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Robust Output Regulation with Nonlinear Exosystems

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***Abstract*— For over a decade, the solvability of the nonlinear robust output regulation problem relies on the assumption that the exosystem is linear and neurally stable. Thus the only exogenous signal that can be accommodated by the existing theory is a combination of finitely many step functions and sinusoidal functions. In this paper, we will show that it is possible to find controllers that can admit exogenous signals produced by nonlinear exosystems. An example with the well known Van der Pol oscillator as the exosystem is given to illustrate our approach.**

I. Introduction

Consider the plant described by

$$\begin{aligned}\dot{x}(t) &= f(x(t),u(t),v(t),w), \quad x(0)=x_0\\ e(t) &= h(x(t),u(t),v(t),w), \quad t\geq 0 \end{aligned} \qquad (1.1)$$

and an exosystem described by

$$\dot{v}(t)=a(v(t)), \quad v(0)=v_0 \qquad (1.2)$$

where $x(t)$ is the n-dimensional plant state, $u(t)$ the m-dimensional plant input, $e(t)$ the p-dimensional plant output representing the tracking error, $v(t)$ the q-dimensional exogenous signal representing the reference input, and w the N-dimensional plant unknown parameter with nominal value 0. For simplicity, all the functions involved in the setup are sufficiently smooth, vanishing at their origins, and $m=p$.

Briefly, the robust output regulation problem is aimed to design a control law such that, for all sufficiently small $v\in\Re^q$ and all sufficiently small $w\in\Re^N$, the solution of the closed-loop is bounded and the tracking error approaches 0 asymptotically. Various versions of this problem have been extensively studied via the dynamic state feedback [10], [11], dynamic output feedback [1], [11], [12], [13], [15], [16], and the dynamic state/output feedback control with the feedforward control [6], and [8].

Among other conditions, a key solvability condition of the robust output regulation problem is that the exosystem is linear and neurally stable, or what is the same,

A0: The exosystem (1.2) is $linear$, i.e., $a(v)=A_1v$ for some matrix A_1, and all eigenvalues of A_1 are simple with zero real parts.

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Clearly, such exosystems can only produce signals which are a combination of finitely many step functions and sinusoidal functions. Thus, assumption A0 severely limits the applicability of the robust output regulation theory. For example, the limit cycle generated by the well known Van der Pol oscillator cannot be handled due to assumption A0.

The major objective of this paper is to replace assumption A0 by some much less restrictive assumption. For this purpose, we will first generalize, in Section II, the notion of the steady state generator and the internal model introduced in [9] to allow the inclusion of the exogenous signal v in the control law. Then, in Section III, we will give the existence conditions of the steady state generator and the internal model. In Section IV, we will establish a set of solvability conditions of the robust output regulation problem when the exosystem is nonlinear. In Section V, we will use an example with the well known Van der Pol oscillator as the exosystem to illustrate our approach. The paper is closed in Section VI with some concluding remarks.

II. Generalization of Steady State Generator and Internal Model

It is known that a necessary condition for the solvability of the robust output regulation problem is the existence of the solution of the following equations [14].

A1: There exist sufficiently smooth functions $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$ with $\mathbf{x}(0,0)=0$ and $\mathbf{u}(0,0)=0$ satisfying the following equations for all $v\in V$, $w\in W$ with V and W open neighborhoods of the origins of respective Euclidean spaces,

$$\begin{aligned}\frac{\partial\mathbf{x}(v,w)}{\partial v}a(v) &= f(\mathbf{x}(v,w),\mathbf{u}(v,w),v,w)\\ 0 &= h(\mathbf{x}(v,w),\mathbf{u}(v,w),v,w). \end{aligned} \qquad (2.1)$$

Equations (2.1) are called regulator equations. Unfortunately, unlike the linear case [4] and [5], the solvability of (2.1) is not sufficient for the solvability of the robust output regulation problem for nonlinear systems. In order to guarantee the solvability of the robust output regulation problem, additional assumptions have to be imposed on the solution of the regulator equations, and on the exosystem. It is shown in [11] that if the solution of the regulator equations is polynomial in $v(t)$, and the exosystem satisfies assumption A0, then the robust output regulation problem is solvable provided that the linear approximation of system (1.1) at the origin is stabilizable and detectable. Other similar conditions are given in [1], [7] and [15], respectively. More recently, a more general framework is established which aims to convert the robust output regulation problem for a given plant into a robust stabilization problem for an augmented system [9]. This framework relies on two important concepts, namely, the steady state generator and the internal model. Existence of the steady state generator and the internal model is guaranteed under assumption A0, and the polynomial assumption on the solution of the regulator equations. In order to remove assumption A0, we will resort to the feedforward control of the exogenous signal v. For this purpose, we need to extend the definitions of the steady state generator and the internal model to the following.

Definition 2.1: Let $g : \Re^{n+m} \mapsto \Re^{l}$ be a mapping for some positive integer $1 \leq l \leq n+m$. Under assumption A1, the nonlinear system (1.1) and (1.2) is said to have a *(generalized) steady state generator* with output $g(x,u)$ if there exists a triple $\{\theta, \alpha, \beta\}$, where $\theta : \Re^{q+N} \mapsto \Re^{s}$, $\alpha : \Re^{s+q} \mapsto \Re^{s}$, and $\beta : \Re^{s} \mapsto \Re^{l}$ for some integer s are sufficiently smooth functions vanishing at the origin, such that, for all trajectories $v(t) \in V$ of (1.2), and all $w \in W$,

$$\begin{aligned} \frac{d\theta\left(v\left(t\right),w\right)}{dt} &= \alpha\left(\theta\left(v\left(t\right),w\right),v\right) \\ g(\mathbf{x}(v(t),w),\mathbf{u}(v(t),w)) &= \beta\left(\theta\left(v\left(t\right),w\right)\right). \end{aligned} \quad (2.2)$$

If, in addition, the pair $\left(\left.\frac{\partial\beta(\theta)}{\partial\theta}\right|_{v=0,w=0}, \left.\frac{\partial\alpha(\theta,v)}{\partial\theta}\right|_{v=0,w=0} \right)$ is observable, then $\{\theta, \alpha, \beta\}$ is called a linearly observable steady state generator with output $g(x,u)$. ∎

Definition 2.2: Assume the nonlinear system (1.1) and (1.2) has a steady state generator with output $g(x,u)$. Let $\gamma : \Re^{s+n+m+p+q} \mapsto \Re^{s}$ be a sufficiently smooth function vanishing at the origin. Then we call the following system

$$\dot{\eta} = \gamma\left(\eta, x, u, e, v\right) \quad (2.3)$$

a *(generalized) internal model* with output $g(x,u)$ if, for all $v \in V$, and all $w \in W$,

$$\gamma\left(\theta\left(v,w\right), \mathbf{x}\left(v,w\right), \mathbf{u}\left(v,w\right), 0, v\right)\right) = \alpha\left(\theta\left(v,w\right), v\right).$$

∎

In the sequel, we assume $g\left(x,u\right) = \left[x_{i_1}, x_{i_2}, \cdots, x_{i_d}, u\right]^T$ where $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ for some integer d satisfying $0 \leq d \leq n$, and, without loss of generality, we can always assume $i_j = j$ for $j = 1, \cdots, d$ since the index of the state variable can be relabelled to have this assumption satisfied.

Remark 2.1: The major difference between Definitions 2.1 and 2.2 and previous definitions given in [9] is that both the steady state generator and the internal model are allowed to depend on the exogenous signal v. It will be seen later that this generalization will allow the exogenous signal v to appear in the control law, too. Such a control scheme can handle the case where the exogenous signal v is a reference input and/or a measurable disturbance. It should be noted that the notion of the steady state generator is a generalization of the notion of the (generalized) system immersion introduced in [1], [2]. The connection of the steady state generator and the system immersion was detailed in [9]. ∎

Now, as in [9], we can attach the internal model to the given plant to yield the following augmented system

$$\begin{aligned} \dot{x} &= f(x,u,v,w) \\ \dot{\eta} &= \gamma\left(\eta, x, u, e, v\right) \\ e &= h(x,u,v,w). \end{aligned} \quad (2.4)$$

Performing on (2.4) the following coordinate and input transformation

$$\begin{aligned} \bar{\eta} &= \eta - \theta\left(v,w\right) \\ \bar{x}_i &= x_i - \beta_i\left(\eta\right), \; i = 1, \cdots, d \\ \bar{x}_i &= x_i - \mathbf{x}_i\left(v,w\right), \; i = d+1, \cdots, n \\ \bar{u} &= u - \left[\beta_{d+1}\left(\eta\right), \cdots, \beta_{d+m}\left(\eta\right)\right]^T = u - \beta_u\left(\eta\right) \end{aligned}$$

gives a new system denoted by

$$\begin{aligned} \dot{\bar{\eta}} &= \bar{\gamma}(\bar{\eta}, \bar{x}, \bar{u}, v, w) \\ \dot{\bar{x}} &= \bar{f}(\bar{\eta}, \bar{x}, \bar{u}, v, w) \\ e &= \bar{h}(\bar{\eta}, \bar{x}, \bar{u}, v, w) \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} \bar{\gamma}(\bar{\eta}, \bar{x}, \bar{u}, v, w) &= \gamma(\eta, x, u, e, v) - \alpha(\theta(v,w), v) \\ \bar{f}_i(\bar{\eta}, \bar{x}, \bar{u}, v, w) &= f_i(x,u,v,w) - \frac{\partial\beta_i(\eta)}{\partial\eta}\gamma(\eta, x, u, e, v), \\ & \quad i = 1, \cdots, d \\ \bar{f}_i(\bar{\eta}, \bar{x}, \bar{u}, v, w) &= f_i(x,u,v,w) - f_i(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w), \\ & \quad i = d+1, \cdots, n \\ \bar{h}(\bar{\eta}, \bar{x}, \bar{u}, v, w) &= h(x,u,v,w) - h(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w). \end{aligned}$$

An important property of system (2.5) is stated as follows.

Theorem 2.1: Suppose A1, and assume the system (1.1) and (1.2) has a steady state generator with output $g(x,u) = \text{col}(x_1, \cdots, x_d, u)$ and an internal model described by (2.3). Then the augmented system described by (2.5) has the property that, for all sufficiently small trajectories $v(t)$ of the exosystem, and all sufficiently small w,

$$\begin{aligned} \bar{\gamma}(0,0,0,v(t),w) &= 0 \\ \bar{f}(0,0,0,v(t),w) &= 0 \\ \bar{h}\left(0,0,0,v(t),w\right) &= 0. \end{aligned} \quad (2.6)$$

Proof: The proof is very similar to what was given in [3], and is omitted here. ∎

The significance of Theorem 2.1 is that it allows the robust output regulation problem of system (1.1) and exosystem (1.2) to be converted into a robust stabilization problem of the augmented system (2.5) as shown by the following result.

Corollary 2.1: Suppose system (1.1) and (1.2) satisfies assumption A1, and has a steady state generator with output $g(x,u) = \text{col}(x_1, \cdots, x_d, u)$ and an internal model described by (2.3). Assume there exists a controller of the form

$$\begin{aligned} \bar{u} &= k\left(\bar{x}_1, \cdots, \bar{x}_d, \xi, e\right) \\ \dot{\xi} &= \zeta\left(\bar{x}_1, \cdots, \bar{x}_d, \xi, e\right) \end{aligned} \quad (2.7)$$

where $\xi \in \Re^z$, and k, ζ are sufficiently smooth functions vanishing at their origins, that locally exponentially stabilizes the equilibrium $(\bar{\eta}, \bar{x}) = (0, 0)$ of the augmented system (2.5). Then the following controller

$$\begin{aligned} u &= \beta_u(\eta) + k\left(x_1 - \beta_1(\eta), \cdots, x_d - \beta_d(\eta), \xi, e\right) \\ \dot{\eta} &= \gamma(\eta, x, u, e, v) \\ \dot{\xi} &= \zeta\left(x_1 - \beta_1(\eta), \cdots, x_d - \beta_d(\eta), \xi, e\right) \end{aligned} \tag{2.8}$$

solves the robust output regulation for system (1.1) and (1.2) in the sense that the equilibrium of the closed-loop system composed of the plant (1.1) and the controller (2.8) is exponentially stable, and for all sufficiently small initial state of the closed-loop system, all sufficiently small $v(t)$, and w, the error e approaches zero asymptotically.

Proof: Consider the closed-loop system composed of the plant (1.1) and the controller (2.8) and denote its state by $x_c = \text{col}(\eta, x, \xi)$. Then

$$\begin{aligned} x_c &= \bar{x}_c + \text{col}(\theta(v,w), \beta_1(\bar{\eta} + \theta(v,w)), \cdots, \\ &\quad \beta_d(\bar{\eta} + \theta(v,w)), \mathbf{x}_{d+1}(v,w), \cdots, \mathbf{x}_n(v,w), 0), \end{aligned}$$

with $\bar{x}_c = \text{col}(\bar{\eta}, \bar{x}, \xi)$. Thus, when $v = 0$ and $w = 0$, the state $\bar{x}_c$ of the closed-loop system composed of (2.5) and (2.7) and the state x_c of the closed-loop system composed of (1.1) and (2.8) are related by a diffeomorphism

$$x_c = \bar{x}_c + \text{col}(0, \beta_1(\bar{\eta}), \cdots, \beta_d(\bar{\eta}), 0, \cdots, 0, 0).$$

Thus the closed-loop system composed of (1.1) and (2.8) is also exponentially stable. Next let $\mathbf{x}_c(v,w) = \text{col}(\theta(v,w), \mathbf{x}(v,w), 0)$. Then it can be verified that $\mathbf{x}_c(v,w)$ is a zeroing output center manifold of the closed-loop system (1.1) and (2.8), thus, the error e approaches zero asymptotically. As a result, the proof is completed. ∎

Remark 2.2: It is known that if the equilibrium of the closed-loop system at the origin is exponentially stable, and the equilibrium of the exosystem at the origin is neurally stable, then the equilibrium of the composite system composed of the closed-loop system and the exosystem is stable. Thus, the trajectory of the closed-loop system exists and is bounded for all sufficiently small $x_c(0)$, $v(0)$, and w. When the exosystem is linear, the neutral stability of the equilibrium of the exosystem is necessary in order to guarantee the boundedness of $x_c(t)$ for all sufficiently small $x_c(0)$, $v(0)$, and w. Nevertheless, when the exosystem is nonlinear, the trajectory of the exosystem may still be bounded even if the equilibrium of the exosystem is unstable. A typical case is the Van Der Pol equation which has an asymptotically stable limit cycle. When the size of the limit cycle is sufficiently small such that the trajectory of the closed-loop system (2.5) and (2.7) starting from sufficiently small $x_c(0)$ and w is contained in domain of the attraction of the closed-loop system [17], Corollary 2.1 can still guarantee the boundedness of $x_c(t)$. Therefore, in the statement of the Corollary 2.1, we have not required the neutral stability of the equilibrium of the exosystem at the origin. ∎

III. Internal Model with Nonlinear Exosystem

As mentioned in the last section, the existence conditions of the steady state generator and the internal model were given in [9] for the case where the exosystem satisfies assumption A0. In this section, we will further present the existence conditions of the steady state generator and the internal model with output $g(x,u) = u$ for the case where the exosystem does not satisfy assumption A0.

Let us first note that the exosystem (1.2) can always be written as follows

$$\dot{v} = A_1 v + \sum_{k=2}^{K} A^{[k]} v a^{[k]}(v) \tag{3.1}$$

for some integer $K \geq 1$, and $A^{[k]} \in \Re^{q \times q}$, $k = 1, \cdots, K$, where $a^{[k]} : \Re^q \mapsto \Re$ is sufficiently smooth function satisfying $a^{[k]}(0) = 0$. In particular, when $a(v)$ is linear, (3.1) is reduced to $\dot{v} = A_1 v$.

Now, as in [9], assume the solution $\mathbf{u}(v(t), w)$ of the regulator equations is a polynomial in $v(t)$. Then, for $i = 1, \cdots, m$, it is known that there exist a set of real numbers $a_1, a_2, \cdots, a_{r_i}$ such that

$$\begin{aligned} L_{A_1 v}^{r_i} \mathbf{u}_i(v,w) &= a_1 \mathbf{u}_i(v,w) + a_2 L_{A_1 v} \mathbf{u}_i(v,w) + \cdots \\ &\quad + a_{r_i} L_{A_1 v}^{r_i - 1} \mathbf{u}_i(v,w), \end{aligned} \tag{3.2}$$

where $L_{A_1 v} \mathbf{u}_i(v,w) = \frac{\partial \mathbf{u}_i(v,w)}{\partial v} A_1 v$, and $L_{A_1 v}^{l} \mathbf{u}_i(v,w) = \frac{\partial L_{A_1 v}^{l-1} \mathbf{u}_i(v,w)}{\partial v} A_1 v$, $l = 2, 3, \cdots$, [1], [8]. Denote

$$\theta_i(v,w) = \text{col}\left(\mathbf{u}_i(v,w), L_{A_1 v} \mathbf{u}_i(v,w), \cdots, L_{A_1 v}^{r_i - 1} \mathbf{u}_i(v,w)\right).$$

Then there exist matrices $\Phi_i = \begin{bmatrix} 0_{(r_i - 1) \times 1} & I_{r_i - 1} \\ a_1 & [a_2 \cdots a_{r_i}] \end{bmatrix}$ and $E_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ such that

$$\begin{aligned} \frac{\partial \theta_i(v,w)}{\partial v} A_1 v &= \Phi_i \theta_i(v,w) \\ \mathbf{u}_i(v,w) &= E_i \theta_i(v,w). \end{aligned} \tag{3.3}$$

The following result gives sufficient conditions for the existence of the steady state generator.

Lemma 3.1: Under assumption A1, if

(i) $\mathbf{u}_i(v(t), w), i = 1, \cdots, m$, is polynomial in $v(t)$ with coefficients depending on w, and

(ii) there exists some matrix $\Phi_i^{[k]}$ satisfying

$$\frac{\partial \theta_i(v,w)}{\partial v} A^{[k]} v = \Phi_i^{[k]} \theta_i(v,w), \; k = 2, \cdots, K, \quad i = 1, \cdots, m \tag{3.4}$$

then the system (1.1) and (1.2) has a linearly observable steady state generator with output u.

Proof: Let $\hat{\theta}_i(v,w) = T_i \theta_i(v,w)$ where T_i is any nonsingular matrix with same dimension as that of Φ_i. Under assumptions (i) and (ii), the Lie derivative of $\hat{\theta}_i(v,w)$ along

(1.2) satisfies

$$
\begin{aligned}
&\frac{\partial\hat{\theta}_i(v,w)}{\partial v}a(v)\\
=&\ T_i\frac{\partial\theta_i(v,w)}{\partial v}\left[A_1 v+\sum_{k=2}^{K}A^{[k]}va^{[k]}(v)\right]\\
=&\ T_i\left[\Phi_i\theta_i(v,w)+\sum_{k=2}^{K}\Phi_i^{[k]}\theta_i(v,w)a^{[k]}(v)\right]\\
=&\ T_i\left[\Phi_i+\sum_{k=2}^{K}\Phi_i^{[k]}a^{[k]}(v)\right]\theta_i(v,w)\\
=&\ T_i\left[\Phi_i+\sum_{k=2}^{K}\Phi_i^{[k]}a^{[k]}(v)\right]T_i^{-1}\left[T_i\theta_i(v,w)\right]\\
=&\ T_i\phi_i(v)T_i^{-1}\hat{\theta}_i(v,w)
\end{aligned}
$$

where $\phi_i(v)=\Phi_i+\phi_i^{[2]}(v)$ and $\phi_i^{[2]}(v)=\sum_{k=2}^{K}\Phi_i^{[k]}a^{[k]}(v)$ with $\phi_i^{[2]}(0)=0$ and $\phi_i(0)=\Phi_i$. On the other hand,

$$\mathbf{u}_i(v,w)=E_i\theta_i(v,w)=E_iT_i^{-1}T_i\theta_i(v,w)=E_iT_i^{-1}\hat{\theta}_i(v,w).$$

It is ready to show that system (1.1) and (1.2) has a steady state generator

$$\left\{\hat{\theta}_i(v,w),\alpha_i(\hat{\theta}_i(v,w),v),\beta_i(\hat{\theta}_i(v,w))\right\}$$

with output u_i, where

$$
\begin{aligned}
\alpha_i(\hat{\theta}_i(v,w),v) &= T_i\phi_i(v)T_i^{-1}\hat{\theta}_i(v,w)\\
\beta_i(\hat{\theta}_i(v,w)) &= E_iT_i^{-1}\hat{\theta}_i(v,w).
\end{aligned}
$$

Moreover, (E_i,Φ_i), hence, $(E_iT_i^{-1},T_i\Phi_iT_i^{-1})$, is observable, that is, the steady state generator is linearly observable. Finally, let $\hat{\theta}=\text{col}(\hat{\theta}_1,\cdots,\hat{\theta}_m)$, $\alpha(\hat{\theta},v)=\text{diag}(\alpha_1(\hat{\theta}_1,v),\cdots,\alpha_m(\hat{\theta}_m,v))$, and $\beta(\hat{\theta})=\text{diag}(\beta_1(\hat{\theta}_1),\cdots,\beta_m(\hat{\theta}_m))$. Then the triple $(\hat{\theta},\alpha,\beta)$ gives a steady state generator for system (1.1) and (1.2) with output u. ∎

Remark 3.1: Condition (i) of Lemma 3.1 is standard in the literatures of the robust output regulation problem [1], [7], and [11]. Condition (ii) automatically holds when the exosystem is linear under condition (i). Of course, the interest of Lemma 3.1 is that conditions (i) and (ii) may hold in many cases even when the exosystem is nonlinear as shown by the example in Section V. ∎

With the steady state generator ready, we can define a nonlinear internal model as follows. For $i=1,\cdots,m$, pick any controllable pair (M_i,N_i) with $M_i\in\Re^{r_i\times r_i}, N_i\in\Re^{r_i\times 1}$, where M_i is Hurwitz and has disjoint spectra with Φ_i. Then there exists a unique, nonsingular matrix T_i satisfying the Sylvester equation

$$T_i\Phi_i-M_iT_i=N_iE_i$$

since the spectra of the matrices Φ_i and M_i are disjoint, and the pair (E_i,Φ_i) is observable.

Consequently, the dynamics

$$
\begin{aligned}
\dot{\eta}_i &= \gamma_i(\eta_i,x,u,e,v)\\
&= M_i\eta_i+T_i\phi_i^{[2]}(v)T_i^{-1}\eta_i+N_iu_i \qquad (3.5)
\end{aligned}
$$

is an internal model of the plant (1.1) and exosystem (1.2) with output u_i, since

$$
\begin{aligned}
&\gamma_i\left(\hat{\theta}_i(v,w),\mathbf{x}(v,w),\mathbf{u}(v,w),0,v\right)\\
=&\ M_i\hat{\theta}_i(v,w)+T_i\phi_i^{[2]}(v)T_i^{-1}\hat{\theta}_i(v,w)+N_iE_iT_i^{-1}\hat{\theta}_i(v,w)\\
=&\ T_i\Phi_iT_i^{-1}\hat{\theta}_i(v,w)+T_i\phi_i^{[2]}(v)T_i^{-1}\hat{\theta}_i(v,w)\\
=&\ T_i\phi_i(v)T_i^{-1}\hat{\theta}_i(v,w)=\alpha_i(\hat{\theta}_i(v,w),v). \qquad (3.6)
\end{aligned}
$$

As a result, system (1.1) and (1.2) has a steady state generator $\left\{\hat{\theta}(v,w),\alpha(\hat{\beta}(v,w),v),\beta(\hat{\theta}(v,w))\right\}$ with output u, where

$$
\begin{aligned}
\hat{\theta}(v,w) &= \text{col}\left(\hat{\theta}_1(v,w),\hat{\theta}_2(v,w),\cdots,\hat{\theta}_m(v,w)\right)\\
\alpha(\hat{\theta}(v,w),v) &= T\phi(v)T^{-1}\hat{\theta}(v,w)\\
\beta(\hat{\theta}(v,w)) &= ET^{-1}\hat{\theta}(v,w),
\end{aligned}
$$

and an internal model

$$\dot{\eta}=\gamma(\eta,x,u,e,v)=M\eta+T\phi^{[2]}(v)T^{-1}\eta+Nu$$

with

$$
\begin{aligned}
\eta &= \text{col}\,(\eta_1,\eta_2,\cdots,\eta_m)\\
T &= \text{diag}\,(T_1,T_2,\cdots,T_m)\\
E &= \text{diag}\,(E_1,E_2,\cdots,E_m)\\
M &= \text{diag}\,(M_1,M_2,\cdots,M_m)\\
N &= \text{diag}\,(N_1,N_2,\cdots,N_m)\\
\phi(v) &= \text{diag}\,(\phi_1(v),\phi_2(v),\cdots,\phi_m(v))\\
\phi^{[2]}(v) &= \text{diag}\left(\phi_1^{[2]}(v),\phi_2^{[2]}(v),\cdots,\phi_m^{[2]}(v)\right).
\end{aligned}
$$

Obviously, $\phi(0)=\Phi=\text{diag}\,(\Phi_1,\Phi_2,\cdots,\Phi_m)$.

IV. Solvability of the Main Problem

In this section, we will give the solvability conditions of the robust output regulation problem without assuming that the exosystem is linear. To this end, let us list two more assumptions.

Define the following notation,

$$
\begin{aligned}
A &= \frac{\partial f}{\partial x}(0,0,0,0),\quad B=\frac{\partial f}{\partial u}(0,0,0,0),\\
C &= \frac{\partial h}{\partial x}(0,0,0,0),\quad D=\frac{\partial h}{\partial u}(0,0,0,0).
\end{aligned}
$$

A2: The pair (A,B) is stabilizable, and the pair (C,A) is detectable.

A3: For all $\lambda\in\sigma(\Phi)$, rank $\begin{bmatrix} A-\lambda I & B\\ C & D\end{bmatrix}=n+m$, where $\sigma(\Phi)=\{\lambda|\det(\Phi-\lambda I)=0\}$.

Theorem 4.1: Suppose assumptions A1 to A3 and the conditions (i) and (ii) in Lemma 3.1, the robust output regulation problem is solvable by an output feedback control law in the sense described in the statement of Corollary 2.1.

Proof: Under the assumption A1 and the conditions (i) and (ii), the system (1.1) and (1.2) has a steady state generator and an internal model with output u. Now, applying the following state and input transformation

$$\begin{aligned} \bar{\eta} &= \mathrm{col}(\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_m) = \eta - \hat{\theta}(v, w) \\ \bar{x} &= x - \mathbf{x}(v, w) \\ \bar{u} &= u - \beta(\eta) \end{aligned}$$

on the augmented system gives a transformed system whose linearization at the origin $(\bar{\eta} = 0, \bar{x} = 0, \bar{u} = 0)$ with v and w being set to zero is

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B\bar{u} + BET^{-1}\bar{\eta} \\ \dot{\bar{\eta}} &= (M + NET^{-1})\bar{\eta} + N\bar{u} \\ e_l &= C\bar{x} + D\bar{u} + DET^{-1}\bar{\eta}. \end{aligned} \tag{4.1}$$

By Corollary 2.1, it suffices to stabilize the equilibrium at the origin of (4.1).

First, consider the decomposition,

$$\begin{aligned} &\begin{bmatrix} A - \lambda I & BET^{-1} & B \\ 0 & M + NET^{-1} - \lambda I & N \end{bmatrix} \\ &= \begin{bmatrix} A - \lambda I & 0 & B \\ 0 & M - \lambda I & N \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & ET^{-1} & I \end{bmatrix}. \end{aligned}$$

Since (A, B) is stabilizable, and M is Hurwitz, we conclude that (4.1) is stabilizable using PBH test.

To show that (4.1) is detectable, first note that $M + NET^{-1} = T\Phi T^{-1}$, and (C, A) is detectable. Thus the following matrix

$$\begin{bmatrix} A - \lambda I & BET^{-1} \\ 0 & M + NET^{-1} - \lambda I \\ C & DET^{-1} \end{bmatrix} \tag{4.2}$$

has full rank for all $\lambda \notin \sigma(\Phi)$ and $\mathrm{Re}\{\lambda\} \geq 0$.

Next consider the case where $\lambda \in \sigma(\Phi)$. Using the decomposition

$$(4.2) = \begin{bmatrix} A - \lambda I & 0 & B \\ 0 & M - \lambda I & N \\ C & 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & ET^{-1} \end{bmatrix},$$

together with A3 and the fact that M is Hurwitz, we conclude that (4.2) has full rank. Thus, the detectability of (4.1) follows from PBH test.

As a result, system (4.1) can be stabilized by a linear control law of the form

$$\begin{aligned} \bar{u} &= -K\xi \\ \dot{\xi} &= L\xi + Qe. \end{aligned} \tag{4.3}$$

By Corollary 2.1, the following output feedback control law

$$\begin{aligned} u &= \beta(\eta) - K\xi \\ \dot{\eta} &= M\eta + T\phi^{[2]}(v)T^{-1}\eta + N(\beta(\eta) - K\xi) \\ \dot{\xi} &= L\xi + Qe \end{aligned} \tag{4.4}$$

solves the robust output regulation problem for the original system (1.1) and (1.2). ∎

Remark 4.1: Condition (i) can be somehow relaxed. Assume the solution $\mathbf{u}(v, w)$ of the regulator equations can be written as follows

$$\mathbf{u}(v, w) = \mathbf{u}_c(v) + \hat{\mathbf{u}}(v, w) \tag{4.5}$$

for some sufficiently smooth function $\mathbf{u}_c(v)$ vanishing at the origin, and some polynomial function $\hat{\mathbf{u}}(v, w)$. Then it is shown in [8] that the regulator equations of the system

$$\begin{aligned} \dot{x}(t) &= f(x(t), \hat{u}(t) + \mathbf{u}_c(v(t)), v(t), w), \quad x(0) = x_0 \\ e(t) &= h(x(t), \hat{u}(t) + \mathbf{u}_c(v(t)), v(t), w), \quad t \geq 0 \end{aligned} \tag{4.6}$$

is given by $\hat{\mathbf{u}}(v, w)$. Clearly, system (4.6) satisfies assumptions A1-A3 if and only if system (1.1) satisfies assumptions A1-A3. Thus, if $\hat{\mathbf{u}}(v, w)$ also satisfies condition (ii), then there exists an output feedback control law

$$\begin{aligned} \hat{u} &= \beta(\eta) - K\xi \\ \dot{\eta} &= M\eta + T\phi^{[2]}(v)T^{-1}\eta + N(\beta(\eta) - K\xi) \\ \dot{\xi} &= L\xi + Qe \end{aligned}$$

that solves the robust output regulation problem for system (4.6). Consequently, the following output feedback control law

$$\begin{aligned} u &= \mathbf{u}_c(v) + \beta(\eta) - K\xi \\ \dot{\eta} &= M\eta + T\phi^{[2]}(v)T^{-1}\eta + N(\beta(\eta) - K\xi) \\ \dot{\xi} &= L\xi + Qe \end{aligned}$$

solves the robust output regulation problem for the original system (1.1). Note that even if $\mathbf{u}(v, w)$ is already a polynomial in v, making use of the decomposition (4.5) sometimes may reduce the order of the steady state generator, thus simplifying the design of control laws as shown by the example in the next section. ∎

V. An Example

Consider the following plant

$$\begin{aligned} \dot{x}_1 &= x_1 + w_1 e^2 + x_2 \\ \dot{x}_2 &= w_2 x_1 + \sin[(1 + w_3)e x_2] + u \\ e &= x_1 - v_1. \end{aligned}$$

where the reference input v_1 is produced by the following Van der Pol oscillator

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ -av_1 + b(1 - v_1^2)v_2 \end{bmatrix} = A_1 v + A^{[2]} v a^{[2]}(v) \tag{5.1}$$

with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix}, \; A^{[2]} = \begin{bmatrix} 0 & 0 \\ 0 & -b \end{bmatrix}, \; a^{[2]}(v) = v_1^2.$$

It is well known that, for all $a > 0$ and $b > 0$, the Van der Pol oscillator will produce an asymptotically stable limit cycle. In what follows, we will design a regulator for the case $a = b = 1$, and the result shows this limit cycle size is small enough to be tracked by the plant output.

First, let us note that the system satisfies assumption A1. In fact, it can be directly verified that the solution of the regulator equations for this system is

$$\begin{aligned} \mathbf{x}_1(v,w) &= v_1 \\ \mathbf{x}_2(v,w) &= -v_1+v_2 \\ \mathbf{u}(v,w) &= -v_2-v_1+(1-v_1^2)v_2-w_2v_1. \end{aligned}$$

Next, it is ready to see that $\mathbf{u}(v,w)$ satisfies conditions (i) and (ii) of Lemma 3.1. In order to reduce the order of the steady state generator, decompose

$$\mathbf{u}(v,w)=\mathbf{u}_c(v)+\hat{\mathbf{u}}(v,w)$$

with $\mathbf{u}_c(v)=-v_2-v_1+(1-v_1^2)v_2$, and $\hat{\mathbf{u}}(v,w)=-w_2v_1$. It can be verified that $\hat{\mathbf{u}}(v,w)$ still satisfies the condition (i) and (ii) of Lemma 3.1 with $K=2$, and $\Phi_1^{[2]}=\left[\begin{array}{cc} 0 & 0 \\ 0 & -1 \end{array}\right]$.

Corresponding to $\hat{\mathbf{u}}(v,w)$, we can obtain a steady state generator with $\theta_1(v,w)=\text{col}(-w_2v_1,-w_2v_2)$,

$$\Phi_1=\left[\begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array}\right],\ E_1=\left[\begin{array}{cc} 1 & 0 \end{array}\right].$$

Clearly, this steady state generator is linearly observable.

To construct an internal model, pick $M_1 = \left[\begin{array}{cc} 0 & 1 \\ -1 & -2 \end{array}\right],\ N_1=\left[\begin{array}{c} 1 \\ 0 \end{array}\right]$. Then, solving the Sylvester equation gives $T_1=\left[\begin{array}{cc} 1.0 & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{array}\right]$.

Noting $\phi_1^{[2]}(v)=\Phi_1^{[2]}a^{[2]}(v)$ gives the internal model as follows

$$\dot{\eta}_1 = M_1\eta_1+T_1\Phi_1^{[2]}T_1^{-1}\eta_1v_1^2+N_1\hat{u}.$$

To verify A2 and A3, note that $A=\left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right]$, $B=\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$, $C=\left[\begin{array}{cc} 1 & 0 \end{array}\right]$ and $D=0$. Thus A2 is obviously satisfied. A3 is also satisfied by noting that $\sigma(\Phi_1)=\{0.5+0.5\sqrt{3}\mathrm{i}, 0.5-0.5\sqrt{3}\mathrm{i}\}$. Thus the output regulation problem of the original system can be solved by an output feedback control law as follows,

$$\begin{aligned} u &= -v_2-v_1+(1-v_1^2)v_2-K\xi+ET^{-1}\eta \\ \dot{\eta} &= M\eta+T\Phi^{[2]}T^{-1}\eta v_1^2+N(-K\xi+ET^{-1}\eta) \\ \dot{\xi} &= L\xi+Qe. \end{aligned}$$

Computer simulation has been used to verify the design with $w_1=0.1, w_2=-0.2, w_3=0.3, v_1(0)=0.1, v_2(0)=0, x_1(0)=-1, x_2(0)=0.2, \eta(0)=0, \xi(0)=0$, $K=[140.50\ \ -98.25\ \ 113.25\ \ 93.75]^T$, $Q=[15.00\ \ 76.75\ \ 45.50\ \ 9.75]^T$, and

$$L=\left[\begin{array}{cccc} -14.00 & 1.00 & 0 & 0 \\ -216.25 & 98.25 & -110.25 & -87.75 \\ -186.00 & 98.25 & -110.25 & -86.75 \\ -9.75 & 0 & -1.00 & -2.00 \end{array}\right].$$

The simulation results are removed due to the space limit.

VI. Conclusion

The paper has studied the solvability of the nonlinear robust output regulation problem without the longstanding restrictive assumption that the exosystem is linear and neurally stable. A set of sufficient conditions for the solvability of the problem are given. The results of this paper allow the robust output regulation problem to admit a much larger class of exogenous signals such as the limit cycle produced by the Van der Pol oscillator.

Our control law relies not only on the error output, but also on the exogenous signals. Such a control law applies to the case where the exogenous signals are reference input and/or measurable disturbance, but not unmeasurable disturbance. It is interesting to further investigate the possibility of constructing dynamic output feedback controllers to solve the problem.

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