Output feedback stabilization and restricted tracking for cascaded systems with bounded control

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Abstract— **In this note we discuss the problems of output feedback stabilization for a class of cascaded systems and of (approximate) output regulation for general nonlinear systems. It is shown that (global) output feedback stabilization for a class of systems in feedforward form can be achieved with a dynamic feedback law, yielding bounded control, and relying on the introduction of a reduced order observer. The above result, together with standard tools borrowed from the nonlinear output regulator theory, is instrumental to construct dynamic control laws achieving (approximate) disturbance rejection and output tracking in the presence of (bounded) disturbance/reference signals generated by a known exosystem.**

I. Introduction

In this note we discuss the problems of global output feedback stabilization and restricted output tracking for a class of cascaded systems in feedforward form. These results are based on recent developments on the global asymptotic stabilization of such systems by means of saturated partial state feedback control, see [17]. The problem of output feedback stabilization of feedforward systems has been widely addressed in the recent literature. Worth noting, to put the present contribution in the proper context, are the results in [22], where the output feedback stabilization problem for a class of cascaded systems is solved using a sort of separation principle, *i.e.* a state feedback control law is combined with a full order observer. Note that the state feedback stabilizing control law in [22] has to be arbitrarily small, so that its *action* on the observer can be neglected. More general results on the (semi-global) output feedback stabilization of feedforward systems have been proposed, see *e.g.* [1].

On the other hand, the problem of output regulation, and robust output regulation, of nonlinear systems has been exhaustingly dealt with, within the last three decades [8], [12], [5], [18], [11], while extensions (to the case of unknown exosystems) and special cases (*e.g.* systems with input saturation) can be found in the more recent work [26], [6] and references therein. On the basis of the seminal works [8] and [12], it is nowadays apparent that, when a disturbance generated by an exosystem is driving the plant to be controlled, a controller that achieves stabilization and disturbance rejection (or output tracking) necessarily needs to *incorporate* a component constructed from a copy (or multiple copies) of the exosystem, known as internal model, together with a stabilizing component.

The application of this principle in the presence of constant disturbances or constant references to be tracked is at the basis of the use of the classical *integral action*, while the extension to non-constant disturbances or references is fairly involved. To begin with it is in general assumed that the exosystem is – at least – Poisson stable. Then, it is noted that the key ingredient in the study of nonlinear regulation problems is the existence, for the system in closed loop with a (dynamic) controller and driven by the exosystem, of an output zeroing manifold. Goal of the controller is to render such a manifold attractive, invariant, and such that the (well-defined) dynamics on the manifold possess specific properties. Necessary conditions for the existence of such a manifold, and of a controller achieving the above listed objectives, have been given by Isidori and Byrnes (see *e.g.* [12]). Note that, unlike the case of linear systems, for general nonlinear systems the construction of the internal model is far from obvious. In the perspective of [9] (see also [7], [11], [10]), the internal model is constructed from the parallel connection of a copy of the exosystem (which has to be linear) and a finite number of linear systems with eigenvalues multiple of the eigenvalues of the exosystem. This internal model, known as k-fold internal model, is able to guarantee approximate regulation, *i.e.* the regulated output will not contain frequency components corresponding to eigenvalues of the exosystem, but will not go asymptotically to zero. In the geometric perspective of Isidori-Byrnes, the exosystem is constructed using an immersion condition. This requires a realizability assumption, which holds for linear exosystem and polynomial systems [26]. Notably, this perspective allows to cope with an unknown exosystem and to address in a simple way robustness issues [3]. Finally, in the perspective of repetitive control (see [29] and references therein) the internal model is constructed using an infinite dimensional component (in general a delay line) in positive feedback with a low pass filter. As this internal model is able to potentially generate all periodic signals of a given fixed frequency, it can be interpreted as an asymptotic version of the k-fold internal model of [9]. Once the internal model is constructed, the main difficulty in the solution of global output regulation problems is the design of the stabilizing component. This design requires the construction of a (dynamic) output feedback stabilizing control law for an extended system, namely the system to be controlled in

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cascaded connection with the internal model. This problem can sometimes be resolved relaxing the global stability requirement and using semi-global stabilization tools [26], or restricting the class of systems under consideration, see *e.g.* [18].

On the basis of the above considerations, the contribution of the present note is twofold. Firstly, the output feedback stabilization problem for a class of feedforward systems is solved by means of a dynamic output feedback control law, yielding bounded control action (as in [22]) and (unlike [22]) relying on a reduced order observer. The stability analysis of the closed loop system is performed using the generalization of the small gain theorem given in [24] and the design tools developed in [17] (see also some preliminary results in [16]). Secondly, the problem of *restricted and approximate* output regulation for general nonlinear systems, satisfying specific properties, is addressed and solved. The term restricted refers to the fact that the disturbances to be rejected or the references to be tracked have to be sufficiently small, so that the control law achieving exact regulation is smaller, in amplitude, than the stabilizing control. The term approximate refers to the fact that the regulated output will not vanish asymptotically, but only certain harmonic components will be cancelled. It is therefore obvious that, in the construction of the internal model we follow the perspective of the k-fold internal model, which allows to pose the output feedback regulation problem in a convenient form (see Section III). The paper is organized as follows. In Section II the global output feedback stabilization problem for a class of cascaded systems is addressed and solved. In Section III the restricted and approximate output regulation problem for a class of nonlinear systems is solved. Note that, for simplicity, we consider only a disturbance rejection problem, as the tracking problem can be posed and solved using similar considerations. Finally, Sections IV and V contain the discussion of a case study and concluding remarks.

II. Output feedback stabilization

Consider a nonlinear cascaded system described by equations of the form

$$\begin{array}{rcl} \dot{z} & = & Jz + h(\xi) + r(\xi)u \\ \dot{\xi} & = & f(\xi) + g(\xi)u, \end{array} \tag{1}$$

with state $(z,\xi) \in \mathbb{R}^p \times \mathbb{R}^n$ and input $u \in \mathbb{R}$. Assume that the origin is an equilibrium point of the system with $u = 0$, that the linear approximation around zero is controllable, that the ξ subsystem is input-to-state-stable (ISS) and LES for $u = 0$, and that J is a stable matrix, *i.e.* $J^T P + PJ \leq 0$ for some $P > 0$. System (1) falls into the category of systems in block feedforward form, and as such can be asymptotically stabilized following one of the available constructive methodologies, see *e.g.* [25], [21], [27], [20]. Besides, under the above assumptions the state feedback stabilization problem for system (1) has been solved in [17] by means of a simple bounded control law, that requires information of the *upper* state z only. This design is based on a generalization of the nonlinear small gain theorem and has, as a starting point, the approximation of system (1) for small $\|\xi\|$.

In this section we extend the result of [17] considering system (1) together with the output

$$y = Dz, \tag{2}$$

and showing that global asymptotic stabilization by dynamic output feedback is achievable provided that a simple (and natural) detectability condition is added to the assumption used to achieve stabilization by (partial) state feedback.

Proposition 2.1: Consider system (1) with the output (2). Assume the following.
(H1) The linear approximation of system (1) around the origin is controllable.
(H2) The system $\dot{\xi} = f(\xi) + g(\xi)u$ is ISS and $\dot{\xi} = f(\xi)$ is locally exponentially stable.
(H3) The pair $\{D, J\}$ is observable.
(H4) $J + J^T \leq 0$[1].
Then there exist matrices L and K and a positive constant $\epsilon^\star$ such that for all $\epsilon \in (0, \epsilon^\star)$ the dynamic controller

$$\begin{array}{rcl} \dot{\hat{z}} & = & J\hat{z} + L(D\hat{z} - y) + Ru \\ u & = & -\epsilon\sigma(\frac{1}{\epsilon}K\hat{z}), \end{array} \tag{3}$$

where $\sigma(\cdot)$ is a saturation function[2] belonging to the sector $(0, 1]$, *globally asymptotically stabilizes* system (1).

Proof: To begin with note that $|u| \leq \epsilon$, hence by Assumption (H2), and selecting ϵ sufficiently small, the state ξ enters, in some finite time t_o, the region $\|\xi\| \leq \delta_\xi$, where $\delta_\xi > 0$ can be made arbitrarily small by choosing a small enough ϵ, and stays therein for all $t \geq t_o$, *i.e.* the region $\|\xi\| \leq \delta_\xi$ is positively invariant and can be arbitrarily shrunk. Consider now the approximation of system (1) in such a region, namely

$$\begin{array}{rcl} \dot{z} & = & Jz + H\xi + Ru \\ \dot{\xi} & = & F\xi + Gu, \end{array} \tag{4}$$

where $H = \left.\frac{\partial h(\xi)}{\partial \xi}\right|_{\xi=0}$, $F = \left.\frac{\partial f(\xi)}{\partial \xi}\right|_{\xi=0}$, $R = r(0)$ and $G = g(0)$, together with the $\dot{\hat{z}}$ equation given in (3), define the error variable $e = \hat{z} - z$ and notice that

$$\dot{e} = (J + LD)e - H\xi.$$

By assumption (H3) there exists a matrix L such that $J + LD$ is Hurwitz. Consider now the system

$$\begin{array}{rcl} \dot{e} & = & (J + LD)e - H\xi \\ \dot{\xi} & = & F\xi + Gu, \end{array} \tag{5}$$

which is globally asymptotically stable for $u = 0$ and ISS with respect to the input u and define the matrices

$$\mathcal{F} = \begin{bmatrix} J + LD & -H \\ 0 & F \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} 0 \\ G \end{bmatrix}, \quad \mathcal{H} = [LD \;\; 0].$$

[1]This assumption can be replaced by $J^T P + PJ \leq 0$, for some $P > 0$.
[2]A number of such functions can be chosen, the simplest of which is $\sigma(s) = \mathrm{sign}(s)\max\{1, |s|\}$.

As a result, system (5) rewrites in compact form as

$$\dot{\zeta} = \mathcal{F}\zeta + \mathcal{G}u,$$

where $\zeta = [e' \;\; \xi']'$. Note that the overall closed loop system with state $(\hat{z}, \zeta)$ is described, in the region $\|\xi\| \leq \delta_\xi$, by the equations

$$\begin{array}{rcl} \dot{\hat{z}} &=& J\hat{z} + \mathcal{H}\zeta + Ru \\ \dot{\zeta} &=& \mathcal{F}\zeta + \mathcal{G}u, \end{array} \tag{6}$$

i.e. it is a system of the same form as system (1), satisfying Assumptions (H1), (H2) and (H4), whereas $\hat{z}$ is now a measured state. As a result, the equilibrium $[\hat{z},\, \zeta] = [0,\, 0]$ can be asymptotically stabilized, as detailed in [17], by a saturated feedback of the state $\hat{z}$, *i.e.*

$$u = -\epsilon\sigma(\frac{1}{\epsilon}K\hat{z}), \tag{7}$$

for some appropriately chosen matrix K. We have thus concluded that the closed loop system (4)-(3) is globally asymptotically and locally exponentially stable.

To complete the proof it is necessary to show that the same properties hold for the closed loop system (1)-(3). This can be shown using standard converse Lyapunov arguments together with Assumption (H2), as detailed in [17]. ■

The result of Proposition 2.1 can be improved showing that system (1) in closed loop with the controller (3), in which u is replaced by $u = -\epsilon\sigma(\frac{1}{\epsilon}K\hat{z}) + v$, is ISS with respect to the new input v and with restriction $|v| < \epsilon$, see [16], [17]. Thus, Proposition 2.1 can be applied iteratively to construct output feedback stabilizing control laws for a more general class of systems in feedforward form, namely systems described by equations of the form

$$\begin{array}{rcl} \dot{z}_1 &=& J_1 z_1 + h_1(z_2, \cdots, \xi) + r_1(z_2, \cdots, \xi)u \\ \dot{z}_2 &=& J_2 z_2 + h_2(z_3, \cdots, \xi) + r_2(z_3, \cdots, \xi)u \\ \vdots && \\ \dot{z}_m &=& J_m z_m + h_m(\xi) + r_m(\xi)u \\ \dot{\xi} &=& f(\xi) + g(\xi)u \\ y_i &=& D_i z_i, \quad i = 1, \cdots, m \end{array} \tag{8}$$

as detailed in the following statement, the proof of which is trivial hence omitted.

Proposition 2.2: Consider the system described by equations (8) and assume the following holds.

(H1′) The linear approximation of system (8) is controllable.

(A2′) The system $\dot{\xi} = f(\xi) + g(\xi)u$ is ISS and $\dot{\xi} = f(\xi)$ is locally exponentially stable.

(A3′) The pairs $\{D_i, J_i\}$, for $i = 1, \cdots, m$, are observable.

(A4′) $J_i + J_i^T \leq 0$[3].

Then, there exist matrices L_i and K_i, for $i = 1, \cdots, m$, and a positive constant $\epsilon^\star$ such that for all $\epsilon \in (0, \epsilon^\star)$ the dynamic controller

$$\begin{array}{rcl} \dot{\hat{z}}_1 &=& J_1\hat{z}_1 + L_1(D_1\hat{z}_1 - y_1) + R_1 u \\ \dot{\hat{z}}_2 &=& J_2\hat{z}_2 + L_2(D_2\hat{z}_2 - y_2) + R_2 u \\ \vdots && \\ \dot{\hat{z}}_m &=& J_m\hat{z}_m + L_m(D_m\hat{z}_m - y_m) + R_m u \\ u &=& -\frac{\epsilon}{2}\sigma(\frac{2}{\epsilon}K_m\hat{z}_m) - \cdots - \frac{\epsilon}{2^m}\sigma(\frac{2^m}{\epsilon}K_1\hat{z}_1), \end{array} \tag{9}$$

globally asymptotically stabilizes system (8).

Remark 2.1: The results of Proposition 2.1 and Proposition 2.2 have to be compared with the results in [22], where a similar problem is solved using bounded control laws and a full order observer. Following the procedure presented therein, the construction of an output feedback stabilizing control law for system (8) requires at each step the construction of a full order observer, *i.e.* the state ξ has to be reconstructed m times, the state z_m $m-1$ times, and so on. This redundancy does not arise in our design. However, similar to [22], the computation of the stabilizing gains K_i in the construction of the observer (9), needs to be performed in steps, *i.e.* the gains K_m to K_i need to be assigned before K_{i-1} can be calculated. Also, the results in [22] apply to a slightly more general class of system, *i.e.* the matrices J_i are allowed to have multiple eigenvalues on the imaginary axis. It is fair to say, though, that the design proposed therein leads to more complex formulae than the one proposed in the present paper.

III. Disturbance rejection with bounded control

In this section we consider systems described by equations of the form

$$\begin{array}{rcl} \dot{x} &=& f(x) + g(x)u + \beta_1(x)w \\ y &=& h(x), \end{array} \tag{10}$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}$, output $y \in \mathbb{R}$ and disturbance $w \in \mathbb{R}^p$, which is generated by an exosystem of the form

$$\dot{w} = Sw, \tag{11}$$

and we address the problem of restricted approximate output regulation by bounded feedback, *i.e.* the problem of determining a bounded control law such that the effect of w on y is *attenuated* (in a way that will be specified) provided that $\|w\|$ is sufficiently small.

For simplicity, throughout the section the following assumptions are made.

(A1) System (10) is ISS with respect to u and w and $\dot{x} = f(x)$ is locally exponentially stable.

(A2) System (11) is *neutrally stable*, *i.e.* without loss of generality, $S + S' = 0$.

Remark 3.1: With assumption (A1) we put emphasis on the problem of disturbance rejection, when stabilization is not an issue. It must be pointed out, however, that the problem of disturbance rejection for more general, *i.e.* unstable, systems can be tackled with the tools presented in the paper.

[3] This assumption can be replaced by $J_i^T P_i + P_i J_i \leq 0$, for some $P_i > 0$.

A. Linear systems

This section is aimed at providing intuitive explanations of the results of the next session. Note that comprehensive results on the output regulation problem for linear systems with bounded control can be found, for example, in [6]. Consider a linear system described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + B_1 w \\ y &= Cx, \end{aligned} \tag{12}$$

where y is considered to be the measured output and at the same time the output to be regulated to zero. It is assumed that A is a Hurwitz matrix, *i.e.* system (12) is globally exponentially stable for $u = 0$ and $w = 0$.

It is well-known that in the presence of a constant disturbance w, integral action leads to dynamic laws that achieve regulation of the output y. Accordingly, for disturbances generated by an exosystem of the form (11) the *internal model* paradigm [12], [14], [4] gives directions for the design of control laws achieving disturbance rejection. In In particular, consider a system driven by the output y,

$$\dot{z} = Sz + Ly, \tag{13}$$

where the matrix L is chosen so that the pair $\{S, L\}$ is controllable, and the control law $u = -Kz$. Note that there exists K such that the system

$$\begin{aligned} \dot{z} &= Sz + LCx \\ \dot{x} &= Ax - BKz + B_1 w \end{aligned} \tag{14}$$

with $w = 0$, is globally exponentially stable. Next, observe that the system (14) is a globally exponentially stable system perturbed by the signal w, which is a linear combination of sinusoidal signals. The perturbed system has a well-defined, unique and attractive steady state response, which can be computed directly considering the (combined) system of (11) and (14) and invoking the center manifold theory, which yields that the steady state response is described by equations of the form

$$z^{ss}(t) = \pi_z(w(t)), \quad x^{ss}(t) = \pi_x(w(t)),$$

where $\pi_z(w)$ and $\pi_x(w)$ are such that $\pi_z(0) = 0$ and $\pi_x(0) = 0$ and satisfy

$$\begin{aligned} \frac{\partial \pi_z(w)}{\partial w} Sw &= S\pi_z(w) + LC\pi_x(w) \\ \frac{\partial \pi_x(w)}{\partial w} Sw &= -BK\pi_z(w) + A\pi_x(w) + B_1 w. \end{aligned} \tag{15}$$

Setting $z(w) = \Pi_z w$ and $x(w) = \Pi_x w$, the set of partial differential equations (15) reduces to the set of algebraic matrix equations

$$\begin{aligned} \Pi_z S &= S\Pi_z + LC\Pi_x \\ \Pi_x S &= -BK\Pi_z + A\Pi_x + B_1 \end{aligned} \tag{16}$$

which has always a solution, provided that the original system does not have transmission zeros which are eigenvalues of the exosystem (11). Moreover, as explained in [13], [19], the first of the equations (16) implies that $C\Pi_x = 0$. Therefore $y^{ss}(t) = Cx^{ss}(t) = C\Pi_x w(t) = 0$. We conclude, by attractivity of the steady state response, that, for any initial condition $w(0)$, $z(0)$ and $x(0)$, $\lim_{t\to\infty} y(t) = 0$.

The above discussion highlights the well-known fact that for linear systems a control law yielding output regulation in the presence of disturbances generated by a known, linear exosystem can be designed using very simple arguments.

B. Nonlinear systems

Consider now the system (10). Assumption (A1) implies that, for each $\delta > 0$, there exist constants $\epsilon_u^\star > 0$ and $\epsilon_w^\star > 0$ such that if $|u(t)| \le \epsilon_u^\star$ and $\|w(t)\| \le \epsilon_w^\star$ for all $t > 0$, then, in finite time, the state of the nonlinear system (10) enters in finite time the region $\|x\| \le \delta$. In this region, system (10) can be approximated by system (12) where

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}, \quad B = g(0), \quad B_1 = \beta_1(0).$$

Moreover, according to the generalized small gain theorem of [24], as exploited in [16], [17], there exists a matrix K such that the system

$$\begin{aligned} \dot{z} &= Sz + LCx \\ \dot{x} &= Ax - \epsilon B\sigma(\tfrac{1}{\epsilon}Kz) \end{aligned}$$

with $\sigma(s) = \mathrm{sign}(s)\min\{1, |s|\}$, is also globally asymptotically (locally exponentially) stable for all $\epsilon > 0$.

The above discussion motivates the following result.

Proposition 3.1: Consider the system (10) and suppose that the disturbance w is generated by the exosystem (11). Suppose that Assumptions (A1) and (A2) hold. Suppose moreover that the linear approximation of the system around the origin has no transmission zeros on the imaginary axis[4].

Then there exist positive constants $\epsilon_w^\star$, $\epsilon_u^\star$ and δ_y and matrices L and K such that for all $\epsilon \in (0, \epsilon_u^\star)$ and for all disturbances such that $\|w(t)\| \le \epsilon_w^\star$ for all $t \ge 0$, the bounded dynamic output feedback control law

$$\begin{aligned} \dot{z} &= Sz + Ly \\ u &= -\epsilon\sigma\left(\tfrac{1}{\epsilon}Kz\right) \end{aligned} \tag{17}$$

yields closed loop trajectories with the following properties.

(P1) $\|y(t)\| \le \delta_y$ for all $t \ge 0$ and for a constant $\delta_y > 0$.

(P2) There exists a periodic function $y_{ss}(t)$, independent from the initial conditions, such that, for any initial condition, one has $\lim_{t\to\infty}(y(t) - y_{ss}(t)) = 0$. Moreover $y_{ss}(t)$ does not contain frequency components corresponding to eigenvalues of the matrix S.

Sketch of the proof. According to the result in [17] there exists a positive constant $\epsilon_u^\star$ and a matrix K such that, for $w = 0$ and all $\epsilon \in (0, \epsilon_u^\star)$, the closed loop system (10)-(17), which is given by the cascade

$$\begin{aligned} \dot{z} &= Sz + Lh(x) \\ \dot{x} &= f(x) - g(x)\epsilon\sigma\left(\tfrac{1}{\epsilon}Kz\right) \end{aligned}$$

[4]This assumption may be relaxed requiring only that the transmission zeros are not eigenvalues of the exosystem.

is globally asymptotically and locally exponentially stable.

By the discussion above note that for small enough disturbances w the well-defined steady state trajectories of the closed loop system (10)-(17) will be given by

$$z_{ss}(t) = \pi_z(w(t)), \quad x_{ss}(t) = \pi_x(w(t)),$$

where the functions $\pi_z(w)$ and $\pi_x(w)$ are such that $\pi_z(0) = 0$ and $\pi_x(0) = 0$ and solve the set of partial differential equations

$$\begin{aligned} \frac{\partial \pi_z(w)}{\partial w} Sw &= S\pi_z(w) + Lh(\pi_x(w)) \\ \frac{\partial \pi_x(w)}{\partial w} Sw &= -g(\pi_x(w))\epsilon\sigma\left(\tfrac{1}{\epsilon}K\pi_z(w)\right) \\ &\quad + f(\pi_x(w)) + b_1(\pi_x(w))w. \end{aligned}$$

The first approximation of the above partial differential equations yields equations (16), hence the claim is a consequence of the discussion in Section III-A and of the general results in [9].

Remark 3.2: The suggested design is based on a linear way of thinking, hence, for general nonlinear systems and as pointed out in [11], there is no guarantee that the *resonant* frequencies of the steady state output will be unaffected by the disturbance. A way to overcome this problem is to use the k-fold internal model of [9]. For example, if the exosystem generating the disturbance is such that

$$S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix},$$

then a controller that uses the dynamic compensator $\dot{z} = S_h z + Ly$ with $S_h = \mathbf{blockdiag}\{S, 2S, 3S\}$. is such that the steady state output response is periodic and has no components at angular velocities ω, 2ω and 3ω.

Remark 3.3: The combination of the results in Proposition 2.1 and 3.1 can provide a solution to the problem of approximate and restricted output regulation by means of error feedback for some class of feedforward systems, namely systems of the form

$$\begin{aligned} \dot{z} &= Jz + h(\xi) + r(\xi)u + r_1(\xi)w \\ \dot{\xi} &= f(\xi) + g(\xi)u + g_1(\xi)w \\ y &= Cz \end{aligned} \tag{18}$$

where y represents the measured output, that needs to be regulated to zero, and $w \in \mathbb{R}^p$ a disturbance that is generated by an exosystem of the form (11). Under the standard controllability and observability assumptions, the assumption on the LES-ISS of the ξ-subsystem with respect to both u and w, and the assumption that $J + J' \leq 0$ and $S + S' \leq 0$, it is possible to achieve approximate and restricted regulation of the output y by means of a bounded dynamic output feedback law of the form

$$\begin{aligned} \dot{\zeta} &= S\zeta + L_2 y \\ \dot{\hat{z}} &= \mathcal{J}\hat{z} - L_1 y + Ru \\ u &= -\epsilon\sigma\left(\frac{1}{\epsilon}K_1\hat{z}\right) - \frac{\epsilon}{2}\sigma\left(\frac{2}{\epsilon}K_2\zeta\right). \end{aligned} \tag{19}$$

IV. Output regulation of the TORA

In this section we illustrate the contribution of the paper considering the disturbance rejection problem for a nonlinear benchmark system: the *Translational Oscillator with a Rotational Actuator*, commonly referred to as TORA [2]. The disturbance rejection problem for such a system has also been studied in [28], whereas various constructive nonlinear control methodologies have been tested on this system, see for example [15], [25], [23]. The system is four-dimensional, with states the translational and angular positions x_d and ϕ, respectively, and the corresponding velocities v_d and ω. In the coordinates

$$\begin{aligned} x_1 &= x_d + \bar{\varepsilon}\sin\phi \\ x_2 &= v_d + \bar{\varepsilon}\omega\cos\phi \\ x_3 &= \phi \\ x_4 &= \psi(x_3)\omega \end{aligned} \tag{20}$$

and for some appropriate function $\psi(x_3)$, the system can be written in *block* feedforward form. The available measurements for control are the positions x_d and ϕ, or equivalently the states x_1 and x_3. In [17] the (global) output feedback stabilization problem has been solved by means of a dynamic controller and a saturated feedback. According to the construction therein, and following [28] for the introduction of disturbances into the closed loop, the model of the “controlled” TORA, subject to a sinusoidal disturbance w acting on the translational movement, is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \bar{\varepsilon}\sin x_3 - \gamma(x_3)\bar{\varepsilon}\cos(x_3)w \\ \dot{x}_3 &= \alpha(x_3)x_4 \\ \dot{x}_4 &= -x_3\alpha(x_3) - x_4 - z - \epsilon\sigma\left(\tfrac{1}{\epsilon}kx_1\right) \\ &\quad +v - \gamma(x_3)\psi(x_3)w \\ \dot{z} &= -\lambda(x_3)z - b\epsilon\sigma\left(\tfrac{1}{\epsilon}kx_1\right) + bv + \gamma(x_3)\psi(x_3)w, \end{aligned} \tag{21}$$

where $\alpha(x_3)$, $\psi(x_3)$, $\lambda(x_3)$ and $\gamma(x_3)$ are known functions, and $\bar{\varepsilon}$, b, k, ϵ are known parameters of the system and of the stabilizing controller. The output to be regulated is the horizontal position of the system given by

$$y = x_d = x_1 - \bar{\varepsilon}\sin(x_3). \tag{22}$$

System (21) is ISS with respect to w and v. To attenuate the effect of the disturbance on the output (22) we introduce an ‘internal model’ based dynamic compensator with a bounded regulator action, namely

$$\dot{z} = Sz + Ly, \quad v = -\frac{\epsilon}{2}\sigma\left(\frac{2}{\epsilon}Kz\right). \tag{23}$$

The response of the closed loop system is shown in Figure 1. Note the asymptotic behavior of the regulated output x_d. In Figure 2 we plot the spectrum decomposition of the signal $y(t)$ when no regulation of the output is engaged (top graph) and when the controller (23) is used (graph in the middle). Note the different scale of the magnitude in the two cases. When the primary oscillation frequency ω is attenuated (by means of the dynamic feedback (23)) it is possible to detect a harmonic at 3ω. This second harmonic can also be attenuated if a 2-fold internal model is used, as explained in Remark 3.2. The result is illustrated in the bottom graph of figure 2.

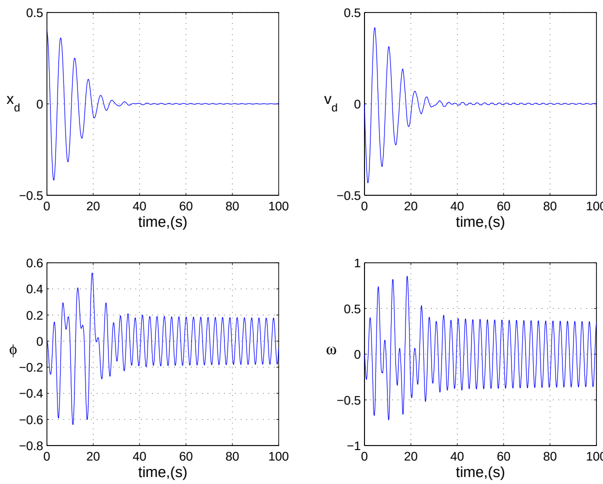

Fig. 1. Response of the TORA system, driven by a sinusoidal disturbance, in closed loop with the control law (23).

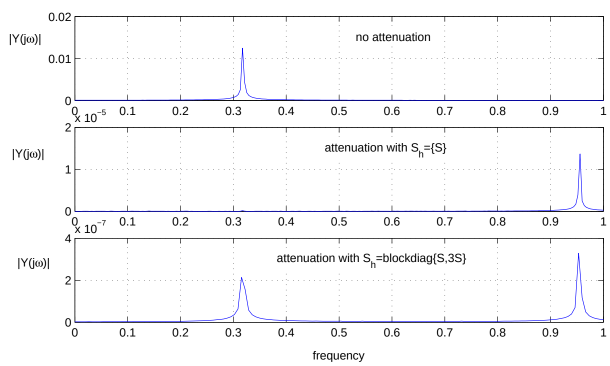

Fig. 2. Spectrum decomposition of the signal $y(t)$ when no regulation of the output is engaged (top graph), when the controller (23) is used (graph in the middle) and when a controller with a 2-fold internal model is used. Note the different scale on the $|Y(j\omega)|$ axis.

V. Conclusions

The problem of global asymptotic stabilization of a class of feedforward systems has been addressed and solved using dynamic output feedback control laws delivering a bounded action. This result has been exploited to solve the problem of approximate and restricted output regulation for general nonlinear systems, provided that the exogenous signals are generated by a linear exosystem.

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