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Several Extensions in Methods for Adaptive Output Feedback Control

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***Abstract*— Several extensions to neural network based adaptive output feedback control of nonlinear systems are developed. An extension that permits the introduction of e-modification in an error observer based approach is given. For the case of non-affine systems, we eliminate a fixed point assumption that has appeared in earlier work, and clarify the role that knowledge of the sign of control effectiveness plays in adaptive control.**

I. INTRODUCTION

This paper presents several extensions for augmenting a nonlinear controller, designed via input/output feedback linearization, with a neural network (NN) based adaptive element, similar to those described in [1], [2]. NN adaptive control combined with feedback linearization [3], [4] is a popular method for control of nonlinear systems. Extensions of the methods from [3], [4] to state observer based output feedback control are treated in [5], [6]. However, these results are limited to systems with full relative degree (vector relative degree = degree of the system) with the added constraint that the relative degree of each output is less than or equal to two. Moreover, since state observers are employed, the dimension of the plant must be known. Therefore, methods that rely on a state observer are vulnerable to unmodeled dynamics. In [1], a direct adaptive approach is developed that does not make use of a state observer and uses linearly-parameterized NNs to compensate for modeling errors. In [2], these same limitations are overcome by employing an error observer, in place of a state observer. The only requirement in the latter two approaches is that the relative degree of the regulated output be known. The adaptive laws in the approaches in [1], [2] have been derived using σ-modification. This paper provides proof of boundedness using e-modification.

Another issue concerns knowledge of the sign of control effectiveness, which is a common assumption in adaptive control. For an affine system with constant control effectiveness it is not difficult to show that knowledge of the sign of control effectiveness is needed to obtain a reasonable adaptive law. In [1], [2], [7], this issue is addressed for the case of non-affine systems by introducing a fixed point assumption for the mapping from the adaptive signal to the modeling error. In [1] it is shown that this mapping is a contraction if and only if the sign of the control effectiveness is known and greater in magnitude than half the actual value. Thus the requirement for knowledge of the sign of the control effectiveness does not appear explicitly in the stability analysis. Furthermore, the contraction mapping assumption may be overly conservative. In [8] stability analysis for a non-affine system for the case of state feedback is performed utilizing the mean value theorem. In this approach the requirement for knowledge of the sign of control effectiveness is explicit. The extension to output feedback employing a high-gain observer is given in [9]. Here we adopt this approach to derive adaptation laws for the formulations given in [1], [2], thereby removing the contraction mapping condition.

The next section states the output feedback control problem and clarifies the role of the sign of control effectiveness in adaptive control of non-affine systems. Section III briefly explains the NN universal approximation property. Section IV provides an extension of the error observer approach along with boundedness proofs. Section V shows simulation results of the proposed methods for a modified Van der Pol oscillator, and Section VI summarizes the results and concludes this study.

II. PLANT DESCRIPTION

Consider the following *observable* and *stabilizable* nonlinear SISO system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, u) \\ y &= h(\boldsymbol{x}) \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is the state of the system on a domain $\mathcal{D}_x \subset \Re^n$, and $u, y \in \Re$ are the control and regulated output variables, respectively. The functions $\boldsymbol{f}$ and h may be unknown.

Assumption II.1 *The functions $\boldsymbol{f} : \mathcal{D}_x \times \Re \to \Re^n$ and $h : \mathcal{D}_x \to \Re$ are input/output feedback linearizable [10], and the output y has relative degree r for all $(\boldsymbol{x}, u) \in \mathcal{D}_x \times \Re$.*

Based on this assumption, the system (1) can be transformed into normal form [11]

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= \boldsymbol{f}_0(\boldsymbol{\xi}, \boldsymbol{\chi}) \\ \dot{\xi}_i &= \xi_{i+1} \qquad i = 1, \cdots, r-1 \\ \dot{\xi}_r &= h_r(\boldsymbol{\xi}, \boldsymbol{\chi}, u) \\ y &= \xi_1 \end{aligned} \tag{2}$$

where $\boldsymbol{\xi} = [\ \xi_1 \quad \dots \quad \xi_r\ ]^T$, $h_r(\boldsymbol{\xi}, \boldsymbol{\chi}, u) = L_f^r h$,and $\boldsymbol{\chi}$ is the state vector associated with the internal dynamics

$$\dot{\boldsymbol{\chi}} = \boldsymbol{f}_0(\boldsymbol{\xi}, \boldsymbol{\chi}) \tag{3}$$

Assumption II.2 *The internal dynamics in (3), with $\boldsymbol{\xi}$ viewed as input, are input-to-state stable. [12]*

Assumption II.3 *$\partial h_r(\boldsymbol{x}, u)/\partial u$ is continuous and non-zero for every $(\boldsymbol{x}, u) \in \mathcal{D}_x \times \Re$ and its sign is known.*

The control objective is to synthesize an output feedback control law such that $y(t)$ tracks a smooth reference model trajectory

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$y_{rm}(t)$ within bounded error. Let $\hat{h}_r(y,u)$ denote an approximate model for $h_r(\boldsymbol{x},u)$ so that

$$h_r(\boldsymbol{x},u) = \hat{h}_r(y,u) + \Delta \tag{4}$$

where the modeling error is $\Delta(\boldsymbol{x},u) = h_r(\boldsymbol{x},u) - \hat{h}_r(y,u)$.

Assumption II.4 *$\partial\hat{h}_r(y,u)/\partial u$ is continuous and non-zero for every $(y,u) \in \mathcal{D}_y \times \Re$.*

Let the approximate function be recast as

$$v = \hat{h}_r(y,u) \tag{5}$$

where v is called pseudo-control. Then the control law can be defined directly from (5)

$$u = \hat{h}_r^{-1}(y,v) \tag{6}$$

The pseudo-control is composed of three signals:

$$v \triangleq y_{rm}^{(r)} + v_{dc} - v_{ad} \tag{7}$$

where $y_{rm}^{(r)}$ is the r^{th} time derivative of $y_{rm}(t)$, v_{dc} is the output of a linear dynamic compensator, and v_{ad} is an adaptive term designed to cancel the modeling error.

The reference model can be expressed in state space form as

$$\begin{aligned} \dot{\boldsymbol{x}}_{rm} &= A_{rm}\boldsymbol{x}_{rm} + \boldsymbol{b}_{rm}y_{com} \\ y_{rm} &= C_{rm}\boldsymbol{x}_{rm} \end{aligned} \tag{8}$$

$$\boldsymbol{x}_{rm} \triangleq \begin{bmatrix} x_{rm} & \dot{x}_{rm} & \cdots & x_{rm}^{(r-1)} \end{bmatrix}^T$$

$$A_{rm} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_r \end{bmatrix}, \boldsymbol{b}_{rm} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a_1 \end{bmatrix},$$

$$C_{rm} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where $\boldsymbol{x}_{rm} \in \Re^r$ is the state vector of the reference model, $y_{com} \in \Re$ is a bounded external command signal, and A_{rm} is Hurwitz.

Assumption II.5 *$y_{com}(t)$ is uniformly bounded so that*

$$\|y_{com}(t)\| \leqslant y_{com}^*, \quad y_{com}^* > 0$$

Let $e \triangleq y_{rm} - y$. Then

$$e^{(r)} = -v_{dc} + v_{ad} - \Delta \tag{9}$$

For the case $r > 1$, the following linear dynamic compensator is introduced to stabilize the dynamics in (9):

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= A_c\boldsymbol{\eta} + \boldsymbol{b}_c e, \quad \boldsymbol{\eta} \in \Re^{n_c} \\ v_{dc} &= \boldsymbol{c}_c\boldsymbol{\eta} + d_c e \end{aligned} \tag{10}$$

where n_c is the order of the compensator. The vector $\mathbf{e} = [\begin{array}{cccc} e & \dot{e} & \cdots & e^{(r-1)} \end{array}]^T$ together with the compensator state $\boldsymbol{\eta}$ will obey the following error dynamics:

$$\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\mathbf{e}} \end{bmatrix} &= \underbrace{\begin{bmatrix} A_c & \boldsymbol{b}_c\boldsymbol{c} \\ -\mathbf{b}\boldsymbol{c}_c & A - \mathbf{b}d_c\boldsymbol{c} \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{e} \end{bmatrix}}_{\boldsymbol{E}} + \underbrace{\begin{bmatrix} \mathbf{0}_{n_c\times 1} \\ \mathbf{b} \end{bmatrix}}_{\bar{\boldsymbol{b}}} (v_{ad} - \Delta) \\ \boldsymbol{z} &= \underbrace{\begin{bmatrix} I_{n_c} & 0_{n_c\times r} \\ 0_{1\times n_c} & \boldsymbol{c} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\eta} \\ e \end{bmatrix} \end{aligned} \tag{11}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \Re^{r\times r}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re^{r\times 1},$$

$$\boldsymbol{c} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \Re^{1\times r},$$

and $\boldsymbol{z}$ is a vector of available signals. With these definitions, the tracking error dynamics in (11) can be rewritten in a compact form:

$$\begin{aligned} \dot{\boldsymbol{E}} &= \bar{A}\boldsymbol{E} + \bar{\boldsymbol{b}}(v_{ad} - \Delta) \\ \boldsymbol{z} &= \bar{C}\boldsymbol{E} \end{aligned} \tag{12}$$

where $A_c, \boldsymbol{b}_c, \boldsymbol{c}_c, d_c$ should be designed such that $\bar{A}$ is Hurwitz.

Reference [1] points out that Δ depends on v_{ad} through (6) and (7) and that v_{ad} is designed to cancel Δ. A contraction assumption is introduced to guarantee the existence and uniqueness of a solution for v_{ad} such that $v_{ad} = \Delta(\cdot, v_{ad})$. As can be seen, the contraction mapping assumption is satisfied if and only if $\text{sgn}(\frac{\partial\hat{h}_r(y,u)}{\partial u}) = \text{sgn}(\frac{\partial h_r(\boldsymbol{x},u)}{\partial u})$ and $\left|\frac{\partial\hat{h}_r}{\partial u}\right| > \frac{1}{2}\left|\frac{\partial h_r}{\partial u}\right|$. However, knowledge of the sign of the control effectiveness is not employed in the stability analysis.

Define the following signals

$$\begin{aligned} v_l &\triangleq y_{rm}^{(r)} + v_{dc} \\ v^* &\triangleq \hat{h}_r(y, h_r^{-1}(\boldsymbol{x}, v_l)) \end{aligned} \tag{13}$$

Invertibility of $h_r(\boldsymbol{x},u)$ with respect to its second argument is guaranteed by Assumption II.3. From (13), it follows that v_l can be written as

$$v_l = h_r(\boldsymbol{x}, \hat{h}_r^{-1}(y, v^*)) \tag{14}$$

and

$$\begin{aligned} v_{ad} - \Delta(\boldsymbol{x},u) &= v_{ad} - h_r(\boldsymbol{x},u) + \hat{h}_r(y,u) \\ &= v_{ad} - h_r(\boldsymbol{x}, \hat{h}_r^{-1}(y,v)) + v_l - v_{ad} \\ &= -h_r(\boldsymbol{x}, \hat{h}_r^{-1}(y,v)) + h_r(\boldsymbol{x}, \hat{h}_r^{-1}(y,v^*)) \end{aligned} \tag{15}$$

Applying the mean value theorem [8], [9] to (15),

$$\begin{aligned} v_{ad} - \Delta &= h_{\bar{v}}(v^* - v) \\ &= h_{\bar{v}}[\hat{h}_r(y, h_r^{-1}(\boldsymbol{x}, v_l)) - v_l + v_{ad}] \\ &= h_{\bar{v}}[v_{ad} - \bar{\Delta}(\boldsymbol{x}, v_l)] \end{aligned} \tag{16}$$

where $\bar{\Delta} = v_l - \hat{h}_r(y, h_r^{-1}(\boldsymbol{x}, v_l))$ and

$$h_{\bar{v}} \triangleq \left.\frac{\partial h_r}{\partial u}\frac{\partial u}{\partial v}\right|_{v=\bar{v}}, \ \bar{v} = \theta v + (1-\theta)v^*, \text{ and } 0 \leqslant \theta(v) \leqslant 1$$

Assumptions II.3 and II.4 indicate that $h_{\bar{v}} = \frac{\partial h_r}{\partial u} / \frac{\partial\hat{h}_r}{\partial u}|_{v=\bar{v}}$ is either strictly positive or strictly negative.

Assumption II.6 *$h_{\bar{v}}$ and $\frac{d}{dt}\left(\frac{1}{h_{\bar{v}}}\right)$ are continuous functions in $\mathcal{D}$.*

According to this assumption, we can define

$$h^B \triangleq \max_{\boldsymbol{x},u\in\mathcal{D}} |h_{\bar{v}}|, \ H \triangleq \max_{\boldsymbol{x},u\in\mathcal{D}} \left|\frac{d}{dt}\left(\frac{1}{h_{\bar{v}}}\right)\right| \tag{17}$$

Now we have the following error dynamics.

$$\dot{\boldsymbol{E}} = \bar{A}\boldsymbol{E} + \bar{\boldsymbol{b}}h_{\bar{v}}(v_{ad} - \bar{\Delta}(\boldsymbol{x}, v_l)) \tag{18}$$

Since $\bar{A}$ is Hurwitz, then for any $Q > 0$, there exists a unique $P > 0$ that solves the Lyapunov equation: $\bar{A}^T P + P\bar{A} = -Q$.

The adaptive term in (18) is designed as

$$v_{ad} = \hat{W}^T \boldsymbol{\sigma}(\hat{V}^T \boldsymbol{\mu}) \tag{19}$$

where $\hat{W}$ and $\hat{V}$ are the NN weights to be updated online in accordance with one of the weight adaptation laws presented in Section IV.

III. NN Approximation of the Inversion Error

The term "artificial NN" has come to mean any architecture that has massively parallel interconnections of simple "neural" processors [13]. Given $\boldsymbol{x} \in \mathcal{D} \subset \Re^{n_1}$, a nonlinearly-parameterized (three layer) NN has an output given by

$$y_i = \sum_{j=1}^{n_2} \left[w_{ji}\ \sigma_j \left(\sum_{k=1}^{n_1} v_{kj} x_k + \theta_{vj} \right) \right] + \theta_{wi}, \quad i = 1, \ldots, n_3 \tag{20}$$

where $\sigma_j(\cdot)$ is an activation function defined as $\sigma_j(z_j) = \frac{1}{1+e^{-a_j z_j}}$, v_{kj} are the first-to-second layer interconnection weights, w_{ji} are the second-to-third layer interconnection weights, and θ_{vj}, θ_{wi} are bias terms. Such an architecture is known to be a universal approximator of continuous nonlinearities with "squashing" activation functions [14], [15], [16]. This implies that a continuous function $\boldsymbol{g}(\boldsymbol{x})$ with $\boldsymbol{x} \in \mathcal{D} \subset \Re^{n_1}$ can be written as

$$\boldsymbol{g}(\boldsymbol{x}) = W^T \boldsymbol{\sigma}(V^T \boldsymbol{x}) + \boldsymbol{\epsilon}(\boldsymbol{x}) \tag{21}$$

where $\mathcal{D}$ is a compact set and $\boldsymbol{\epsilon}(\boldsymbol{x})$ is the function reconstruction error (also called "representation error" or "approximation error"). In general, given a constant real number $\epsilon^* > 0$, $\boldsymbol{g}(\boldsymbol{x})$ is within ϵ^* range of the NN if there exist constant weights V, W, such that for all $\boldsymbol{x} \in \mathcal{D} \subset \Re^{n_1}$, (21) holds with $\|\boldsymbol{\epsilon}\| < \epsilon^*$. The following theorem extends the results found in [14], [15], [16] to map the unknown dynamics of an *observable* plant from available input/output history.

Theorem III.1 *[17] Given $\epsilon^* > 0$ and the compact set $\mathcal{D} \subset \mathcal{D}_x \times \Re$, there exists a set of bounded weights V, W and n_2 sufficiently large such that a continuous function $\bar{\Delta}(\boldsymbol{x}, v_l)$ can be approximated by a nonlinearly-parameterized NN*

$$\begin{aligned} &\bar{\Delta}(\boldsymbol{x}, v_l) = W^T \boldsymbol{\sigma}(V^T \boldsymbol{\mu}) + \epsilon(\boldsymbol{\mu}, d), \\ &\quad \|W\|_F < W^*, \quad \|V\|_F < V^*, \quad \|\epsilon(\boldsymbol{\mu}, d)\| < \epsilon^* \end{aligned} \tag{22}$$

using the input vector

$$\boldsymbol{\mu}(t) = \begin{bmatrix} 1 & v_l & \boldsymbol{v}_d^T(t) & \boldsymbol{y}_d^T(t) \end{bmatrix}^T \in \Re^{2N_1 - r + 2}, \quad \|\boldsymbol{\mu}\| \leqslant \mu^*$$

where

$$\begin{aligned} \boldsymbol{v}_d(t) &= \begin{bmatrix} v(t) & v(t-d) & \cdots & v(t-(N_1-r-1)d) \end{bmatrix}^T \\ \boldsymbol{y}_d(t) &= \begin{bmatrix} y(t) & y(t-d) & \cdots & y(t-(N_1-1)d) \end{bmatrix}^T \end{aligned}$$

with $N_1 \geqslant n$ and $d > 0$.

Remark III.1 *In the case of full relative degree $(r = n)$, the input to the NN need not include the pseudo control signal since the states can be reconstructed without the use of control input, and $\bar{\Delta}$ is not dependent on v. It should be noted that for the case of $r < n$, although there is no need to solve a fixed point solution for v_{ad} to cancel $\bar{\Delta}$, there exists a fixed point solution problem for the NN output since v is needed to reconstruct the state $\boldsymbol{x}$. This problem can be avoided by removing the current time step pseudo control signal at the expense of increased NN approximation error bound. A NN approximation bound can be derived when $\boldsymbol{\mu} = \begin{bmatrix} 1 & v_l & \boldsymbol{v}_d^T(t-d) & \boldsymbol{y}_d^T(t) \end{bmatrix}$ is used as an input to the NN [18].*

IV. The Error Observer Approach

In the case of full state feedback [19], [4], Lyapunov-like stability analysis of the error dynamics in (18) results in update laws for the adaptive control parameters in terms of the error vector $\boldsymbol{E}$. In [5], [6] an adaptive state observer is developed for a nonlinear plant to provide state estimates needed in the adaptation laws. However, the stability analysis was limited to second order systems with position measurements. To relax these assumptions, we make use of a simple linear observer for the tracking error dynamics in (18) [2]. This observer provides estimates of the unavailable error signals for the update laws of the adaptive parameters that will be presented in (26).

Consider the following full-order linear observer for the tracking error dynamics in (18):

$$\begin{aligned} \dot{\hat{\boldsymbol{E}}} &= \bar{A}\hat{\boldsymbol{E}} + K(\boldsymbol{z} - \hat{\boldsymbol{z}}) \\ \hat{\boldsymbol{z}} &= \bar{C}\hat{\boldsymbol{E}}, \end{aligned} \tag{23}$$

where K should be chosen in a way to make $\bar{A} - K\bar{C}$ asymptotically stable. The following remarks will be useful in the sequel.

Remark IV.1 *One can also design a minimal order estimator that treats the $\boldsymbol{\eta}$ component of z as a noiseless measurement [20].*

Remark IV.2 *Additional measurements (if available) may also be used in the inversion control, in the compensator design and in the observer design [7].*

The stability of the closed-loop system should be considered along with the observer error dynamics. Let

$$\tilde{A} \triangleq \bar{A} - K\bar{C}, \quad \tilde{\boldsymbol{E}} \triangleq \hat{\boldsymbol{E}} - \boldsymbol{E}. \tag{24}$$

Then the observer error dynamics can be written:

$$\dot{\tilde{\boldsymbol{E}}} = \tilde{A}\tilde{\boldsymbol{E}} - \bar{\boldsymbol{b}} h_{\bar{v}} \left[v_{ad} - \bar{\Delta} \right]. \tag{25}$$

and there exists a positive definite matrix $\tilde{P}$ solving the Lyapunov equation: $\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} = -\tilde{Q}$ for arbitrary $\tilde{Q} > 0$.

Introduce the largest convex compact set which is contained in $\mathcal{D}_{\boldsymbol{\zeta}}$ such that $B_R \triangleq \{\boldsymbol{\zeta} \ : \ \|\boldsymbol{\zeta}\| \leqslant R\}, \quad R > 0$ where $\boldsymbol{\zeta} = \begin{bmatrix} \boldsymbol{E}^T & \tilde{\boldsymbol{E}}^T & \tilde{W}^T & (vec\tilde{V})^T \end{bmatrix}^T \in \mathcal{D}_{\zeta}$. We want to ensure that a Lyapunov function level set Ω_β is a positive invariant set for the error $\boldsymbol{\zeta}$ in $\mathcal{D}_{\boldsymbol{\zeta}}$ by showing that the level set Ω_β inside B_R contains a compact set Γ outside which a time derivative of the Lyapunov function candidate is negative. A Lyapunov function level set Ω_α is introduced to ensure that Ω_β is contained in B_R, and a ball B_C is introduced to provide that Ω_β contains Γ. Before we state theorems, we give assumptions that will be used in proofs of the theorems.

The update law which we use in this section is a modification of backpropagation. The algorithm was first proposed by Lewis et.al.[19] in a state feedback setting. In [2], the error observer was implemented in an output feedback setting to generate the estimated error vector used as a teaching signal to the NN when $r \geqslant 2$, and the adaptive law was incorporated with σ-modification. The drawback of σ-modification is that when the tracking error becomes small, $\dot{\hat{W}}, \dot{\hat{V}}$ are dominated by the σ-modification term in (26) and $\hat{W}, \hat{V}$ are driven towards zero. Therefore, even if the NN reconstruction error and Taylor series expansion higher order terms are eliminated, the errors do not converge to zero. This drawback motivates the use of another variation called e-modification, which was suggested by Narendra and Annaswamy [21], [22]. The idea is to multiply the σ-modification term by the norm of the tracking error so that it tends to zero with the tracking error. The adaptive law with e-modification is given by

$$\begin{aligned}\dot{\hat{W}} &= -\Gamma_W\left[\mathrm{sgn}(h_{\bar{v}})(\hat{\boldsymbol{\sigma}} - \hat{\sigma}'\hat{V}^T\boldsymbol{\mu})\hat{\boldsymbol{E}}^T P\bar{\boldsymbol{b}} + k_e\|\hat{\boldsymbol{E}}\|\hat{W}\right]\\ \dot{\hat{V}} &= -\Gamma_V\left[\mathrm{sgn}(h_{\bar{v}})\boldsymbol{\mu}\hat{\boldsymbol{E}}^T P\bar{\boldsymbol{b}}\hat{W}^T\hat{\sigma}' + k_e\|\hat{\boldsymbol{E}}\|\hat{V}\right]\end{aligned} \tag{26}$$

where $\Gamma_V, \Gamma_W > 0$ and $k_e > 0$. Note that knowledge of the sign of control effectiveness is explicit in the adaptive law.

For the boundedness proof we need the Taylor series expansion of $W^T\boldsymbol{\sigma}(V^T\boldsymbol{\mu})$ at $W = \hat{W}$ and $V = \hat{V}$. Define

$$\tilde{W} \triangleq \hat{W} - W, \quad \tilde{V} \triangleq \hat{V} - V\,, \quad \tilde{Z} \triangleq \begin{bmatrix} \tilde{W} & 0 \\ 0 & \tilde{V}\end{bmatrix} \tag{27}$$

For the stability proof we will need the following representation:

$$\begin{aligned}v_{ad} - \Delta &= \hat{W}^T\hat{\boldsymbol{\sigma}} - W^T\left(\hat{\boldsymbol{\sigma}} + \hat{\sigma}'(V^T\boldsymbol{\mu} - \hat{V}^T\boldsymbol{\mu}) + \mathcal{O}^2\right) - \epsilon\\ &= \tilde{W}^T\left(\hat{\boldsymbol{\sigma}} - \hat{\sigma}'\hat{V}^T\boldsymbol{\mu}\right) + \hat{W}^T\hat{\sigma}'\tilde{V}^T\boldsymbol{\mu} + \bar{w}\end{aligned} \tag{28}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}(V^T\boldsymbol{\mu}),\ \hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\hat{V}^T\boldsymbol{\mu})$, the disturbance term $\bar{w} = \tilde{W}^T\hat{\sigma}'V^T\boldsymbol{\mu} - W^T\mathcal{O}^2 - \epsilon$ and $\mathcal{O}^2 = \mathcal{O}(-\tilde{V}^T\boldsymbol{\mu})^2 = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} + \hat{\sigma}'\tilde{V}^T\boldsymbol{\mu}$. The following bounds are useful to prove the stability of the proposed adaptive scheme:

$$\|W^T\boldsymbol{\sigma}\| \leqslant \sqrt{n_2+1}\|W\|, \tag{29}$$

$$|z_i\sigma_i'(z_i)| \leqslant \delta = 0.224, \tag{30}$$

$$\|W^T\hat{\sigma}'\hat{V}^T\boldsymbol{\mu}\| \leqslant \delta\sqrt{n_2+1}\|W\| \tag{31}$$

The equality in (30) holds when $a_i z_i = 1.543$ [23]. Using the above bounds, a bound for $\bar{w}$ over the compact set $\mathcal{D}_\mu$ can be expressed:

$$\begin{aligned}\|\bar{w}\| &= \|W^T\tilde{\boldsymbol{\sigma}} - W^T\hat{\sigma}'\hat{V}^T\boldsymbol{\mu} + \hat{W}^T\hat{\sigma}'V^T\boldsymbol{\mu} - \epsilon\|\\ &\leqslant 2\sqrt{n_2+1}W^* + \delta\sqrt{n_2+1}W^* + \|\hat{W}\|\frac{a^*}{4}V^*\mu^* + \epsilon^*\\ &\leqslant \gamma_1\|\tilde{Z}\|_F + \gamma_2\end{aligned}$$

where $\gamma_1 = \frac{a^*}{4}Z^*\mu^*$, $\gamma_2 = ((2+\delta)\sqrt{n_2+1} + \gamma_1)W^* + \epsilon^*$. $v_{ad} - \Delta$ can be shown to be bounded by

$$\begin{aligned}\|v_{ad} - \Delta\| &= \|\hat{W}^T\hat{\boldsymbol{\sigma}} - W^T\boldsymbol{\sigma} - \epsilon\|\\ &\leqslant \alpha_1\|\tilde{Z}\| + \alpha_2\end{aligned} \tag{32}$$

where $\alpha_1 = \sqrt{n_2+1}$ and $\alpha_2 = 2\sqrt{n_2+1}W^* + \epsilon^*$

Assumption IV.1 *Let $R > C\sqrt{\frac{\lambda_{max}(T)}{\lambda_{min}(T)}} \geqslant C$, where $\lambda_{max}(T)$ and $\lambda_{min}(T)$ are the maximum and minimum eigenvalues of the following matrix:*

$$T \triangleq \begin{bmatrix} P & 0 & 0 & 0\\ 0 & \tilde{P} & 0 & 0\\ 0 & 0 & \Gamma_W^{-1} & 0\\ 0 & 0 & 0 & \Gamma_V^{-1}\end{bmatrix}, \tag{33}$$

which will be used in a Lyapunov function candidate as $L = \boldsymbol{\zeta}^T T\boldsymbol{\zeta}$, and $C \triangleq \max\left(\frac{2}{\bar{q}}\Upsilon, \quad \tilde{Z}^*\right)$, where

$$\begin{aligned}\kappa_2 &= \gamma_1\|P\bar{\boldsymbol{b}}\| + \alpha_1(\|\tilde{P}\bar{\boldsymbol{b}}\|h^B + \|P\bar{\boldsymbol{b}}\|)\\ \kappa_3 &= 2\gamma_2\|P\bar{\boldsymbol{b}}\| + 2\alpha_2(\|\tilde{P}\bar{\boldsymbol{b}}\|h^B + \|P\bar{\boldsymbol{b}}\|) + k_e{Z^*}^2\\ \tilde{Z}^* &= \sqrt{2{Z^*}^2 + \frac{4(n_2+1)\|P\bar{\boldsymbol{b}}\|^2}{k_e^2}}\\ \Upsilon &\triangleq 2\kappa_2\tilde{Z}^* + \kappa_3\end{aligned} \tag{34}$$

Let α be the minimum value of the Lyapunov function L on the boundary of B_R and β be the maximum value of the Lyapunov function L on the boundary of B_C. The following compact sets are defined as

$$\begin{aligned}\Omega_\alpha &= \{\boldsymbol{\zeta} \in B_R \mid L \leqslant \alpha = R^2\lambda_{min}(T)\}\\ \Omega_\beta &= \{\boldsymbol{\zeta} \in B_R \mid L \leqslant \beta = C^2\lambda_{max}(T)\}\end{aligned} \tag{35}$$

Theorem IV.1 *Let Assumptions II.1, II.2, II.3, II.4, II.5, IV.1 hold. If the initial errors belong to the compact set Ω_α defined in (35), then the feedback control law given by (6) and the adaptation law (26) ensure that the signals $\boldsymbol{E}, \tilde{\boldsymbol{E}}, \tilde{W}$ and $\tilde{V}$ in the closed-loop system are ultimately bounded with the ultimate bound $C\sqrt{\frac{\lambda_{max}(T)}{\lambda_{min}(T)}}$.*

Proof: Boundedness of all the error signals is shown in two steps. First, boundedness of weight error signals is shown employing a Lyapunov analysis, and then this result is used to show boundedness of the tracking and observer error signals.

Consider the following candidate Lyapunov function for the weight error signals

$$L_w = \frac{1}{2}\tilde{W}^T\Gamma_W^{-1}\tilde{W} + \frac{1}{2}\mathrm{tr}(\tilde{V}^T\Gamma_V^{-1}\tilde{V})$$

The time derivative of L_w is

$$\begin{aligned}\dot{L}_w = &-\tilde{W}^T\left[\mathrm{sgn}(h_{\bar{v}})(\hat{\boldsymbol{\sigma}} - \hat{\sigma}'\hat{V}^T\boldsymbol{\mu})\hat{\boldsymbol{E}}^T P\bar{\boldsymbol{b}} + k_e\|\hat{\boldsymbol{E}}\|\hat{W}\right]\\ &- \mathrm{tr}\{\tilde{V}^T\left[\mathrm{sgn}(h_{\bar{v}})\boldsymbol{\mu}\hat{\boldsymbol{E}}^T P\bar{\boldsymbol{b}}\hat{W}^T\hat{\sigma}' + k_e\|\hat{\boldsymbol{E}}\|\hat{V}\right]\}\\ = &-\mathrm{sgn}(h_{\bar{v}})\tilde{W}^T\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{E}}^T P\bar{\boldsymbol{b}} - k_e\|\hat{\boldsymbol{E}}\|\{\tilde{W}^T\hat{W} + \mathrm{tr}(\tilde{V}^T\hat{V})\}\end{aligned}$$

Using $-2\mathrm{tr}(\tilde{Z}^T\hat{Z}) \leqslant -\|\tilde{Z}\|^2 + {Z^*}^2$ and $\sqrt{n_2+1}\|\tilde{W}\|\|P\bar{\boldsymbol{b}}\| \leqslant \frac{k_e}{4}\|\tilde{W}\|^2 + \frac{(n_2+1)\|P\bar{\boldsymbol{b}}\|^2}{k_e}$

$$\begin{aligned}\dot{L}_w &\leqslant \sqrt{n_2+1}\|\tilde{W}\|\|\hat{\boldsymbol{E}}\|\|P\bar{\boldsymbol{b}}\| - \frac{k_e}{2}\|\hat{\boldsymbol{E}}\|(\|\tilde{Z}\|^2 - {Z^*}^2)\\ &\leqslant -\|\hat{\boldsymbol{E}}\|\left\{\frac{k_e}{4}\|\tilde{Z}\|^2 - \frac{k_e}{2}{Z^*}^2 - \frac{(n_2+1)\|P\bar{\boldsymbol{b}}\|^2}{k_e}\right\}\\ &< 0 \quad \text{if} \quad \|\tilde{Z}\| > \sqrt{2{Z^*}^2 + \frac{4(n_2+1)\|P\bar{\boldsymbol{b}}\|^2}{k_e^2}}\end{aligned}$$

Hence $\tilde{Z}$ is bounded and its bound is denoted as $\|\tilde{Z}\| \leqslant \tilde{Z}^*$

Consider the following Lyapunov function candidate for the entire error system

$$L = \left|\frac{1}{h_{\bar{v}}}\right|\boldsymbol{E}^T P\boldsymbol{E} + \tilde{\boldsymbol{E}}^T\tilde{P}\tilde{\boldsymbol{E}} + 2L_w$$

The time derivative of L is

$$\begin{aligned}\dot{L} &= \frac{d}{dt}\left|\frac{1}{h_{\bar{v}}}\right| \boldsymbol{E}^T P \boldsymbol{E} \\ &+ \left|\frac{1}{h_{\bar{v}}}\right| \left(-\boldsymbol{E}^T Q \boldsymbol{E} + 2\boldsymbol{E}^T P \bar{\boldsymbol{b}} h_{\bar{v}} (v_{ad} - \bar{\Delta})\right) \\ &- \tilde{\boldsymbol{E}}^T \tilde{Q} \tilde{\boldsymbol{E}} - 2\tilde{\boldsymbol{E}}^T \tilde{P} \bar{\boldsymbol{b}} h_{\bar{v}} (v_{ad} - \bar{\Delta}) \\ &+ 2\tilde{W}^T \Gamma_W^{-1} \dot{\tilde{W}} + 2\mathrm{tr}(\tilde{V}^T \Gamma_V^{-1} \dot{\tilde{V}})\end{aligned}$$

Applying the adaptive law (26) and the representation (28),

$$\begin{aligned}\dot{L} = \frac{d}{dt}\left|\frac{1}{h_{\bar{v}}}\right| \boldsymbol{E}^T P \boldsymbol{E} &+ \left|\frac{1}{h_{\bar{v}}}\right| \left(-\boldsymbol{E}^T Q \boldsymbol{E} + 2\hat{\boldsymbol{E}}^T P \bar{\boldsymbol{b}} h_{\bar{v}} \bar{w}\right) \\ &- \tilde{\boldsymbol{E}}^T \tilde{Q} \tilde{\boldsymbol{E}} - 2\tilde{\boldsymbol{E}}^T \tilde{P} \bar{\boldsymbol{b}} h_{\bar{v}} (v_{ad} - \bar{\Delta}) \\ &- 2\mathrm{sgn}(h_{\bar{v}}) \tilde{\boldsymbol{E}}^T P \bar{\boldsymbol{b}} (v_{ad} - \bar{\Delta}) \\ &- 2k_e \|\hat{\boldsymbol{E}}\| (\tilde{W}^T \hat{W} + \mathrm{tr}(\tilde{V}^T \hat{V}))\end{aligned}$$

Utilizing Assumption II.6,

$$\begin{aligned}\dot{L} \leqslant H\lambda_{max}(P)\|\boldsymbol{E}\|^2 &- \frac{1}{h^B}\lambda_{min}(Q)\|\boldsymbol{E}\|^2 \\ &+ 2\|\hat{\boldsymbol{E}}\|\|P\bar{\boldsymbol{b}}\|(\gamma_1\|\tilde{Z}\| + \gamma_2) - \lambda_{min}(\tilde{Q})\|\tilde{E}\|^2 \\ &+ 2\|\tilde{E}\|(h^B\|\tilde{P}\bar{\boldsymbol{b}}\| + \|P\bar{\boldsymbol{b}}\|)(\alpha_1\|\tilde{Z}\| + \alpha_2) \\ &- 2k_e\|\hat{\boldsymbol{E}}\|\mathrm{tr}(\tilde{Z}^T\hat{Z})\end{aligned}$$

Using $\bar{q} \triangleq \min[\frac{\lambda_{\min}(Q)}{h^B} - H\lambda_{max}(P),\ \lambda_{\min}(\tilde{Q})]$, $-2\mathrm{tr}(\tilde{Z}^T\hat{Z}) \leqslant -\|\tilde{Z}\|^2 + {Z^*}^2 \leqslant {Z^*}^2$, and $\|\tilde{Z}\| \leqslant \tilde{Z}^*$.

$$\begin{aligned}\dot{L} \leqslant &- \frac{\bar{q}}{2}(\|\boldsymbol{E}\| + \|\tilde{\boldsymbol{E}}\|)^2 + 2\kappa_2(\|\boldsymbol{E}\| + \|\tilde{\boldsymbol{E}}\|)\tilde{Z}^* \\ &+ \kappa_3(\|\boldsymbol{E}\| + \|\tilde{\boldsymbol{E}}\|)\end{aligned}$$

where $\kappa_2 = \gamma_1\|P\bar{\boldsymbol{b}}\| + \alpha_1(\|\tilde{P}\bar{\boldsymbol{b}}\|h^B + \|P\bar{\boldsymbol{b}}\|)$, $\kappa_3 = 2\gamma_2\|P\bar{\boldsymbol{b}}\| + 2\alpha_2(\|\tilde{P}\bar{\boldsymbol{b}}\|h^B + \|P\bar{\boldsymbol{b}}\|) + k_e{Z^*}^2$. Combining terms to obtain

$$\dot{L} \leqslant -(\|\boldsymbol{E}\| + \|\tilde{\boldsymbol{E}}\|)\left[\frac{\bar{q}}{2}(\|\boldsymbol{E}\| + \|\tilde{\boldsymbol{E}}\|) - 2\kappa_2\tilde{Z}^* - \kappa_3\right]$$

The following condition renders $\dot{L} < 0$.

$$\|\boldsymbol{E}\| + \|\tilde{\boldsymbol{E}}\| > \frac{2}{\bar{q}}\Upsilon$$

where $\Upsilon = 2\kappa_2\tilde{Z}^* + \kappa_3$. Therefore $\boldsymbol{\zeta}$ remains in Ω_β after a finite time period. ∎

V. Numerical Example

The efficacy of the adaptive output feedback controllers developed in Section IV is demonstrated using a modified Van der Pol oscillator model treated in [1] with an additional constant disturbance term in $\dot{x}_2$ equation.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2(x_1^2 - 1)x_2 - x_1 + u + 1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3 - 0.2x_4 + x_1 \\ y &= x_1 + x_3\end{aligned} \tag{36}$$

The initial condition for the plant is $x_1(0) = 0.5, x_2(0) = 2.5, x_3(0) = 0, x_4(0) = 0$. The output has relative degree $r = 2$. We assume that we have an approximate model as

$$\ddot{\hat{y}} = u \tag{37}$$

A second order reference model is selected with a natural frequency of 1 rad/sec and damping ratio of 0.707. The linear controller is designed such that the closed-loop poles of the approximate model in (37) are placed at $-3, -2 \pm 2i$.

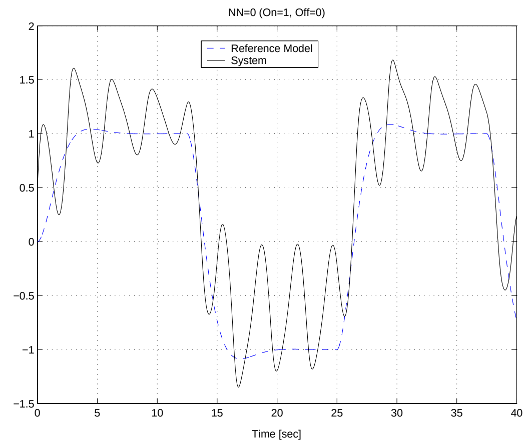

Fig. 1. Output response with a linear compensator

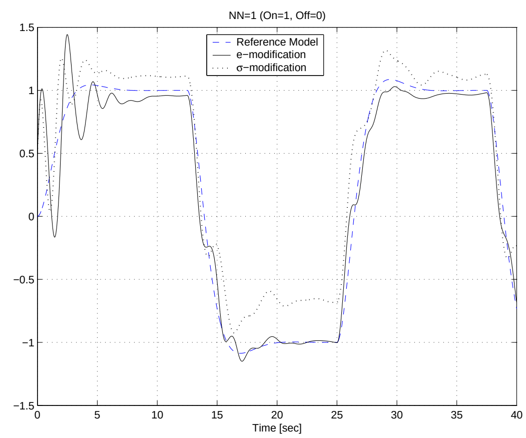

Fig. 2. Tracking performance with error observer approach

$$v_{dc} = \frac{20(s + 1.2)}{s + 7}\tilde{y} \tag{38}$$

The approximate inversion law becomes

$$u = v = \ddot{y}_{rm} + v_{dc} - v_{ad} \tag{39}$$

Then the error dynamics becomes,

$$\begin{aligned}\tilde{y} &= G(s)(v_{ad} - \Delta), \\ G(s) &= \frac{s + 7}{s^3 + 7s^2 + 20s + 24}\end{aligned} \tag{40}$$

where $\Delta = -2(x_1^2 - 1)x_2 - x_1 + 1$. The output response without NN augmentation in Fig. 1 exhibits a limit-cycle-like oscillation due to unmodeled dynamics. The eigenvalues of $\tilde{A}$ in (24) have been placed to be four times faster than those of $\bar{A}$ in (18). Five hidden neurons were implemented in the NN design with activation potentials chosen to be $\begin{bmatrix}1 & 0.8 & 0.6 & 0.4 & 0.2\end{bmatrix}$. Simulation results for the application used two different NN weight update laws. One is with e-modification in (26), and the other is with σ-modification in [2]. The e-modification gain k_e and the σ-modification gain k_σ in the NN update laws were selected to be 0.01 and 0.4, respectively. The adaptation gains have been set to $\Gamma_W = 3I$, $\Gamma_V = 4I$ for e-modification and

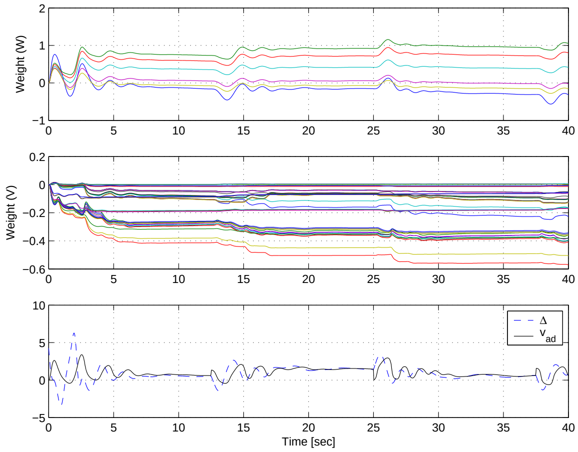

Fig. 3. Weight histories and Δ vs. v_{ad} with e-modification

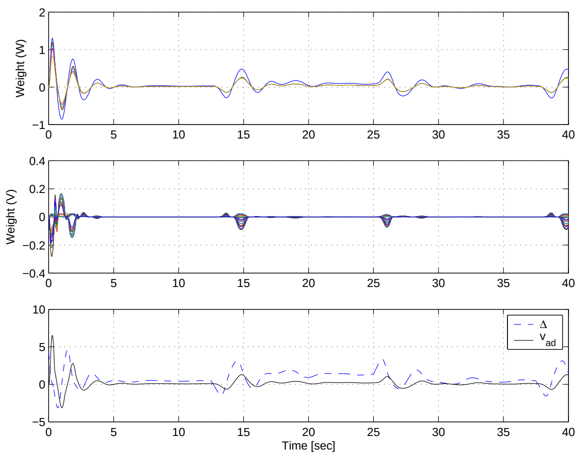

Fig. 4. Weight histories and Δ vs. v_{ad} with σ-modification

$\Gamma_W = 10I$, $\Gamma_V = 50I$ for σ-modification, respectively. Fig. 2 shows the tracking performance of the error observer approach with e-modification and σ-modification. With NN augmentation, the oscillation is removed after about a seven second adaptation period. Although adaptation with e-modification takes longer to adapt, its steady state tracking error is smaller than that of the σ-modification. Figs. 3 and 4 show the NN weight histories and the NN output (v_{ad}) with the inversion error (Δ) using e-modification and σ-modification, respectively. NN weight histories in Fig. 3 show that NN weights approach nearly constant values that are non-zero, in contrast to the NN weight histories in Fig. 4 that tend to return to zero.

VI. Conclusion

This paper presents extensions for adaptive output feedback control of nonlinear systems. The error observer approach is extended so that e-modification can be used in the adaptive law. The mean value theorem is used throughout to avoid the assumption of a fixed point solution, and to make explicit the role of the sign of control effectiveness in the boundedness analysis and in the adaptive law.

References

[1] Calise, A., Hovakimyan, N., and Idan, M., "Adaptive Output Feedback Control of Nonlinear Systems using Neural Networks," *Automatica*, Vol. 37, No. 8, 2001, pp. 1201–1211.

[2] Hovakimyan, N., Nardi, F., Calise, A., and Kim, N., "Adaptive Output Feedback Control of Uncertain Systems using Single Hidden Layer Neural Networks," *IEEE Transactions on Neural Networks*, Vol. 13, No. 6, Nov. 2002.

[3] Rysdyk, R. and Calise, A. J., "Adaptive model inversion flight control for tilt-rotor aircraft," *Journal of Guidance,Control and Dynamics*, Vol. 22, No. 3, Feb. 1999, pp. 402–407.

[4] Calise, A., Lee, S., and Sharma, M., "Development of a reconfigurable flight control law for a tailless aircraft," *Journal of Guidance,Control and Dynamics*, Vol. 24, No. 5, Sep.-Oct. 2001, pp. 896–902.

[5] Kim, Y. and Lewis, F., High Level Feedback Control with Neural Networks, chap. 3, World Scientific, Singapore, 1998.

[6] Hovakimyan, N., Rysdyk, R., and Calise, A., "Dynamic neural networks for output feedback control," *Proceedings of the IEEE Conference for Decision and Control*, Vol. 2, 1999, pp. 1685–1690.

[7] Kim, N., Calise, A., Hovakimyan, N., Prasad, J., and Corban, E., "Adaptive Output Feedback for High-Bandwidth Flight Control," *Journal of Guidance,Control and Dynamics*, Vol. 25, No. 6, 2002, pp. 993–1002.

[8] Zhang, T., Ge, S., and Hang, C., "Direct Adaptive Control of Non-affine Nonlinear Systems Using Multilayer Neural Networks," *Proceedings of the IEEE American Control Conference*, June 1998, pp. 515–519.

[9] Ge, S. and Zhang, J., "Neural-network control of nonaffine nonlinear system with zero dynamics by state and output feedback," *IEEE Transactions on Neural Networks*, Vol. 14, No. 4, 2003, pp. 900–918.

[10] Khalil, H., Nonlinear Systems, chap. 13, Prentice Hall, New Jersey, 3rd ed., 2002.

[11] Isidori, A., Nonlinear Control Systems, chap. 4, Springer, Berlin, 1995.

[12] Khalil, H., Nonlinear Systems, chap. 4, Prentice Hall, New Jersey, 3rd ed., 2002.

[13] Narendra, K. S., "Neural networks for control: Theory and practice," *Proceedings of the IEEE*, Vol. 84, No. 10, Oct. 1996, pp. 1385–1406.

[14] Cybenko, G., "Approximation by Superpositions of Sigmoidal Function," *Mathematics of Control, Signals, and Systems*, Vol. 2, No. 4, 1989, pp. 303–314.

[15] Funahashi, K., "On the Approximate Realization of Continuous Mappings by Neural Networks," *IEEE Transactions on Neural Networks*, Vol. 2, 1989, pp. 183–192.

[16] Hornik, K., Stinchcombe, M., and White, H., "Multi-layer Feedforward Networks are Universal Approximators," *IEEE Transactions on Neural Networks*, Vol. 2, 1989, pp. 359–366.

[17] Lavretsky, E., Hoavakimyan, N., and Calise, A., "Upper bounds for approximation of continuous-time dynamics using delayed outputs and feedforward neural networks," *IEEE Transactions on Automatic Control*, Vol. 48, No. 9, Sept. 2003, pp. 1606–1610.

[18] Kim, N., *Improved methods in Neural Network-Based Adaptive Output Feedback Control, with Applications to Flight Control*, Ph.D. thesis, Georgia Institute of Technology, Sept. 2003.

[19] Lewis, F., Yesildirek, A., and Liu, K., "Multilayer Neural-Net Robot Controller with Guaranteed Tracking Performance," *IEEE Transactions on Neural Networks*, Vol. 7, No. 2, 1996, pp. 388–399.

[20] Kutay, A. T., Calise, A. J., and Hovakimyan, N., "Adaptive Output Feedback Control with Reduced Sensitivity to Sensor Noise," *Proceedings of the IEEE American Control Conference*, ACC03-AIAA0031, June 2003.

[21] Narendra, K. and Annaswamy, A., "A New Adaptive Law for Robust Adaptation Without Persistent Excitation," *IEEE Transactions on Automatic Control*, Vol. 32, No. 2, 1987, pp. 134–145.

[22] Narendra, K. and Annaswamy, A., *Stable Adaptive Systems*, chap. 8, Prentice Hall, 1989.

[23] Zhang, T., Ge, S., and Hang, C., "Design and performance analysis of a direct adaptive controller for nonlinear systems," *Automatica*, Vol. 35, 1999, pp. 1809–1817.