \frac{n-k}{n}x_2^*$ for $k = 0 \ldots n$. Since J_δ is convex, $z_k^* \in J_\delta$ for all $k = 0 \ldots n$. Further, $|z_{k+1}^* - z_k^*| \leq \delta$ for $k = 0 \ldots n-1$. Then, in view of (a), we reach $\Omega_{x_2^*}$ in n steps by going from $x(t) \in \Omega_{x_1^*}$ to $\Omega_{z_1^*}$, then to $\Omega_{z_2^*}$, and so on to $\Omega_{z_n^*}$. ■

Proposition 4.2 *x^* satisfies $(G+M)x^* + (|G|+H)b \leq g$ if and only if Ω_{x^*} is a controlled invariant.*

Proof: "⇒" Let $x \in \Omega_{x^*}$. From $(G+M)x^* + (|G| + H)b \leq g$ and $G(x - x^*) \leq |G|b$ we get $Gx + Hb + Mx^* \leq g$. Since $Gx + Hb + Mx^* \leq g$ is the projection of

$$\left\{ \begin{array}{lll} Ax + Bu + d^+ & \leq & x^* + b \\ Ax + Bu - d^- & \geq & x^* - b \end{array} \right. \tag{16}$$

that removes the variable $u \in \mathcal{U}$, it follows that there is $u \in \mathcal{U}$ such that when $x(t) = x \in \Omega_{x^*}$, $\forall d \in \mathcal{D}$: $x(t+1) \in \Omega_{x^*}$.

"⇐" If Ω_{x^*} is a controlled invariant, then (3) is satisfied for $x_1^* = x_2^* = x^*$. This is also true of (4) and (6) with $\beta = 0$. So the conclusion follows. ■

Proposition 4.3 *The set $J = \{x : (G+M)x + (|G|+H)b < g\}$ is a controllable invariant.*

Proof: By Proposition 4.2, Ω_x is a controlled invariant for all $x \in J$. It remains to show that for any $x_1^*, x_2^* \in J$, $\Omega_{x_2^*}$ can be reached from any $x \in \Omega_{x_1^*}$. Let $x_1^*, x_2^* \in J$. Note that $\exists \delta_1, \delta_2 > 0$: $(G+M)x_1^* + (|G|+H)b + |M|\delta_1 \leq g$ and $(G+M)x_2^* + (|G|+H)b + |M|\delta_2 \leq g$. Let $\delta = \min(\delta_1, \delta_2)$. It follows that $x_1^*, x_2^* \in J_\delta$, and so the conclusion follows by Proposition 4.1. ■

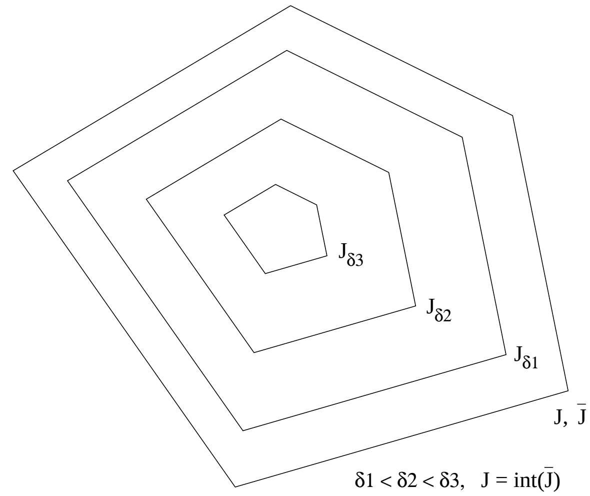

Fig. 3. Illustration of the (inclusion) relation among the sets J_δ, J, and $\overline{J}$.

Proposition 4.4 *All controllable invariant sets of set $\Omega = \{x : |x| \leq b\}$ are subsets of the set $\overline{J} = \{x : (G+M)x + (|G|+H)b \leq g\}$.*

Proof: For any controllable invariant set I, the set Ω_x for $x \in I$ should be a controlled invariant. Therefore, the conclusion follows immediately from Proposition 4.2. ■

Propositions 4.3 and 4.4 indicate that the construction of the controllable set J in (9) is nearly optimal, as all controllable invariant sets I of set Ω satisfy $I \subseteq \overline{J}$. Further, if the maximal controllable set J_m exists, it satisfies $J \subseteq J_m \subseteq \overline{J}$. Note that J is the interior of $\overline{J}$. So J is a very tight approximation of the optimum.

The computation of the set J has been done assuming $Inv(q) = \mathbb{R}^n$. In the general case, a controllable invariant set can be computed as follows. Let $W = \{x \in Inv(q) : \Omega_x \subseteq Inv(q)\}$. Assuming W to be connected, note that a controllable invariant set is $J_0 = J \cap W$. This construction ensures that regardless of the current state x, as long as $x \in \Omega_{x'}$ for some $x' \in J_0$, the state is inside $Inv(q)$.

Note that the computation of the controllable invariant sets J is not recursive (there are no iterations, and so no termination issues). In contrast, the computation of the maximal controlled invariant sets is recursive [3], [10]. Since controllable invariants are also controlled invariants, we could use the approach of this section for a nonrecursive computation of controlled invariants. However, if we are only interested in the computation of controlled invariants, a better (larger) controlled invariant than J and its closure $\overline{J}$ can be obtained by eliminating b from $(G+M)x + (|G| + H)b \leq g$ via FME.

A question to be addressed is what happens in our approach when no nonempty controllable invariant set exists.

To this end, we show that $\overline{J} = \emptyset$ if and only if there is no nonempty controllable invariant of set $\Omega = \{x : |x| \leq b\}$. This result has the weakness that the class of nonempty controllable invariants include the singletons $I = \{x\}$ such that Ω_x is a controlled invariant. Future work is to find conditions in terms of nontrivial controllable invariants, where a controllable invariant set is nontrivial if containing more than one element.

Proposition 4.5 *A nonempty controllable invariant of set $\Omega = \{x : |x| \leq b\}$ exists if and only if $\overline{J} = \{x : (G + M)x + (|G| + H)b \leq g\} \neq \emptyset$.*

Proof: If $\overline{J} \neq \emptyset$ there is $x \in \overline{J}$, and so $I = \{x\}$ is a nonempty (but trivial) controllable invariant set, by definition and Proposition 4.2. On the other hand, if $\overline{J} = \emptyset$, there is no nonempty controllable invariant of set Ω, by Proposition 4.4. ■

The relation between controllability and the existence of nonempty controllable invariant sets is also of interest. First we show that controllability is neither sufficient nor necessary.

Proposition 4.6 *The controllability of (A, B) is neither sufficient nor necessary for the existence of a nonempty controllable invariant set.*

Proof: The proof is by examples. For the nonsufficiency proof, the system (10) is considered with arbitrary sets Ω and "large" disturbances. Then we show that not even for controllable invariants of sets $\Omega = \{x : |x| \leq b\}$ is controllability necessary.

Nonsufficiency: The proof is by contradiction. Assume a nonempty controllable invariant set I exists. Let $x^* \in I$ and $x(0) \in \Omega_{x^*}$ such that $x(0) \neq 0$. Without loss of generality, assume $x(0) > 0$. Now consider the system (10) with $a > 1$ and $d_0 > u_0$. The system $(a, 1)$ is controllable. Assume $d(k) = d_0$ for all k. Then, $x(t) = a^t x(0) + \sum_{k=1}^{t} a^{t-k}(u(k-1) + d(k-1))$, and so $x(t) \to \infty$ as $k \to \infty$, regardless of $u(k)$. It follows the input cannot keep the state in Ω_{x^*}, so I cannot be a controllable invariant.

Non-necessity: Assume a system consisting of two state variables x and x', where x obeys (10) and $x'(t+1) = a'x'(t) + d'(t)$, $-d_0 \leq d'(t) \leq d_0$ for all t, and $|a'| < 1$. Clearly, the system is not controllable, as u has no effect on x'. Note that for $b' \geq d_0/(1-|a'|)$ the set $\Omega' = [-b', b']$ is an invariant of $x'(t+1) = a'x'(t) + d'(t)$. It follows that if I is a controllable invariant of set Ω for the system (10), a controllable invariant for our system is $I' = I \times \{0\}$ of set $\Omega \times \Omega'$. Moreover, nonempty controllable invariants I of set $\Omega = \{x : |x| \leq b\}$ can be constructed, as shown in Example 4.1. ■

The fact that controllability is neither sufficient nor necessary may be surprising. Intuitively, nonsufficiency results from the fact that large enough disturbances can render controllability ineffective. On the other hand, in case of partial controllability, a controllable invariant set may be the Cartesian product of the origin and of a portion of the subspace of the state space that can be affected by the control input. Naturally, this would suggest the uncontrollable part of the system should be stable. The next results shows that under common circumstances the existence of a nonempty controllable invariant requires the uncontrollable eigenvalues of the system (A, B) to be in or on the unity circle. Recall, the pair (A, B) can be transformed by a similarity transformation to $(\hat{A}, \hat{B})$ such that

$$\hat{A} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{17}$$

and (A_1, B_1) is controllable [1]. Thus, the eigenvalues of A_2 are called the uncontrollable eigenvalues of (A, B).

Proposition 4.7 *Assume $0 \in \mathcal{D}$. Then a nonempty controllable invariant exists only if all uncontrollable eigenvalues λ of (A, B) satisfy $|\lambda| \leq 1$.*

Proof: Assume a nonempty controllable invariant I of set Ω exists. Let Q be a similarity transformation transforming (A, B) to the standard form (17). We have $\hat{A} = Q^{-1}AQ$, $\hat{B} = Q^{-1}B$, $\hat{E} = Q^{-1}E$ and $\hat{x} = Qx$. Note that (17) has the nonempty controllable invariant $\hat{I} = \{\hat{x} : Q^{-1}\hat{x} \in I\}$ of set $\hat{\Omega} = \{\hat{x} : Q^{-1}\hat{x} \in \Omega\}$. We can write $\hat{x}(t+1) = \hat{A}\hat{x}(t) + \hat{B}u(t) + \hat{E}d(t)$ as $x_1(t+1) = A_1x_1(t) + A_{12}x_2(t) + B_1u(t) + E_1d(t)$ and $x_2(t+1) = A_2x_2(t) + E_2d(t)$ for $\hat{x} = [x_1^T, x_2^T]^T$ and $\hat{E} = [E_1^T, E_2^T]^T$. Let $x^* \in \hat{I}$. Then $\hat{\Omega}_{x^*}$ is a controlled invariant. Let $x(t) \in \Omega_{x^*}$ such that $x_2(t) \neq 0$. Assume $d_2(t+i) = 0\ \forall i = 0 \ldots k$. We have that $x_2(t+k) = A_2^k x_2(t)$. Let λ be an eigenvalue of A_2 and w its left eigenvector. Then $wx_2(t+k) = \lambda^k wx_2(t)$. Since $\hat{\Omega}_{x^*}$ is a controlled invariant, in order to have $x(t+k) \in \hat{\Omega}_{x^*}$ we need a bounded $wx_2(t+k)$ at all k, and so $|\lambda| \leq 1$. ■

V. CONCLUSIONS

This paper has introduced the controllable invariant sets, as a subclass of the controlled invariant sets. A nearly optimal method for the computation of the controllable invariant sets has been proposed. The computation approach assumes linear discrete-time dynamics with bounded disturbances. This approach is very dissimilar to the approaches used for the computation of controlled invariant sets in that it involves no iterations, and so has guaranteed termination. Extensions to classes of nonlinear dynamics are possible, and may be considered in the future work.

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