Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

A High Gain Multiple Time-Scaling Technique for Global Control of Nonlinear Systems

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Abstract: We consider a general high-gain scaling technique for global control of strict-feedback-like systems. Unlike previous results, the scaling utilizes arbitrary powers (instead of requiring successive powers) of the high gain parameter with the powers chosen to satisfy certain inequalities depending on system nonlinearities. The scaling induces a weak-Cascading Upper Diagonal Dominance (w-CUDD) structure on the dynamics. The analysis is based on our recent results on the w-CUDD property and uniform solvability of coupled state-dependent Lyapunov equations. The proposed scaling provides extensions in both state-feedback and output-feedback cases. The state-feedback problem is solved for a class of systems with certain ratios of nonlinear terms being polynomially bounded. The controller has a simple form being essentially linear with state-dependent dynamic gains and does not involve recursive computations. In the output-feedback case, the scaling technique is applied to the design of the observer which is then coupled with a backstepping controller. The results relax the assumption in our earlier papers on cascading dominance of upper diagonal terms. However, since the required upper diagonal cascading dominance in observer and controller contexts are dual, it is not possible to use a dual high-gain observer/ controller in the proposed design preventing the bounds on uncertain functions from being of the more general form in our earlier work. A topic of further research is to examine the possibility of a scaling (perhaps utilizing more than one high gain parameter) that achieves bidirectional cascading dominance.

I. Introduction

The class of systems considered in this paper is

$$\begin{aligned}\dot{x}_i &= \phi_i(t, y, x_2, \ldots, x_i) + \phi_{(i,i+1)}(y)x_{i+1} \ , \ i = 1, \ldots, s-1 \\ x_{s+i} &= \phi_{s+i}(t, y, x_2, \ldots, x_{s+i}) + \phi_{(s+i,s+i+1)}(y)x_{s+i+1} \\ &\quad + \mu_i(y)u \ , \quad i = 0, \ldots, n-s \ ; \ x_{n+1} = 0 \qquad (1)\end{aligned}$$

where $x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n$ is the state and $u \in \mathcal{R}$ is the input. $\phi_{(i,i+1)}$ and μ_i are continuous functions with $\phi_{(n,n+1)} \equiv 0$. ϕ_i are time-varying uncertain functions.[1] s is the relative degree of the system. We consider both state ($y = x$) and output-feedback ($y = x_1$) problems.

Among the control design methodologies developed in the literature for various classes of nonlinear systems ([1–3] and references therein), backstepping and its robust and adaptive variants are particularly suited for lower triangular systems. While (1) is not, in general, in lower triangular form in the state-feedback case ($y = x$), a variant of robust backstepping under certain assumptions on ϕ_i and $\phi_{(i,i+1)}$ can be applied.

Assuming that ϕ_i are known and linear in the unmeasured states, i.e., $\phi_i = \phi_{(i,1)}(x_1)+\sum_{j=2}^{i}\phi_{(i,j)}(x_1)x_j$, the *generalized output-feedback canonical form* [4] is obtained which reduces to the *standard output-feedback canonical* form [1–3] if $\phi_{(i,j)}, j \geq 2$ are constants. In [5], this system class was considered with $\phi_{(i,j)}$'s being bounded. This restriction was removed in [6] using an observer of order $n(n+3)/2$ with gains generated through a matrix Differential Riccati Equation (DRE). In [4], assuming that a constant positive-definite matrix can be found to satisfy a certain inequality, a solution was proposed of dynamic order $(n-1)$ and the observer structure recovered the standard linear reduced-order observer for linear systems. A sufficient condition for the existence of a matrix satisfying the required inequality was shown to be the CUDD condition requiring the *upper diagonal* terms $\phi_{(i,i+1)}$ to be larger than $\phi_{(i,j)}$ and $\phi_{(i+1,i+2)}$ (to within constant factors) [4]. Adaptive extensions of the results in [4] and [6] were considered in [7] assuming that parametric uncertainties appear in output-dependent terms. In [8], assuming that $\phi_{(i,i+1)} = 1$ for $i = 1, \ldots, n-1$, and that $\phi_i, i = 1, \ldots, n$, are incrementally linearly bounded in unmeasured states, the matrix DRE in [6] was collapsed to a scalar DRE driven by x_1 and governing a scalar parameter r appearing as a dynamic high gain scaling. The scaling in [8] differs from earlier high gain[9–11] results in two features: 1) the dynamics of r are a scalar DRE driven by y guaranteeing boundedness of r if y remains bounded (which is not guaranteed by the classical dynamic high gain scaling with $\dot{r} = y^2$), and 2) an additional scaling $\frac{1}{r^b}$ is introduced in the scaled observer error definition. In [12], the high gain scaling was shown to essentially amplify the upper diagonal terms thus inducing CUDD [4]. Furthermore, the additional scaling r^b was removed in [12] through the solution of a pair of coupled Lyapunov equations. Motivated by duality considerations, a dynamic high gain scaling based state-feedback controller was designed in [13] and a dual high gain observer/controller architecture was proposed in [14,15]. The removal of r^b in our earlier result proves important in the dual design.

In this paper, we consider a generalization of the scaling technique by introducing arbitrary powers of the high gain parameter. The standard scaling $\frac{x_i}{r^{ai+b}}$ with constants a and b can scale the functions ϕ_i relative to the upper diagonal terms $\phi_{(i,i+1)}$ as in [12]. However, this scaling does not modify the relative magnitudes of the upper diagonal terms since all upper diagonal terms are scaled by r^a. Hence, it was necessary to assume cascading dominance of upper diagonal terms [13,14]. In this paper, it is seen that a scaling $\frac{x_i}{r^{q_i}}$ with arbitrary constants q_i can scale ratios of upper diagonal terms. Moreover, choosing constants q_i appropriately, nonlinear functions ϕ_i can be scaled to obtain control designs under weaker assumptions. The scaling technique and basic results are presented in Section II. In Section III, polynomial bounds on ratios of certain terms in the system dynamics are shown to be sufficient to design a state-feedback high gain based controller of an algebraically simple structure, being essentially a linear feedback with state-dependent dynamic gains and involving no recursive computations. The associated Lyapunov function is simply quadratic in the scaled states. The output-feedback problem is considered in Section IV. The cascading dominance assumption on upper diagonal terms in [14] is not required in this design since the cascading dominance is essentially achieved by the scaling. The obtained output-feedback controller is applicable to the largest class of systems for which output-feedback results are currently available and includes the class of systems in [6] as a special case. While [6] required a matrix DRE resulting in a controller of order $n(n+3)/2$, the proposed solution is of order n. This confirms the expectation noted in the conclusion of [4] that an n^{th} order system linear in unmeasured states should be output-feedback stabilizable with a dynamic observer/controller of order $O[n]$.

[1] ϕ_i can depend on all states and input. However, ϕ_i are shown in (1) to be functions only of subsets of the state to emphasize the state dependence of the bounds to be introduced on ϕ_i.

II. Definitions and Basic Theorems

Definition 1: Let ρ be a positive constant. An $n\times n$ matrix A is said to be w-CUDD(ρ) if the following hold:
1) A is in lower Hessenberg form, i.e.,[2] $A_{(i,j)} \equiv 0$ for $j \geq i+2$.
2) The upper diagonal elements of A are non-zero, i.e., $A_{(i,i+1)} \neq 0$, $i = 1,\ldots,n-1$.
3) The following inequalities hold:

$$\begin{aligned}\frac{|A_{(i,j)}|}{\sqrt{|A_{(i,i+1)}||A_{(j-1,j)}|}} &\leq \rho\ ,\ i=2,\ldots,n-1, j=2,\ldots,i\\ \frac{|A_{(n,j)}|}{\sqrt{|A_{(n-1,n)}||A_{(j-1,j)}|}} &\leq \rho\ ,\ j=2,\ldots,n\\ \frac{|A_{(i,i+1)}|}{|A_{(i-1,i)}|} &\leq \rho\ ,\ i=2,\ldots,n-1. \end{aligned} \quad (2)$$

Definition 2: Let ρ be a positive constant. An $n\times n$ matrix A is said to be dual w-CUDD(ρ) if the following hold:
1) A is in lower Hessenberg form, i.e., $A_{(i,j)} \equiv 0$ for $j \geq i+2$.
2) The upper diagonal elements of A are non-zero, i.e., $A_{(i,i+1)} \neq 0$, $i = 1,\ldots,n-1$.
3) The following inequalities hold:

$$\begin{aligned}\frac{|A_{(i,j)}|}{\sqrt{|A_{(i,i+1)}||A_{(j-1,j)}|}} &\leq \rho\ ,\ i=2,\ldots,n-1, j=2,\ldots,i\\ \frac{|A_{(i,1)}|}{\sqrt{|A_{(i,i+1)}||A_{(1,2)}|}} &\leq \rho\ ,\ i=1,\ldots,n-1\\ \frac{|A_{(i-1,i)}|}{|A_{(i,i+1)}|} &\leq \rho\ ,\ i=2,\ldots,n-1. \end{aligned} \quad (3)$$

The w-CUDD and dual w-CUDD concepts are dual in the sense that a matrix A is w-CUDD(ρ) if and only if the matrix $\tilde{A} = QA^TQ$ is dual w-CUDD(ρ) where Q is the matrix with 1's on the anti-diagonal and zeros elsewhere. The w-CUDD and dual w-CUDD concepts appear in the observer and controller contexts, respectively, for systems of form (1). The duality between these concepts can be used to map theorems from one context to the other and essentially results in a unified observer and controller design procedure.

Theorem 1: Let θ be a variable ranging over some set Θ. Let $A(\theta)$ be an $n\times n$ matrix function of θ. Let $D_1(\theta),\ldots,D_n(\theta)$ be scalar real-valued functions of θ. Assume that positive constants $\underline{D}$ and $\overline{D}$ exist such that $\underline{D} \leq D_i(\theta) \leq \overline{D}, i = 1,\ldots,n$, for all $\theta\in\Theta$. Define $D = \mathrm{diag}[D_1(\theta),\ldots,D_n(\theta)]^T$. Let $C = [1,0,\ldots,0]$ be a $1\times n$ vector. Assume that a positive constant ρ exists such that $A(\theta)$ is w-CUDD(ρ) for every $\theta\in\Theta$. Assume also that for each $i\in\{1,\ldots,n-1\}$, the sign of $A_{(i,i+1)}(\theta)$ is independent of θ. Then, an $n\times 1$ vector $G(\theta)$, a constant symmetric positive-definite matrix P, and positive constants ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ exist such that for all $\theta\in\Theta$,

$$P[A(\theta)+G(\theta)C]+[A(\theta)+G(\theta)C]^TP \leq -\nu_1|A_{(n-1,n)}(\theta)|I \quad (4)$$

$$\underline{\nu}_2 I \leq PD(\theta)+D(\theta)P \leq \overline{\nu}_2 I. \quad (5)$$

Let $\underline{A}_{(i,i+1)}(\theta)$ and $\overline{A}_{(i,i+1)}(\theta), i = 1,\ldots,n-1$, be positive functions of θ satisfying $\underline{A}_{(i,i+1)}(\theta) \leq |A_{(i,i+1)}(\theta)| \leq \overline{A}_{(i,i+1)}(\theta)$, $i=1,\ldots,n-1$, for all $\theta\in\Theta$. Let $\overline{A}_{(i,1)}(\theta), i = 1,\ldots,n$, be positive functions of θ satisfying $|A_{(i,1)}(\theta)| \leq \overline{A}_{(i,1)}(\theta)$ for all $\theta\in\Theta$. Then, the choice of $G(\theta)$ to satisfy (4) needs to depend only on $\underline{A}_{(i,i+1)}(\theta)$ and $\overline{A}_{(i,i+1)}(\theta)$, $i = 1,\ldots,n$, and $\overline{A}_{(i,1)}(\theta), i = 1,\ldots,n$. Furthermore, $G(\theta)$ can be picked such that a positive constant $\overline{G}$ exists satisfying

$$|G(\theta)| \leq \overline{G}\left[\max\{\overline{A}_{(1,2)}(\theta),\ldots,\overline{A}_{(n-1,n)}(\theta)\} + \frac{\max\{\overline{A}^2_{(1,1)}(\theta),\ldots,\overline{A}^2_{(n,1)}(\theta)\}}{\min\{\underline{A}_{(1,2)}(\theta),\ldots,\underline{A}_{(n-1,n)}(\theta)\}}\right]. \quad (6)$$

Theorem 2: Let θ be a variable ranging over some set Θ. Let $A(\theta)$ be an $n\times n$ matrix function of θ. Let $D_1(\theta),\ldots,D_n(\theta)$ be scalar real-valued functions of θ. Assume that positive constants $\underline{D}$ and $\overline{D}$ exist such that $\underline{D} \leq D_i(\theta) \leq \overline{D}, i = 1,\ldots,n$, for all $\theta\in\Theta$. Define $D = \mathrm{diag}[D_1(\theta),\ldots,D_n(\theta)]^T$. Let $B = [0,\ldots,0,1]^T$ be an $n\times 1$ vector. Assume that a positive constant ρ exists such that $A(\theta)$ is dual w-CUDD(ρ) for every $\theta\in\Theta$. Assume also that for each $i\in\{1,\ldots,n-1\}$, the sign of $A_{(i,i+1)}(\theta)$ is independent of θ. Then, a $1\times n$ vector $K(\theta)$, a constant symmetric positive-definite matrix P, and positive constants ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ exist such that for all $\theta\in\Theta$,

$$P[A(\theta)+BK(\theta)]+[A(\theta)+BK(\theta)]^TP \leq -\nu_1|A_{(1,2)}(\theta)|I \quad (7)$$

$$\underline{\nu}_2 I \leq PD(\theta)+D(\theta)P \leq \overline{\nu}_2 I. \quad (8)$$

Let $\underline{A}_{(i,i+1)}(\theta)$ and $\overline{A}_{(i,i+1)}(\theta)$, $i = 1,\ldots,n-1$, be positive functions of θ satisfying $\underline{A}_{(i,i+1)}(\theta) \leq |A_{(i,i+1)}(\theta)| \leq \overline{A}_{(i,i+1)}(\theta)$, $i = 1,\ldots,n-1$, for all $\theta\in\Theta$. Let $\overline{A}_{(n,j)}(\theta), j = 1,\ldots,n$ be positive functions of θ satisfying $|A_{(n,j)}(\theta)| \leq \overline{A}_{(n,j)}(\theta)$ for all $\theta\in\Theta$. Then, the choice of $K(\theta)$ to satisfy (7) needs to depend only on $\underline{A}_{(i,i+1)}(\theta)$ and $\overline{A}_{(i,i+1)}(\theta)$, $i = 1,\ldots,n$, and $\overline{A}_{(n,j)}(\theta), j = 1,\ldots,n$. Furthermore, $K(\theta)$ can be picked such that a constant $\overline{K} > 0$ exists satisfying

$$|K(\theta)| \leq \overline{K}\left[\max\{\overline{A}_{(1,2)}(\theta),\ldots,\overline{A}_{(n-1,n)}(\theta)\} + \frac{\max\{\overline{A}^2_{(n,1)}(\theta),\ldots,\overline{A}^2_{(n,n)}(\theta)\}}{\min\{\underline{A}_{(1,2)}(\theta),\ldots,\underline{A}_{(n-1,n)}(\theta)\}}\right]. \quad (9)$$

Remark 1: The proofs of Theorems 1 and 2 can be found in [16] and are omitted for brevity. In the particular case where A has nonzero entries only on the upper diagonal, solvability of the coupled Lyapunov equations was proved in [12]. The solvability of the single Lyapunov equation (4) assuming CUDD was shown in [4]. Theorem 1 can be thought of as a result on assignability of observer gains so that the system matrix $A(\theta)+G(\theta)C$ is uniformly stable in the sense that a common quadratic Lyapunov function demonstrating stability exists for the entire family $A(\theta)+G(\theta)C$, $\theta\in\Theta$. Theorem 2 which is a dual of Theorem 1 can be similarly interpreted in the context of assignment of controller gains.

Theorem 3: Let $A(\theta,\beta)$ be an $n\times n$ matrix function of $\theta = [\theta_1,\ldots,\theta_n]\in\mathcal{R}^n$ and $\beta\in\mathcal{R}^m$. Assume that a positive function $f(\beta)$ exists such that $A(\theta,\beta)$ is w-CUDD($f(\beta)$) for all $\theta\in\mathcal{R}^n, \beta\in\mathcal{R}^m$. Let ρ be any positive constant. Then, positive constants $q_1,\ldots,q_n$ exist such that for every $\theta\in\mathcal{R}^n$ and $\beta\in\mathcal{R}^m$, the matrix $\tilde{A}(r,\theta,\beta) = T(r)A(\theta,\beta)T^{-1}(r)$ is w-CUDD(ρ) for all $r \geq \frac{f(\beta)}{\rho}$ where $T(r) = \mathrm{diag}(\frac{1}{r^{q_1}},\ldots,\frac{1}{r^{q_n}})$.

Proof of Theorem 3: Define

$$q_i = 1+\frac{1}{2}(n-1)n-\frac{1}{2}(n-i)(n-i+1)\ ,\ i=1,\ldots,n. \quad (10)$$

The matrix $\tilde{A}$ is easily seen to satisfy Properties 1 and 2 in Definition 1. To verify Property 3, note that the $(i,j)^{th}$ element of $\tilde{A}$ is given by $\tilde{A}_{(i,j)} = r^{q_j-q_i}A_{(i,j)}$. Hence,

$$\begin{aligned}\frac{|\tilde{A}_{(i,j)}|}{\sqrt{|\tilde{A}_{(i,i+1)}||\tilde{A}_{(j-1,j)}|}} &= r^{\frac{q_j+q_{j-1}-q_i-q_{i+1}}{2}}\frac{|A_{(i,j)}|}{\sqrt{|A_{(i,i+1)}||A_{(j-1,j)}|}}\\ &= r^{\frac{(j-i-1)(2n-i-j+1)}{2}}\frac{|A_{(i,j)}|}{\sqrt{|A_{(i,i+1)}||A_{(j-1,j)}|}}\\ &\leq \frac{f(\beta)}{r}\ ,\ i=2,\ldots,n-1, j=2,\ldots,i\end{aligned}$$

[2] If A is a matrix, $A_{(i,j)}$ denotes its $(i,j)^{th}$ element.

$$
\begin{aligned}
\frac{|\tilde{A}_{(n,j)}|}{\sqrt{|\tilde{A}_{(n-1,n)}||\tilde{A}_{(j-1,j)}|}} &= r^{\frac{q_j+q_{j-1}+q_{n-1}-3q_n}{2}} \frac{|A_{(n,j)}|}{\sqrt{|A_{(n-1,n)}||A_{(j-1,j)}|}} \\
&= \frac{1}{r^{\frac{1+(n-j+1)^2}{2}}} \frac{|A_{(n,j)}|}{\sqrt{|A_{(n-1,n)}||A_{(j-1,j)}|}} \\
&\leq \frac{f(\beta)}{r} \ , \ j = 2, \dots, n \\
\frac{|\tilde{A}_{(i,i+1)}|}{|\tilde{A}_{(i-1,i)}|} &= r^{(q_{i+1}+q_{i-1}-2q_i)} \frac{|A_{(i,i+1)}|}{|A_{(i-1,i)}|} \\
&= \frac{1}{r} \frac{|A_{(i,i+1)}|}{|A_{(i-1,i)}|} \leq \frac{f(\beta)}{r}, i = 2, \dots, n-1. \quad (11)
\end{aligned}
$$

If $r \geq \frac{f(\beta)}{\rho}$, it readily follows that $\tilde{A}$ is w-CUDD(ρ).

Theorem 4: Let $A(\theta, \beta)$ be an $n \times n$ matrix function of $\theta = [\theta_1, \dots, \theta_n] \in \mathcal{R}^n$ and $\beta \in \mathcal{R}^m$. Assume that a positive function $f(\beta)$ and non-negative constants $\zeta_{(i,j,k)}, i = 2, \dots, n, j = 2, \dots, i, k = 1, \dots, i$, exist such that the following inequalities hold[3] for all $\theta \in \mathcal{R}^n$ and $\beta \in \mathcal{R}^m$:

$$
\frac{|A_{(i,j)}|}{\sqrt{|A_{(i,i+1)}||A_{(j-1,j)}|}} \leq f(\beta)[1 + \sum_{k=1}^{i} |\theta_k|^{\zeta_{(i,j,k)}}], i = 2, \dots, n-1, \quad j = 2, \dots, i
$$

$$
\frac{|A_{(i,1)}|}{\sqrt{|A_{(i,i+1)}||A_{(1,2)}|}} \leq f(\beta)[1 + \sum_{k=1}^{i} |\theta_k|^{\zeta_{(i,1,k)}}] \ , \ i = 1, \dots, n-1
$$

$$
\frac{|A_{(i-1,i)}|}{|A_{(i,i+1)}|} \leq f(\beta)[1 + \sum_{k=1}^{i} |\theta_k|^{\zeta_{(i,i+1,k)}}] \ , \ i = 2, \dots, n-1. \quad (12)
$$

Let ρ be any positive constant. Then, positive constants $q_1, \dots, q_n$, and a positive function $R(\theta, \beta)$ exist such that for every $\theta \in \mathcal{R}^n$ and $\beta \in \mathcal{R}^m$, the matrix $\tilde{A}(r, \theta, \beta) = T(r)A(T^{-1}(r)\theta, \beta)T^{-1}(r)$ is dual w-CUDD(ρ) for all $r \geq R(\theta, \beta)$ where $T(r) = \mathrm{diag}(\frac{1}{r^{q_1}}, \dots, \frac{1}{r^{q_n}})$.

Proof of Theorem 4: The matrix $\tilde{A}$ is easily seen to satisfy Properties 1 and 2 in Definition 2. Note that the $(i,j)^{th}$ element of $\tilde{A}$ is given by $\tilde{A}_{(i,j)}(r, \theta, \beta) = r^{q_j - q_i} A_{(i,j)}(T^{-1}(r)\theta, \beta)$. Pick q_1 to be any arbitrary constant. $q_2, \dots, q_n$ can be recursively obtained to satisfy the inequalities

$$
\begin{aligned}
q_{i+1} \geq \max \Big\{ & \Big[2 - q_i + q_j + q_{j-1} + 2\max\{q_k \zeta_{(i,j,k)} | k = 1, \dots, i\} \Big], \\
& \Big[2 - q_i - q_2 + 3q_1 + 2\max\{q_k \zeta_{(i,1,k)} | k = 1, \dots, i\} \Big], \\
& \Big[1 + 2q_i - q_{i-1} + \max\{q_k \zeta_{(i,i+1,k)} | k = 1, \dots, i\} \Big] \Big\}. \quad (13)
\end{aligned}
$$

Defining

$$
\begin{aligned}
R(\theta, \beta) = \max \Big\{ 1, \frac{f(\beta)}{\rho} \max\{[1 + \sum_{k=1}^{i} \theta_k^{\zeta_{(i,j,k)}}] \,|\, i = 2, \dots, n, \\
j = 2, \dots, i, k = 2, k = 1, \dots, i\} \Big\}, \quad (14)
\end{aligned}
$$

we see through inequalities similar to (11) that $\tilde{A}(r, \theta, \beta)$ is dual w-CUDD(ρ) for all $r \geq R(\theta, \beta)$ and all $\theta \in \mathcal{R}^n, \beta \in \mathcal{R}^m$.

III. State-Feedback Control

A. Assumptions

Assumption A1: Observability, controllability, and uniform relative degree of (1), i.e.,

$$
|\phi_{(i,i+1)}(x)| \geq \sigma > 0 \ , \ 1 \leq i \leq n-1 \ ; \ |\mu_0(x)| \geq \sigma > 0 \, \forall x \in \mathcal{R}^n. \quad (15)
$$

Furthermore, the sign of each $\phi_{(i,i+1)}, i = 1, \dots, n-1$, is independent of its argument.

Assumption A2: The *inverse dynamics* of (1), i.e.,

$$
\begin{aligned}
\dot{z}_i &= \phi_{s+i}(t, \Upsilon, [v_1, y_2]^T, v_2, \dots, v_s, z_1, \dots, z_i) \\
&+ \phi_{(s+i,s+i+1)}([v_1, y_2]^T) z_{i+1} - \frac{\mu_i([v_1, y_2]^T)}{\mu_0([v_1, y_2]^T)} \phi_{(s,s+1)}([v_1, y_2]^T) z_1 \\
&+ \mu_i([v_1, y_2]^T) v_0 \ , \ 1 \leq i \leq n-s \quad (16)
\end{aligned}
$$

with $z_{n-s+1} = 0$ being a dummy variable are Bounded-Input-Bounded-State (BIBS) stable with states $(z_1, \dots, z_{n-s})$ and inputs $(\Upsilon, v_0, \dots, v_s)$ where[4] $y_2 = [z_1, \dots, z_{n-s}]^T$.

Assumption A3: The functions ϕ_i can be bounded as

$$
|\phi_i(t, y, x_2, \dots, x_i)| \leq \sum_{j=1}^{i} \phi_{(i,j)}(x)|x_j| \ , \ i = 1, \dots, s \quad (17)
$$

with $\phi_{(i,j)}, i = 1, \dots, s, j = 1, \dots, i$, being known continuous nonnegative functions. Nonnegative functions $\gamma_{(i,j)}(x_1)$ and nonnegative constants $\zeta_{(i,j,k)}, i = 2, \dots, n, j = 2, \dots, i, k = 1, \dots, i$, exist such that for all $x \in \mathcal{R}^n$

$$
\frac{|\phi_{(i,j)}(x)|}{\sqrt{|\phi_{(i,i+1)}(x)||\phi_{(j-1,j)}(x)|}} \leq \gamma_{(i,j)}(x_1) \sum_{k=1}^{i} |x_k|^{\zeta_{(i,j,k)}}, i = 2, \dots, n-1, \quad j = 2, \dots, i
$$

$$
\frac{|\phi_{(i,1)}(x)|}{\sqrt{|\phi_{(i,i+1)}(x)||\phi_{(1,2)}(x)|}} \leq \gamma_{(i,1)}(x_1) \sum_{k=1}^{i} |x_k|^{\zeta_{(i,1,k)}}, i = 1, \dots, n-1
$$

$$
\frac{|\phi_{(i-1,i)}(x)|}{|\phi_{(i,i+1)}(x)|} \leq \gamma_{(i-1,i)}(x_1) \sum_{k=1}^{i} |x_k|^{\zeta_{(i,i+1,k)}} \ , \ i = 2, \dots, n-1. \quad (18)
$$

Furthermore, a positive function $\hat{\phi}_{(1,2)}(x_1)$ and nonnegative constants $\epsilon_{(i,j)}, i = 1, \dots, n-1, j = 1, 2$, exist such that

$$
|\phi_{(i,1)}(x)| \leq \epsilon_{(i,1)} \hat{\phi}_{(1,2)}(x_1) \sqrt{|\phi_{(1,2)}(x)||\phi_{(2,3)}(x)|}, i = 2, \dots, n-1
$$

$$
|\phi_{(i,2)}(x)| \leq \epsilon_{(i,2)} \hat{\phi}_{(1,2)}(x_1) \sqrt{|\phi_{(1,2)}(x)||\phi_{(2,3)}(x)|}, i = 1, \dots, n-1. \quad (19)
$$

B. Controller Design

Define $\xi_2 = x_2 + \zeta(x_1)$; $\xi_i = x_i$, $i = 3, \dots, s$; $\xi = [\xi_2, \dots, \xi_s]^T$ where $\zeta(x_1) = \zeta_1(x_1)x_1$ is a design freedom to be chosen later. The control input u is designed to be

$$
u = \frac{1}{\mu_0(x)} \Big[-\phi_{(s,s+1)}(x) x_{s+1} - K(r,x)\xi \Big]. \quad (20)
$$

where $r \geq 1$ is a dynamic high gain scaling parameter whose dynamics will be designed later and $K(r,x)$ is an $(n-1) \times 1$ gain vector to be specified later. Under the action of the control law, (20), the dynamics of ξ are given by

$$
\dot{\xi} = [A_c + B_c K + H C_c]\xi + \Phi + \Xi \quad (21)
$$

$$
A_c = \begin{bmatrix} 0 & \phi_{(2,3)} & 0 & \dots & 0 \\ 0 & 0 & \phi_{(3,4)} & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & \phi_{(s-1,s)} \\ 0 & 0 & \dots & & 0 \end{bmatrix} \quad (22)
$$

$$
B_c = [0, \dots, 0, 1]^T \ , \ C_c = [1, 0, \dots, 0] \ , \ \Phi = [\phi_2, \dots, \phi_s]^T \quad (23)
$$

$$
H = [\{\zeta_1'(x_1)x_1 + \zeta_1(x_1)\}\phi_{(1,2)}(x), 0, \dots, 0]^T \quad (24)
$$

$$
\Xi = \Big[\{[\zeta_1'(x_1)x_1 + \zeta_1(x_1)][\phi_1 - \phi_{(1,2)}(x)\zeta_1(x_1)x_1]\}, 0, \dots, 0 \Big]^T. \quad (25)
$$

$\zeta_1'(x_1)$ denotes the partial derivative evaluated at x_1 of ζ_1. The dynamic high gain scaling parameter r is initialized greater than 1. The dynamics of r designed during the stability analysis will ensure that r is non-decreasing.

C. Stability Analysis

Define

$$
M_i = [\epsilon_{(i,1)} + \epsilon_{(i,2)}|\zeta_1(x_1)|]\hat{\phi}_{(1,2)}(x_1) \ ; \ M = [M_2, \dots, M_s] \quad (26)
$$

[3] For notational convenience, we drop the arguments of functions whenever no confusion will result.

[4] In the output-feedback case, the required assumption on the inverse dynamics is Assumption A2 with y_2 being the empty vector.

$$\overline{\Phi} = \begin{bmatrix} \phi_{(2,2)} & 0 & 0 & \dots & 0 \\ \phi_{(3,2)} & \phi_{(3,3)} & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ \phi_{(s-1,2)} & & \dots & \phi_{(s-1,s-1)} & 0 \\ \phi_{(s,2)} & & \dots & & \phi_{(s,s)} \end{bmatrix} \quad (27)$$

$$\overline{\Xi} = \Bigg[\frac{1}{\sqrt{|\phi_{(1,2)}(x)|}} \{ |\zeta_1'(x_1)x_1 + \zeta_1(x_1)| [\phi_{(1,1)}(x) + |\phi_{(1,2)}(x)\zeta_1(x_1)|] \}, 0, \dots, 0 \Bigg]^T. \quad (28)$$

It is easily seen that the matrix $A(\xi, x_1) = (A_c + \overline{\Phi} + HC_c)$ satisfies the assumptions of Theorem 4. Hence, if ρ is any given positive constant, nonnegative constants $q_1, \dots, q_{s-1}$, and a positive function $R_\zeta(T(r)\xi, x_1) \geq 1$ exist such that $T(r)A(\xi, x_1)T^{-1}(r)$ is dual w-CUDD(ρ) for all $r \geq R_\zeta(T(r)\xi, x_1)$ where $T(r) = \mathrm{diag}(\frac{1}{r^{q_1}}, \dots, \frac{1}{r^{q_{s-1}}})$. As noted in Theorem 4, q_1 can be picked arbitrarily. Hence, without loss of generality, we set $q_1 = 1$. Also, since the construction in Theorem 4 recursively assigns lower bounds on $q_2, \dots, q_{s-1}$, we set $q_2 \geq 4$. Note that R_ζ depends on the choice of the function ζ and this dependence is indicated in the subscript. Define $\eta = T(r)\xi$. The dynamics of η are

$$\dot{\eta} = T(r)[A_c + B_cK + HC_c]T^{-1}(r)\eta + T(r)\Phi + T(r)\Xi - \frac{\dot{r}}{r}D_c\eta \quad (29)$$

where $D_c = \mathrm{diag}(q_1, \dots, q_{s-1})$. Using Theorem 2, a $1\times(s-1)$ vector $\tilde{K}(r, x)$, a symmetric positive-definite matrix P_c, and positive constants ν_c, $\underline{\nu}_c$, and $\overline{\nu}_c$ exist such that for all $r \geq R_\zeta(T(r)\xi, x_1)$ and all $x \in \mathcal{R}^n$

$$P_c\{T(r)[A_c + Q_1\overline{\Phi}Q_2 + HC_c]T^{-1}(r) + B_c\tilde{K}\} + \{T(r)[A_c+Q_1\overline{\Phi}Q_2+HC_c]T^{-1}(r)+B_c\tilde{K}\}^T P_c \leq -\frac{\nu_c}{r^{q_1-q_2}}|\phi_{(2,3)}(x)|I$$
$$\underline{\nu}_c I \leq P_cD_c + D_cP_c \leq \overline{\nu}_c I \quad (30)$$

where Q_1 and Q_2 are arbitrary $(s-1)\times(s-1)$ diagonal matrices with each diagonal entry $+1$ or -1. Furthermore, by Theorem 2, the choice of $\tilde{K}$ does not need to depend on Q_1 and Q_2. $K(r, x)$ is defined as $K(r, x) = r^{q_{s-1}}\tilde{K}(r, x)T(r)$ so that $B_c\tilde{K} = T(r)B_cKT^{-1}(r)$. Closed-loop stability can be demonstrated using the Lyapunov function $V = \frac{1}{2}x_1^2+\eta^TP_c\eta$ whose derivative satisfies

$$\dot{V} = x_1[\phi_1+\phi_{(1,2)}x_2] - \frac{\dot{r}}{r}\eta^T(P_cD_c+D_cP_c)\eta + 2\eta^TP_c(T\Phi+T\Xi) + \eta^T\Big\{P_cT[A_c+B_cK+HC_c]T^{-1}+T^{-1}[A_c+B_cK+HC_c]^TTP_c\Big\}\eta. \quad (31)$$

The term $2\eta^TP_cT(r)\Phi$ can be upper bounded as[5]

$$|\Phi|_e \leq_e M\sqrt{|\phi_{(1,2)}(x)||\phi_{(2,3)}(x)|}|x_1| + \overline{\Phi}T^{-1}(r)|\eta|_e$$
$$\begin{aligned} 2\eta^TP_cT(r)\Phi &\leq |\phi_{(2,3)}(x)|[|\eta^TP_c|_eT(r)M]^2 + |\phi_{(1,2)}(x)|x_1^2 \\ &\quad +2\eta^TP_cQ_1T(r)\overline{\Phi}T^{-1}(r)Q_2\eta \\ &\leq \frac{1}{r^2}\lambda_{max}^2(P_c)|M|^2|\phi_{(2,3)}(x)||\eta|^2 + |\phi_{(1,2)}(x)|x_1^2 \\ &\quad +2\eta^TP_cT(r)Q_1\overline{\Phi}Q_2T^{-1}(r)\eta \end{aligned} \quad (32)$$

where Q_1 and Q_2 are $(s-1)\times(s-1)$ diagonal matrices with each diagonal entry $+1$ or -1 such that $|P_c\eta|_e = Q_1P_c\eta$ and $|\eta|_e = Q_2\eta$. The last inequality in (32) follows using $r \geq 1$, $T(r)$ is a diagonal matrix containing negative powers of r, and two diagonal matrices commute. Similarly,

$$|\Xi|_e \leq_e \overline{\Xi}\sqrt{|\phi_{(1,2)}(x)|}|x_1|$$
$$2\eta^TP_cT(r)\Xi \leq \frac{1}{r^2}\lambda_{max}^2(P_c)|\overline{\Xi}|^2|\eta|^2 + |\phi_{(1,2)}(x)|x_1^2. \quad (33)$$

Using (32) and (33), (31) reduces to

$$\begin{aligned} \dot{V} &\leq -x_1^2\phi_{(1,2)}(x)\zeta_1(x_1) + |\phi_{(1,2)}(x)|\gamma_{(1,1)}(x_1)|x_1|^{(2+\zeta_{(1,1,1)})} \\ &\quad -\frac{\dot{r}}{r}\eta^T(P_cD_c + D_cP_c)\eta + r^2\epsilon_{(1,2)}^2\hat{\phi}_{(1,2)}^2(x_1)|\phi_{(2,3)}(x)||\eta|^2 \\ &\quad +\eta^T\Big\{P_cT(r)[A_c + B_cK + HC_c + Q_1\overline{\Phi}Q_2]T^{-1}(r) \\ &\quad +T^{-1}(r)[A_c + B_cK + HC_c + Q_1\overline{\Phi}Q_2]^TT(r)P_c\Big\}\eta \\ &\quad +3|\phi_{(1,2)}(x)|x_1^2 + \frac{1}{r^2}\lambda_{max}^2(P_c)[|M|^2|\phi_{(2,3)}(x)| + |\overline{\Xi}|^2]|\eta|^2 \end{aligned} \quad (34)$$

Note that $|\overline{\Xi}| \leq \overline{\overline{\Xi}}\sqrt{|\phi_{(2,3)}(x)|}$ with

$$\overline{\overline{\Xi}} = |\zeta_1'(x_1)x_1+\zeta_1(x_1)|\epsilon_{(1,2)}\hat{\phi}_{(1,2)}(x_1)[\gamma_{(1,1)}|x_1|^{\zeta_{(1,1,1)}}+|\zeta_1(x_1)|] \quad (35)$$

Define

$$\overline{R}_\zeta(\eta, x_1) = \max\left\{R_\zeta, \frac{4\epsilon_{(1,2)}^2\hat{\phi}_{(1,2)}^2}{\nu_c}, \left(\frac{4\lambda_{max}^2(P_c)[|M|^2 + \overline{\overline{\Xi}}^2]}{\nu_c}\right)^{\frac{1}{5}}\right\}. \quad (36)$$

By the definition of R_ζ, the function $\overline{R}_\zeta$ is not smaller than unity for any argument. Recalling that $q_2-q_1 \geq 3$, it follows from (30) that if $r \geq \overline{R}_\zeta(\eta, x_1)$, then

$$\begin{aligned} &\eta^T\Big\{P_cT(r)[A_c + B_cK + HC_c + Q_1\overline{\Phi}Q_2]T^{-1}(r) \\ &+T^{-1}(r)[A_c + B_cK + HC_c + Q_1\overline{\Phi}Q_2]^TT(r)P_c\Big\}\eta \\ &+r^2\epsilon_{(1,2)}^2\hat{\phi}_{(1,2)}^2(x_1)|\phi_{(2,3)}(x)||\eta|^2 \\ &+\frac{1}{r^2}\lambda_{max}^2(P_c)[|M|^2|\phi_{(2,3)}(x)|+|\overline{\Xi}|^2]|\eta|^2 \leq -\frac{\nu_c}{2}r^3|\phi_{(2,3)}(x)||\eta|^2. \end{aligned} \quad (37)$$

The design freedom ζ_1 is picked to satisfy[6]

$$\zeta_1(x_1) = \frac{1}{\sigma}\mathrm{sign}(\phi_{(1,2)})[3\sigma+\sigma\gamma_{(1,1)}(x_1)|x_1|^{\zeta_{(1,1,1)}}+\zeta^*(x_1)] \quad (38)$$

where ζ^* is a function of x_1 with $\inf_{x_1\in\mathcal{R}}\zeta^*(x_1) > 0$. The dynamics of the high gain parameter r are designed as

$$\dot{r} = \begin{cases} \Delta(r, x) & \text{if } r < \overline{R}_\zeta(\eta, x_1) \\ 0 & \text{if } r \geq \overline{R}_\zeta(\eta, x_1) \end{cases} \quad (39)$$

$$\Delta(r, x) = \frac{r}{\underline{\nu}_c}\Bigg\{r^* + 2\lambda_{max}(P_c)\frac{r^{q_{s-1}}}{r^{q_1}}\Big[||A_c+B_cK+HC_c|| + ||\overline{\Phi}||\Big] + r^2\epsilon_{(1,2)}^2\hat{\phi}_{(1,2)}^2|\phi_{(2,3)}| + \frac{1}{r^2}\lambda_{max}^2(P_c)\Big[|M|^2+\overline{\overline{\Xi}}^2\Big]|\phi_{(2,3)}|\Bigg\} \quad (40)$$

with r^* being a positive constant. Using (34), (38), (39), and (40), if $r < \overline{R}_\zeta(\eta, x_1)$, the derivative of the Lyapunov function V satisfies $\dot{V} \leq -x_1^2\zeta^*(x_1)-r^*|\eta|^2$. If $r \geq \overline{R}_\zeta(\eta, x_1)$, we have $\dot{V} \leq -x_1^2\zeta^*(x_1) - \frac{\nu_c}{2}r^3|\phi_{(2,3)}(x)||\eta|^2$. Hence, V is a non-increasing function of time implying that x_1 and η are bounded. Thus, $\overline{R}_\zeta(\eta, x_1)$ is bounded. Hence, r remains bounded and since r is initialized greater than 1, $r(t) \geq 1$ for all time. The boundedness of ξ and hence, $x_2, \dots, x_s$ follows from boundedness of r. Invoking the BIBS Assumption A2, all closed-loop states remain bounded. Also, x_1 and η, and hence $x_1, \dots, x_s$ go to zero asymptotically. Strengthening Assumption A2 to include a minimum-phase assumption, all closed-loop states (except r) go to zero asymptotically.

Theorem 5: Under Assumptions A1-A3, the proposed dynamic state-feedback compensator achieves global stability of system (1) and $x_1, \dots, x_s$ go to zero asymptotically. Furthermore, under a minimum-phase assumption on the inverse dynamics, $x_1, \dots, x_n$, and u go to zero asymptotically.

Remark 2: To make the dynamics of r smooth upto any number of derivatives l, dynamics (39) can be replaced by $\dot{r} = q(\overline{R}_\zeta - r)\Delta(r, x)$ where q is a nonnegative l-times continuously differentiable function such that $q(b) = 1$ if $b > 0$ and $q(b) = 0$ for $b < -\epsilon_r$ with ϵ_r being any positive constant.

IV. Output-Feedback Control

In this section, we consider output-feedback control of system (1) with output $y = x_1$. The output-feedback design in this section is based on Assumptions A1, A2, and A3'.

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[5] $|\beta|_e$ denotes a matrix of the same dimension as β with each element replaced by its absolute value. $\leq_e$ denotes an element-wise inequality between two matrices of equal dimension. $\lambda_{max}(P)$ with P being a square symmetric matrix denotes the maximum eigenvalue of P.

[6] By assumption, $\phi_{(1,2)}(x)$ assumes the same sign for all $x \in \mathcal{R}^n$.

A. Assumptions

Assumption A3': Continuous functions $\hat{\phi}_i(t, x_1, \ldots, x_i)$ and nonnegative functions $\phi_{(i,j)}$ are known such that

$$|\phi_i(t, x_1, 0, \ldots, 0)| \le \phi_{(i,1)}(x_1)|x_1| \ , \ 1 \le i \le n$$

$$|\hat{\phi}_i(t, x_1, \hat{x}_2, \ldots, \hat{x}_i) - \phi_i(t, x_1, x_2, \ldots, x_i)| \le \phi_{(i,1)}(x_1)|x_1| + \sum_{j=2}^{i} \phi_{(i,j)}(x_1)|\hat{x}_j - x_j| \ , \ 2 \le i \le n. \quad (41)$$

Remark 3: As seen from Theorems 1 and 2, the cascading dominance of upper diagonal terms required in observer and controller contexts are dual. The observer design requires upper diagonal terms nearer to the output to be larger while the controller design requires upper diagonal terms closer to the input to be larger. Hence, it is not, in general, possible to design a high gain observer and high gain controller using the proposed scaling technique. The output-feedback design in this section uses the proposed scaling technique for the observer which is then coupled with a backstepping controller. This constrains the functions ϕ_i to be incrementally linear in unmeasured states and prevents them from having the more general bound in [15]:

$$|\hat{\phi}_i(t, x_1, \hat{x}_2, \ldots, \hat{x}_i) - \phi_i(t, x_1, x_2 \ldots, x_i)| \ \le \ \phi_{(i,1)}(x_1)|x_1| + \sum_{j=2}^{i} \phi_{(i,j)}(x_1)[|\hat{x}_j| + \sum_{j=2}^{i} |\hat{x}_j - x_j|] \ , \ i = 2, \ldots, n. \ (42)$$

The term $\sum_{j=2}^{i} |\hat{x}_j|$ which would need to be compensated for by the controller design cannot be handled in a backstepping-based design because, while backstepping can efficiently assign gains to the output, the gains to other states cannot be assigned arbitrarily since states farther from the output appear in increasingly complicated combinations in the recursive design procedure and generated Lyapunov function. In contrast, the high gain controller of [15] which is similar to the state-feedback controller in Section III utilizes a Lyapunov function quadratic in the state estimates and provides robustness both to uncertainties ϕ_i satisfying (42) and to appended ISS dynamics driven by all states.

B. Observer Design

A reduced-order observer for the system (1) is given by[7]

$$\begin{aligned} \dot{\hat{x}}_i \ = \ & \hat{\phi}_i(t, x_1, \hat{x}_2 + f_2(r, x_1), \ldots, \hat{x}_i + f_i(r, x_1)) \\ & + \phi_{(i,i+1)}(x_1)[\hat{x}_{i+1} + f_{i+1}(r, x_1)] + \mu_{i-s}(x_1)u \\ & - g_i(r, x_1)[\hat{x}_2 + f_2(r, x_1)] - \dot{r}h_i(r, x_1) \ , \ 2 \le i \le n \end{aligned} \quad (43)$$

where r is a dynamic high gain scaling parameter, $f_i(r, x_1)$ are design functions of x_1, and

$$g_i(r, x_1) = \phi_{(1,2)}(x_1)\frac{\partial f_i(r, x_1)}{\partial x_1} \ , \ h_i(r, x_1) = \frac{\partial f_i(r, x_1)}{\partial r}. \quad (44)$$

The dynamics of r are of the form $\dot{r} = w(r, x_1)$ with w being s-times continuously differentiable. Defining the observer errors

$$e_i \ = \ \hat{x}_i + f_i(r, x_1) - x_i \ , \ 2 \le i \le n, \quad (45)$$

the observer error dynamics are, $2 \le i \le n$,

$$\dot{e}_i \ = \ \tilde{\phi}_i - \phi_i + \phi_{(i,i+1)}(x_1)e_{i+1} + g_i(r, x_1)\frac{\phi_1}{\phi_{(1,2)}(x_1)} - g_i(r, x_1)e_2 \quad (46)$$

with $e_{n+1} = 0$ being a dummy variable where, for notational convenience, we have introduced

$$\tilde{\phi}_i = \hat{\phi}_i(t, x_1, \hat{x}_2 + f_2(r, x_1), \ldots, \hat{x}_i + f_i(r, x_1)) \ , \ i = 2, \ldots, n. \quad (47)$$

In matrix form, the dynamics of $e = [e_2, \ldots, e_n]^T$ are

$$\dot{e} \ = \ \tilde{\Phi} + [A_o + GC]e \quad (48)$$

$$\tilde{\Phi} \ = \ [\tilde{\Phi}_2, \ldots, \tilde{\Phi}_n]^T \ ; \ \tilde{\Phi}_i = \tilde{\phi}_i - \phi_i + g_i(r, x_1)\frac{\phi_1}{\phi_{(1,2)}} \quad (49)$$

$$A_o \ = \ \begin{bmatrix} 0 & \phi_{(2,3)} & 0 & \ldots & 0 \\ 0 & 0 & \phi_{(3,4)} & \ldots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & \phi_{(n-1,n)} \\ 0 & 0 & \ldots & & 0 \end{bmatrix} \quad (50)$$

$$G(r, x_1) \ = \ [g_2(r, x_1), \ldots, g_n(r, x_1)]^T \ , \ C = [1, 0, \ldots, 0]. \quad (51)$$

C. Controller Design

Define $\xi_i = \hat{x}_i + f_i(r, x_1)$, $i = 2, \ldots, s$. The controller is designed through backstepping[2] using the subsystem with states $(x_1, \xi_2, \ldots, \xi_s)$ whose dynamics are

$$\begin{aligned} \dot{x}_1 \ &= \ -\phi_{(1,2)}(x_1)e_2 + \phi_1(t, x_1) + \phi_{(1,2)}(x_1)\xi_2 \\ \dot{\xi}_i \ &= \ -g_i(r, x_1)e_2 + \hat{\phi}_i(t, x_1, \xi_2, \ldots, \xi_i) + g_i(r, x_1)\frac{\phi_1(t, x_1)}{\phi_{(1,2)}(x_1)} \\ & \quad + \phi_{(i,i+1)}(x_1)\xi_{i+1} \ , \ i = 2, \ldots, s-1 \\ \dot{\xi}_s \ &= \ -g_s(r, x_1)e_2 + \hat{\phi}_s(t, x_1, \xi_2, \ldots, \xi_s) + g_s(r, x_1)\frac{\phi_1(t, x_1)}{\phi_{(1,2)}(x_1)} \\ & \quad + \phi_{(s,s+1)}(x_1)[\hat{x}_{s+1} + f_{s+1}(r, x_1)] + \mu_0(x_1)u. \end{aligned} \quad (52)$$

Step 1: The backstepping is commenced using the Lyapunov function $V_1 = \frac{1}{2}\eta_1^2$ with $\eta_1 = x_1$ yielding

$$\begin{aligned} \dot{V}_1 \ &= \ -x_1\phi_{(1,2)}e_2 + x_1\phi_1 + x_1\phi_{(1,2)}\xi_2 \\ &\le \ -\alpha(r, x_1)x_1^2 - \zeta_1\eta_1^2 + \phi_{(1,2)}\eta_1\eta_2 + \frac{e_2^2}{4r^2} \end{aligned} \quad (53)$$

where $\zeta_1 > 0$ is a constant, α is a smooth positive function, $\eta_2 = \xi_2 - \xi_2^*(r, x_1)$, and

$$\xi_2^*(r, x_1) \ = \ -\frac{[r^2 x_1\phi_{(1,2)}^2(x_1) + \phi_{(1,1)}(x_1)x_1 + \zeta_1 x_1 + \alpha(r, x_1)x_1]}{\phi_{(1,2)}(x_1)}. \quad (54)$$

Step i ($2 \le i \le s-1$): Assume that at step $(i-1)$, a Lyapunov function V_{i-1} has been designed such that

$$\dot{V}_{i-1} \ \le \ -\alpha(r, x_1)x_1^2 - (1 - \frac{i-2}{2s})\zeta_1\eta_1^2 - \sum_{j=2}^{i-1}\zeta_j\eta_j^2 + \phi_{(i-1,i)}\eta_{i-1}\eta_i + \frac{(i-1)e_2^2}{4r^2} \quad (55)$$

where $\eta_j = \xi_j - \xi_j^*(r, x_1, \xi_2, \ldots, \xi_{j-1}), j = 2, \ldots, i$, with ξ_j^* being functions designed in the previous steps of backstepping. Defining $V_i = V_{i-1} + \frac{1}{2}\eta_i^2$,

$$\dot{V}_i \le -\alpha(r, x_1)x_1^2 - (1 - \frac{i-1}{2s})\zeta_1\eta_1^2 - \sum_{j=2}^{i}\zeta_j\eta_j^2 + \phi_{(i,i+1)}\eta_i\eta_{i+1} + \frac{ie_2^2}{4r^2}$$

where $\eta_{i+1} = \xi_{i+1} - \xi_{i+1}^*(r, x_1, \xi_2, \ldots, \xi_i)$ and

$$\begin{aligned} \xi_{i+1}^* = -\frac{1}{\phi_{(i,i+1)}(x_1)}\Bigg\{ & \zeta_i\eta_i + \phi_{(i-1,i)}\eta_{i-1} + \hat{\phi}_i(t, x_1, \xi_2, \ldots, \xi_i) \\ & - \frac{\partial \xi_i^*}{\partial r}w(r, x_1) - \frac{\partial \xi_i^*}{\partial x_1}\phi_{(1,2)}(x_1)\xi_2 - \sum_{j=2}^{i-1}\frac{\partial \xi_i^*}{\partial \xi_j}[\hat{\phi}_j(t, x_1, \xi_2, \ldots, \xi_j) \\ & + \phi_{(j,j+1)}(x_1)\xi_{j+1}] + r^2\eta_i\Big[-g_i(r, x_1) + \frac{\partial \xi_i^*}{\partial x_1}\phi_{(1,2)}(x_1) \\ & + \sum_{j=2}^{i-1}\frac{\partial \xi_i^*}{\partial \xi_j}g_j(r, x_1)\Big]^2 + \frac{s}{2\zeta_1}\eta_i\Big[\frac{g_i(r, x_1)}{\phi_{(1,2)}(x_1)} - \frac{\partial \xi_i^*}{\partial x_1} \\ & - \frac{1}{\phi_{(1,2)}(x_1)}\sum_{j=2}^{i-1}\frac{\partial \xi_i^*}{\partial \xi_j}g_j(r, x_1)\Big]^2\phi_{(1,1)}^2(x_1)\Bigg\}. \end{aligned} \quad (56)$$

Step s: At this step, the control input u is designed as

$$u \ = \ \frac{\phi_{(s,s+1)}(x_1)}{\mu_0(x_1)}\Big[\xi_{s+1}^*(r, x_1, \xi_2, \ldots, \xi_s) - \hat{x}_{s+1} - f_{s+1}(r, x_1)\Big] \quad (57)$$

where ξ_{s+1}^* is obtained from (56) with $i = s+1$. The Lyapunov function $V_s = \frac{1}{2}\sum_{j=1}^{s}\eta_i^2$ satisfies

$$\dot{V}_s \ \le \ -\alpha(r, x_1)x_1^2 - \frac{1}{2}\zeta_1\eta_1^2 - \sum_{j=2}^{s}\zeta_j\eta_j^2 + \frac{se_2^2}{4r^2}. \quad (58)$$

D. Stability Analysis

Define

$$M_i \ = \ \phi_{(i,1)}(x_1) + |g_i(r, x_1)|\frac{\phi_{(1,1)}(x_1)}{|\phi_{(1,2)}(x_1)|} \ ; \ M = [M_2, \ldots, M_n]^T$$

[7]For simplicity of notation, we introduce the dummy variables $\hat{x}_{n+1} = f_{n+1} = g_{n+1} = 0$ and $\mu_i \equiv 0$ for $i < 0$.

$$\overline{\Phi} = \begin{bmatrix} \phi_{(2,2)} & 0 & 0 & \dots & 0 \\ \phi_{(3,2)} & \phi_{(3,3)} & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ \phi_{(n-1,2)} & & \dots & \phi_{(n-1,n-1)} & 0 \\ \phi_{(n,2)} & & \dots & & \phi_{(n,n)} \end{bmatrix} \quad (59)$$

The matrix $A_o + \overline{\Phi}$ satisfies the assumptions of Theorem 3. Hence, given any positive constant ρ, nonnegative constants $q_1, \dots, q_{n-1}$, and a positive function $R(x_1) \geq 1$ exist such that $T(r)[A_o + \overline{\Phi}]T^{-1}(r)$ is w-CUDD(ρ) for all $r \geq R(x_1)$ where $T(r) = \mathrm{diag}(\frac{1}{r^{q_1}}, \dots, \frac{1}{r^{q_{n-1}}})$. From the construction in Theorem 3, q_1 can be taken to be 1 and $q_n - q_{n-1} = 1$. Using Theorem 1, a $(n-1) \times 1$ vector $\tilde{G}(r, x_1)$, a symmetric positive-definite matrix P_o, and positive constants $\nu_o, \underline{\nu}_o$, and $\overline{\nu}_o$ exist such that for all $r \geq R(x_1)$ and all $x_1 \in \mathcal{R}$

$$P_o\{T(r)[A_o + Q_1\overline{\Phi}Q_2]T^{-1}(r) + \tilde{G}C\}$$
$$+\{T(r)[A_o+Q_1\overline{\Phi}Q_2]T^{-1}(r)+\tilde{G}C\}^T P_o \leq -\frac{\nu_o}{r^{q_{n-1}-q_n}}|\phi_{(n-1,n)}|I$$
$$\underline{\nu}_o I \leq P_o D_o + D_o P_o \leq \overline{\nu}_o I \quad (60)$$

where $D_o = \mathrm{diag}(q_1, \dots, q_{n-1})$ and Q_1 and Q_2 are arbitrary diagonal $(n-1) \times (n-1)$ matrices with each diagonal entry $+1$ or -1. By Theorem 1, the choice of $\tilde{G}$ does not need to depend on Q_1 and Q_2. $G(r, x_1) = [g_2(r, x_1), \dots, g_n(r, x_1)]^T$ is defined as $G(r, x_1) = r^{-q_1}T^{-1}(r)\tilde{G}(r, x_1)$ so that $\tilde{G}C = T(r)GCT^{-1}(r)$. $f_i, i = 2, \dots, n$, are obtained as

$$f_i(r, x_1) = \int_0^{x_1} \frac{g_i(r, \pi)}{\phi_{(1,2)}(\pi)} d\pi. \quad (61)$$

The dynamics of $\epsilon \overset{\triangle}{=} T(r)e$ are

$$\dot{\epsilon} = T(r)\tilde{\Phi} + T(r)[A_o + GC]T^{-1}(r)\epsilon - \frac{\dot{r}}{r}D_o\epsilon. \quad (62)$$

The derivative of the Lyapunov function $V_o = \epsilon^T P_o \epsilon$ satisfies

$$\dot{V}_o = 2\epsilon^T P_o T(r)\tilde{\Phi} + \epsilon^T\{P_o T(r)[A_o + GC]T^{-1}(r)$$
$$+T^{-1}(r)[A_o + GC]^T T(r)P_o\}\epsilon - \frac{\dot{r}}{r}\epsilon^T[P_o D_o + D_o P_o]\epsilon. \quad (63)$$

The term $2\epsilon^T P_o T(r)\tilde{\Phi}$ can be upper bounded as

$$|\tilde{\Phi}_i| \leq \sum_{j=2}^{i} \phi_{(i,j)}|e_j| + [\phi_{(i,1)} + |g_i|\frac{\phi_{(1,1)}}{|\phi_{(1,2)}|}]|x_1|$$
$$|\tilde{\Phi}|_e \leq_e \overline{\Phi}|e|_e + M|x_1|$$
$$2\epsilon^T P_o T(r)\tilde{\Phi} \leq 2|\epsilon^T P_o|_e T(r)[\overline{\Phi}T^{-1}(r)|\epsilon|_e + M|x_1|]$$
$$\leq 2\epsilon^T P_o T(r)Q_1\overline{\Phi}Q_2T^{-1}(r)\epsilon + \frac{\nu_o\sigma}{2}|\epsilon|^2$$
$$+\frac{2}{\nu_o\sigma}\lambda^2_{max}(P_o)|M|^2x_1^2 \quad (64)$$

where Q_1 and Q_2 are diagonal matrices with each diagonal entry $+1$ or -1 such that $|P_o\epsilon|_e = Q_1P_o\epsilon$ and $|\epsilon|_e = Q_2\epsilon$.

The dynamics of the high gain parameter are designed as

$$\dot{r} = q(R - r)\Delta(r, x_1) \text{ with initial value } r(0) \geq 1 \quad (65)$$

$$\Delta(r, x_1) = \frac{r}{\underline{\nu}_o}\left\{r^* + 2\lambda_{max}(P_o)\frac{r^{q_{n-1}}}{r^{q_1}}\left[||A_o + GC|| + ||\overline{\Phi}||\right] + \frac{\nu_o\sigma}{2}\right\} \quad (66)$$

with r^* being any positive constant. q is a nonnegative s-times continuously differentiable function such that $q(b) = 1$ if $b > 0$ and $q(b) = 0$ if $b < -\epsilon_r$ with ϵ_r being a positive constant. Using (60), (64), (65), and (66), (63) reduces to

$$\dot{V}_o \leq -\min(\frac{\nu_o\sigma}{2}, r^*)|\epsilon|^2 + \frac{2}{\nu_o\sigma}\lambda^2_{max}(P_o)|M|^2x_1^2. \quad (67)$$

The controller design freedom $\alpha(r, x_1)$ is picked to satisfy

$$\alpha(r, x_1) \geq \frac{2s^*}{\nu_o\sigma}\lambda^2_{max}(P_o)|M|^2$$
$$= \frac{2s^*}{\nu_o\sigma}\lambda^2_{max}(P_o)\sum_{i=2}^{n}\left[\phi_{(i,1)}(x_1) + |g_i(r, x_1)|\frac{\phi_{(1,1)}(x_1)}{|\phi_{(1,2)}(x_1)|}\right]^2 \quad (68)$$

where

$$s^* = \frac{\frac{s}{4} + 1}{\min(\frac{\nu_o\sigma}{2}, r^*)} \quad (69)$$

Defining the overall Lyapunov function $V = V_s + s^*V_o$, differentiating, and noting that $e_2 = r^{q_1}\epsilon_2 = r\epsilon_2$,

$$\dot{V} \leq -|\epsilon|^2 - \frac{1}{2}\zeta_1\eta_1^2 - \sum_{j=2}^{s}\zeta_j\eta_j^2. \quad (70)$$

Using (70), ϵ, x_1, $\eta_2, \dots, \eta_s$ are bounded. Boundedness of x_1 implies boundedness of $R(x_1)$, and hence of r and $e = T^{-1}(r)\epsilon$. Boundedness of $\hat{x}_i, i = 2, \dots, s$, follows from boundedness of $\eta_2, \dots, \eta_s, x_1$, and r. Invoking the BIBS Assumption A2, $x_{s+1}, \dots, x_n$ are bounded. Hence, all closed-loop states are bounded. Moreover, from (70), noting that $\hat{\phi}_i(t, 0, \dots, 0) = 0$, it follows that $e_2, \dots, e_n, x_1, x_2, \dots, x_s$ go to zero asymptotically. If Assumption A2 is strengthened to include a minimum-phase assumption, it is seen that all closed-loop states (except r) go to zero asymptotically.

Theorem 6: Under Assumptions A1, A2, and A3', the proposed output-feedback compensator globally stabilizes system (1) and makes observer errors and states $x_1, \dots, x_s$ go to zero asymptotically. Moreover, under a minimum-phase condition on inverse dynamics, the observer errors, states $x_1, \dots, x_n$, and control input u go to zero asymptotically.

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