Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Nonlinear Model Predictive Control based on Predicted State Error Convergence

John T. Wen
Electrical, Computer, & Systems Eng.
Rensselaer Polytechnic Institute
Troy, NY 12180
wen@cat.rpi.edu

Sooyong Jung
School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, GA
sjung@sununo.me.gatech.edu

***Abstract*— Systems subject to nonholonomic constraints or nonintegrable conservation laws may be modeled as driftless nonlinear control systems. Such systems are not stabilizable using continuous time-invariant state feedback. Time varying or piecewise continuous feedback are the only stabilization strategies currently available. This paper presents a new feedback stabilization algorithm for this class of systems by choosing the control action based on a gradient matrix to reduce the predicted state error. This approach is similar to the model predictive control but does not consider optimality nor constraints explicitly. As a result, the on-line computation load is greatly reduced. Existing stability analysis techniques for model predictive control are not directly applicable due to the lack of local stabilizability. We show that under the full rank condition of a gradient matrix, the closed loop system is globally asymptotically stable. Examples of several nonlinear control systemsare included to illustrate the proposed method.**

I. Introduction

Control systems subject to nonholonomic constraints, such as the wheel no-slip constraint in vehicles, or non-integrable conservation laws, such as the conservation of angular momentum in multi-body systems in space or free fall, may be modeled as a continuous time driftless nonlinear control system [1]:

$$\dot{x} = g(x)u, \quad x \in \mathcal{R}^n, u \in \mathcal{R}^m. \tag{1}$$

For $n > m$, this system linearized about any state is not controllable, which means that it cannot be stabilized by a linear controller. Furthermore, there does not even exist a continuous time-invariant stabilizing feedback control law [2]. For these reasons, this class of nonlinear systems is considered to be challenging in terms of control design.

In this paper, we consider the finite difference approximation of (1):

$$x_{k+1} = x_k + t_s g(x_k)u_k, \tag{2}$$

where t_s is the sampling period. We present a new feedback stabilization scheme that chooses the control action to reduce the predicted state error. This approach is similar to nonlinear model predictive control (NMPC) but does not consider optimization nor constraints explicitly. As a result, the on-line computation requirement is much reduced. The motivation of this approach is based on path planning for nonholonomic systems using a gradient-type of iteration [3]–[5]. Though the linearized system about an equilibrium is not controllable, the system linearized about a finite trajectory (resulting in a linear time varying system) is almost always controllable. This scheme was extended to a model predictive implementation in [6], in which the look-ahead control vector is updated by a Newton descent step based on the predicted error. Using linearized time varying system in a Newton iteration is also employed in an NMPC scheme introduced in [7]. However, here we only require improvement of the predicted state error instead of convergence of this error to zero.

Existing stability analysis techniques in NMPC are not directly applicable to our gradient based control scheme since the optimization index is not positive definite (it is zero over the entire prediction horizon) and the system is not locally stabilizable. The goal of this paper is to show globally asymptotically stability of the desired state and to demonstrate the effectiveness of this controller through a number of nonlinear examples. The only assumptions that are required are the uniform full rank condition of a gradient matrix (equivalent to the controllability of the system linearized about the predicted state trajectory) and the boundedness of the evolution of the predicted control sequence by the predicted state error.

To illustrate the proposed nonlinear control scheme, we have included several nonlinear control examples: kinematic control of a unicycle (used in [8]), a so-called double-chain system with 5 states and 3 inputs, an under-actuated satellite orientation control using two angular velocities as the control inputs, and a nonlinear affine system used in [9] to demonstrate the lack of robustness in NMPC with the terminal constraint and a short horizon. Due to the space limitation, all proofs are omitted. The complete version of the paper including all proofs may be found in [10].

0-7803-8335-4/04/$17.00 ©2004 AACC

Notation: We use $\|x\|$ to denote the Euclidean norm. All matrix norms are induced 2-norms (maximum singular value). The notation of I_m is used for the $m \times m$ identity matrix and 0_m denotes the $m \times m$ zero matrix. Given $f : \mathcal{R}^n \to \mathcal{R}^m$, $\frac{\partial f(x)}{\partial x}$ denotes the $m \times n$ gradient matrix, and $\frac{\partial f(z)}{\partial x}$ denotes $\left.\frac{\partial f(x)}{\partial x}\right|_{x=z}$.

II. Main Result

A. Overview of Algorithm

We will state our algorithm for a general discrete time-invariant nonlinear control system:

$$x_{k+1} = f(x_k, u_k) \tag{3}$$

where $x_k \in \mathcal{R}^n$, $u_k \in \mathcal{R}^m$, and f is smooth in both variables. Let $x_d \in \mathcal{R}^n$ be the desired closed loop equilibrium state that satisfies

$$x_d = f(x_d, u_d) \tag{4}$$

for some input vector $u_d \in \mathcal{R}^m$. Note that for discretized driftless nonlinear systems (2), $u_d = 0$.

We shall address the following full state feedback stabilization problem:

> For each $k \geq 0$, find $u_k \in \mathcal{R}^m$ in terms of current and past state vectors, $\{x_i : i \leq k\}$, such that x_d is an asymptotically stable equilibrium.

At time k, given the state x_k and an M-step-ahead control sequence, $\underline{u}_{k,M} \in \underbrace{\mathcal{R}^m \times \ldots \times \mathcal{R}^m}_{M \text{ times}}$,

$$\underline{u}_{k,M} = \begin{bmatrix} u_1^{(k)T} & \ldots & u_M^{(k)T} \end{bmatrix}^T, \tag{5}$$

the M-step-ahead error is defined as

$$e_{k,M} = \phi_M(x_k, \underline{u}_{k,M}) - x_d \tag{6}$$

where ϕ_M denotes the state at the end of M steps with the initial state x_k and input sequence $\underline{u}_{k,M}$.

The proposed algorithm is simple to describe: at the kth time step, the M-step-ahead control sequence, $\underline{u}_{k,M}$, is refined to $\underline{v}_{k,M}$ based on the M-step-ahead prediction error, $e_{k,M}$ (e.g., by using Newton descent, steepest descent, Levinson-Marquardt, etc.); the first m elements of $\underline{v}_{k,M}$ are used as the actual control, u_k; then $\underline{v}_{k,M}$ is shifted forwarded with the last m elements replaced by an appropriately chosen vector, $\tilde{u}_k$. A more detailed description is now given below.

Algorithm 1: Choose an initial M-step-ahead control sequence, $\underline{u}_{0,M}$.

For $k = 0, 1, \ldots$,

1) *Refine the M-step-ahead control sequence $\underline{u}_{k,M}$ to $\underline{v}_{k,M} = \begin{bmatrix} v_1^{(k)T} & \ldots & v_M^{(k)T} \end{bmatrix}^T$ based on the M-step-ahead predicted state error, $e_{k,M}$:*

$$\underline{v}_{k,M} = \underline{u}_{k,M} + \Delta \underline{u}_{k,M} \tag{7}$$

where $\Delta \underline{u}_{k,M}$ is an (mM)-vector to be chosen.

2) *Generate the control action u_k from $\underline{v}_{k,M}$:*

$$u_k = v_1^{(k)} = \Lambda \underline{v}_{k,M} \tag{8}$$

where $\Lambda := \begin{bmatrix} I_m & 0_m & \ldots & 0_m \end{bmatrix}$.

3) *Update of M-step-ahead sequence vector:*

$$\underline{u}_{k+1,M} = \begin{bmatrix} u_1^{(k+1)} \\ \vdots \\ u_{M-1}^{(k+1)} \\ u_M^{(k+1)} \end{bmatrix} = \begin{bmatrix} v_2^{(k)} \\ \vdots \\ v_M^{(k)} \\ \tilde{u}_k \end{bmatrix} = \Gamma \underline{v}_{k,M} + \Phi \tilde{u}_k \tag{9}$$

where $\tilde{u}_k$ is an m-vector to be chosen, and $\Gamma : \mathcal{R}^{m \cdot M} \to \mathcal{R}^{m \cdot M}$ and $\Phi : \mathcal{R}^m \to \mathcal{R}^{m \cdot M}$ are defined as

$$\Gamma := \begin{bmatrix} 0_m & I_m & 0_m & \ldots & 0_m \\ & 0_m & I_m & \ddots & 0_m \\ \vdots & & \ddots & \ddots & \vdots \\ & & & 0_m & I_m \\ 0_m & & \ldots & & 0_m \end{bmatrix}, \quad \Phi := \begin{bmatrix} 0_m \\ \vdots \\ 0_m \\ I_m \end{bmatrix}.$$

If the Newton algorithm is used in Step 1, then the control sequence update law is given by

$$\Delta \underline{u}_{k,M} = -\eta_k D_k^{\dagger} e_{k,M}, \tag{10}$$

where $D_k := \frac{\partial \phi_M(x_k, \underline{u}_{k,M})}{\partial \underline{u}_{k,M}} \in \mathcal{R}^{n \times (mM)}$ is the gradient of the predicted state with respect to the M-step-ahead control sequence, $D_k^{\dagger}$ is the Moore-Penrose pseudo-inverse of D_k, and η_k will be determined to satisfy Assumption 1 in the next section. If (3) is a linear time invariant (LTI) system, then Algorithm 1 with (10) is an (mM)th order linear compensator which degenerates to a static full state feedback for $\eta_k = 1$. If $M = n$, it becomes a dead beat controller with gain given by the Ackerman formula placing all closed loop poles at the origin.

If D_k is of full row rank (i.e., rank n), then $D_k^{\dagger}$ is just the right inverse of D_k:

$$D_k^{\dagger} = D_k^T (D_k D_k^T)^{-1}. \tag{11}$$

In terms of implementation, the update law (10) can be performed more efficiently by solving a matrix equation instead of computing the pseudo-inverse explicitly [11]:

$$D_k \Delta \underline{u}_{k,M} = -\eta_k e_{k,M}. \tag{12}$$

If D_k is very large (e.g., due to a long prediction horizon or large input and state dimensions), other efficient numerical methods such as the matrix-free method used in finite element analysis [12] may be employed to avoid explicit calculation and storage of D_k.

The gradient matrix D_k is related to the system linearized about the predicted state trajectory (which is a linear time varying system):

$$\begin{aligned} \delta x_{k+i+1} &= A_i \delta x_{k+i} + B_i \delta u_{i+1}^{(k)}, \ \delta x_k = 0, \quad (13) \\ A_i &:= \frac{\partial f(x_{k+i}, u_{i+1}^{(k)})}{\partial x}, \ B_i := \frac{\partial f(x_{k+i}, u_{i+1}^{(k)})}{\partial u} \\ D_k &= [A_{M-1} \dots A_1 B_0 | \dots | B_{M-1}]. \quad (14) \end{aligned}$$

The full row rank condition on D_k is equivalent to the controllability of the above linear time varying system. In fact, $D_k D_k^T$ is precisely the controllability gramian of this system. Given x_k, the control sequences at which D_k loses rank are exactly the singular controls of the optimal control problem for (3) with only the terminal state cost. These singularities have been shown to be non-generic [5] for continuous time driftless systems in the sense that the non-singular controls form a dense set in C^∞. In [13], a sufficient condition, called the strong bracket generating condition, is derived for a driftless system to have only singular controls that are identically zero. This condition is satisfied for the so-called unicycle model (see Sec. III-A), but is too stringent in general. Singular controls for certain classes of driftless and affine nonlinear systems (including multi-chain systems) are completely characterized in [14]. Singular controls have also been considered in the context of mobile robots in [15].

With Algorithm 1, the closed loop system may be represented as an interconnected system shown in Figure 1:

$$\begin{aligned} P: \quad & x_{k+1} = f(x_k, u_k), \quad u_k = \Lambda \underline{v}_{k,M} \quad (15) \\ K: \quad & \hat{x}_{j+1}^{(k)} = f(\hat{x}_j^{(k)}, \Lambda \underline{\hat{u}}_j^{(k)}), \quad \hat{x}_0^{(k)} = x_k \quad (16) \\ & \underline{\hat{u}}_{j+1}^{(k)} = \Gamma \underline{\hat{u}}_j^{(k)}, \quad \underline{\hat{u}}_0^{(k)} = \underline{u}_{k,M} \quad (17) \\ & \underline{v}_{k,M} = \underline{u}_{k,M} + \Delta \underline{u}_{k,M}(e_{k,M}), \quad (18) \\ & e_{k,M} = \hat{x}_M^{(k)} - x_d \quad (19) \\ Q: \quad & \underline{u}_{k+1,M} = \Gamma \underline{v}_{k,M} + \Phi \tilde{u}_k(e_{k,M}). \quad (20) \end{aligned}$$

We will use the following steps to show the closed loop asymptotic stability:

1) We first show that under Assumption 1, which is essentially a full rank condition on D_k, $\Delta u_{k,M}$ and $\tilde{u}_k$ may be chosen so that $e_{k,M}$ converges asymptotically and monotonically.
2) We next show that under Assumption 2, which imposes bounds on $\Delta \underline{u}_{k,M}$ and $\tilde{u}_k$ as a function of $e_{k,M}$, u_k converges to u_d, and $\underline{u}_{k,M}$ and $\underline{v}_{k,M}$ converge to $\underline{u}_d := \begin{bmatrix} u_d^T & \dots & u_d^T \end{bmatrix}^T$.
3) From step 2, u_k and $\Lambda \underline{\hat{u}}_j^{(k)}$ will be arbitrarily close for k sufficiently large. By the continuity of f, it follows from (15)–(16) that x_{k+M} converges to $\hat{x}_M^{(k)}$ which in turn converges to x_d from step 1.
4) Finally, with $\underline{u}_{0,M}$ chosen to be continuous with respect to x_0, x_d is stable in the sense of Lyapunov. Combining with the global asymptotic convergence in step 3, x_d is a global asymptotically stable equilibrium.

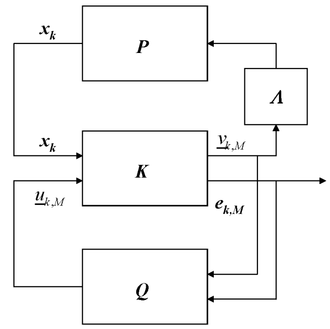


Fig. 1. Closed Loop System under Algorithm 1

B. Asymptotic Convergence of the Predicted State Error

The first step of the stability proof is to show that the M-step-ahead prediction error, $e_{k,M}$, converges to zero under the following assumption, which states that the combination of M-step-ahead control refinement and an appropriate choice of $\tilde{u}_k$ will result in a decrease of the M-step-ahead prediction error.

Assumption 1: For all $k \geq 0$, the following conditions hold:
1. The M-step-ahead error under the refined M-step-ahead sequence, $\underline{v}_{k,M}$, satisfies the bound

$$\left\| \phi_M(x_k, \underline{v}_{k,M}) - x_d \right\| \leq \lambda_k \left\| \phi_M(x_k, \underline{u}_{k,M}) - x_d \right\| \quad (21)$$

where λ_k is a positive constant, $P > 0$.
2. A control vector $\tilde{u}_k$ (in (9)) can be found to satisfy the following bound:

$$\left\| \phi_{M+1}(x_k, (\underline{v}_{k,M}, \tilde{u}_k)) - x_d \right\| \leq \alpha_k \left\| \phi_M(x_k, \underline{v}_{k,M}) - x_d \right\|, \quad (22)$$

where α_k is a positive constant.
2. The constants λ_k and α_k from (21)–(22) satisfy

$$\begin{aligned} \lim_{k \to \infty} \prod_{j=0}^{k} \lambda_j \alpha_j &= 0 \quad (23) \\ \alpha_k \lambda_k &\leq 1, \quad \textit{for all } k \geq 0. \quad (24) \end{aligned}$$

For discretized driftless systems (2), $\alpha_k \leq 1$. In fact, we could also just choose $\tilde{u}_k = u_d = 0$, and $\alpha_k = 1$. In this case, Assumption 1 reduces to the uniform full row rank condition on D_k (i.e., the minimum singular value of D_k is uniformly positive). If D_k is of uniform full row rank, then η_k may be updated by using a line search to ensure $\lambda_k < 1$ uniformly. This implies that Assumption 1.3 is satisfied with $\alpha_k \lambda_k < 1$ uniformly.

It follows from Assumption 1 that the M-step-ahead error is non-increasing and converges to zero asymptotically.

Lemma 1: Given the initial M-step-ahead control sequence $\underline{u}_{0,M}$ and the control law in Algorithm 1 that satisfies Assumptions 1. Then $e_{k,M}$ converges to 0 as $k \to \infty$ and satisfies the bound, for all $k, j \geq 0$:

$$\|e_{k+j,M}\| \leq \left(\prod_{i=0}^{j-1} \alpha_{k+i} \lambda_{k+i} \right) \|e_{k,M}\| . \quad (25)$$

As discussed earlier, for discretized driftless nonlinear systems, if D_k is of uniform full row rank, then $e_{k,M}$ converges to 0 exponentially.

Instead of the two-step approach in Assumption 1, i.e., choosing $\Delta \underline{u}_{k,M}$ based on (21) and $\tilde{u}_k$ based on (22), we can choose $(\underline{u}_{k,M}, \tilde{u}_k)$ together to satisfy the monotonically decreasing requirement on $e_{k,M}$.

C. Asymptotic Convergence of the Control Signal

We next show that the convergence of the predicted error implies the convergence of $\underline{u}_{k,M}$, $\underline{v}_{k,M}$, and u_k. To this end, we assume that $\Delta \underline{u}_{k,M}$ and $\tilde{u}_k$ in (19)–(20) are bounded by the prediction error, $e_{k,M}$.

Assumption 2: There exist positive constants β and ρ such that for all $k \geq 0$

$$\begin{aligned} \left\| \Delta \underline{u}_{k,M} \right\| &= \left\| \underline{v}_{k,M} - \underline{u}_{k,M} \right\| \leq \beta \|e_{k,M}\| \quad (26) \\ \|\tilde{u}_k - u_d\| &\leq \rho \|e_{k,M}\| . \quad (27) \end{aligned}$$

For the Newton update (10), the boundedness assumption on $\Delta \underline{u}_{k,M}$, (26), is equivalent to the uniform full row rank condition on D_k (which is also needed for Assumption 1) since $\left\| D_k^\dagger \right\| = 1/\sigma_{\min}(D_k)$, where $\sigma_{\min}(D_k)$ is the minimum singular value of D_k. Under Assumptions 1–2, we can show that u_k converges to u_d, and $\underline{u}_{k,M}$ and $\underline{v}_{k,M}$ both converge to $\underline{u}_d$, as $k \to \infty$.

D. Asymptotic Convergence of the State

Given that $\underline{u}_{k,M}$ converges to $\underline{u}_d$ and u_k converges to u_d, we can show that x_{k+M} converges to $\hat{x}_M^{(k)}$, by using the continuity of f. This in turn shows that x_k converges to x_d since $e_{k,M} \to 0$ from Lemma 1.

Lemma 2: Given the control law in Algorithm 1. If Assumptions 1–2 hold, then for all $x_0 \in \mathcal{R}^n$ and $\underline{u}_{0,M} \in \mathcal{R}^{mM}$, $x_k \to x_d$ as $k \to \infty$.

E. Closed Loop Asymptotic Stability

The asymptotic stability of the desired state x_d requires stability in the sense of Lyapunov in addition to the asymptotic convergence. This can be shown if the initial M-step-ahead control sequence, $\underline{u}_{0,M}$, is appropriately chosen.

Theorem 1: Consider the control law given in Algorithm 1 with $\underline{u}_{0,M}$ continuous in x_0, and $\underline{u}_{0,M} = \underline{u}_d$ if $x_0 = x_d$. Suppose that Assumptions 1-2 hold. Then x_d is a globally asymptotically stable equilibrium.

If the constant control sequence u_d is not a singular control, we can just choose $u_j^{(0)} = u_d$ for all x_0. Otherwise, we may use $u_j^{(0)} = u_d + a n_j$ where n_j is a random vector bounded by $\|n_j\| \leq 1$ and a is proportional to $\|x_0 - x_d\|$.

III. Simulation Results

We will present the results of applying Algorithm 1 with the Newton update (10) to several nonlinear examples, including the kinematic control of a unicycle, a double-chain nonholonomic system, an under-actuated satellite orientation control, and a discrete time system used in [9] to show the lack of robustness of conventional NMPC schemes.

In all the nonholonomic examples, we will consider the unconstrained case as well as imposing an input saturation constraint. Suppose a hard constraint $|u| \leq u_{\max}$ is required for a given input channel, we will use the following transformation

$$u = s(v) = \frac{\pi}{2} u_{\max} \tan^{-1} \left(\frac{2v}{\pi u_{\max}} \right), \quad (28)$$

and apply our algorithm with v as the input. Note that with this transformation, an LTI system would become nonlinear.

The step size η_k in (10) is chosen with the Amijo's rule [16]. We start with an initial step size h_0. The step size is repeatedly halved until either $\left\| \phi_M(x_k, \underline{u}_{k,M} - h D_k^+ e_{k,M}) \right\| < \left\| \phi_M(x_k, \underline{u}_{k,M}) \right\|$ or if $h < h_{\min}$ in which case h is set to zero. When $h = 0$, the system essentially switches to the open loop control using the control vectors in $\underline{u}_{k,M}$.

A. Kinematic Model of Unicycle

The kinematic model of a planar unicycle is commonly used to represent mobile robots. Under the assumption that the wheel does not slip, the kinematic model is given by

$$\dot{x} = \cos\theta \; u_1, \quad \dot{y} = \sin\theta \; u_1, \quad \dot{\theta} = u_2 \quad (29)$$

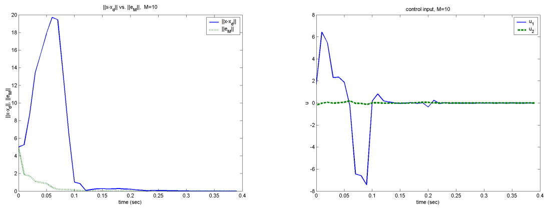
where (x, y, θ) represent the position and orientation of the unicycle and (u_1, u_2) the driving and steering velocities.

We consider the finite difference discretization of the equation of motion with $t_s = 0.01$sec. By calculating the gradient matrix D_k in (14), it is straightforward to show that the singularity corresponds to u_1 and u_2 being identically zero for the entire prediction horizon. Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$ are uniformly bounded in x, our control scheme is globally asymptotically stable if the initial control sequence is not identically zero. We therefore choose $\underline{u}_{0,M}$ as a random vector with a small amplitude.

To illustrate the parallel parking maneuver, the initial state is chosen to be $x_0 = (0, 5, 0)$.

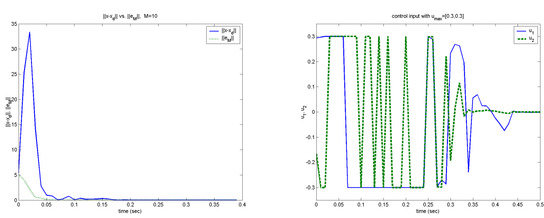
First we consider the unconstrained problem with the look-ahead horizons chosen to be $M = 10$. As shown in Figure 2, the predicted state error converges monotonically. The actual state error increases initially but eventually converges. The corresponding input trajectory is also shown. In general, when a smaller horizon is used, the convergence is faster but with a larger control effort.

Without the input constraint, the input can be very large and renders the finite difference discretization invalid. We therefore apply the transformation (28) to impose the input saturation constraint. With this constraint, the control signal becomes oscillatory, but the state still converges to zero, though over a longer period (see Figure 3).



state input

Fig. 2. Unicycle Example: Convergence of predicted State vs. actual state and input trajectory, $M = 10$



state input

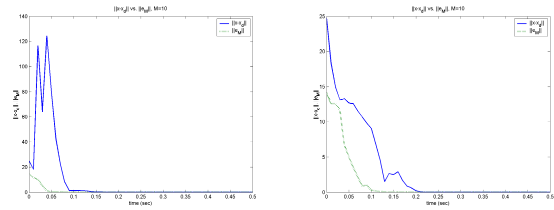
Fig. 3. Unicycle Example: Input trajectories for $u_{\max} = [0.3, 0.3]$, $M = 10$

B. Double-Chain System

Certain nonholonomic systems occurring in practice can be transformed to the chain form (e.g., a multi-trailer system). We consider the following double-chain system where the depth of each chain is 2:

$$\dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = x_2 u_1, \dot{x}_4 = u_3, \dot{x}_5 = x_4 u_1. \tag{30}$$

The singularity corresponds to when u_1 is identically zero for the entire prediction horizon. We again discretize the equation using finite difference with $t_s = 0.01$sec. The initial state is arbitrarily chosen to be $x_0 = (-5, 5, 10, -20, 8)$. The initial control vector is set to zero. Though this is a singular control, the initial predicted error, $e_{0,M}$, is not in the null space of $D_0^\dagger$, therefore, the algorithm is able to proceed. We consider both the unconstrained case and the input constrained case with the saturation level at $[1, 1, 1]$. The predicted state error vs. the actual state error plots for the two cases are shown in Figure 4. In both cases, the predicted error converges monotonically, and the actual error converges asymptotically. The convergence in the constrained input case takes about twice as long but the maximum state error is much smaller.



unconstrained input-constrained

Fig. 4. Double Chain Example: Convergence of predicted State vs. actual state, $M = 10$, no input constraint vs. $u_{\max} = [1, 1, 1]$

C. Under-actuated Satellite System

Consider the orientation control of satellite using only two of the angular velocities. Using the vector quaternion representation, the continuous time equation of motion is given by

$$\dot{x} = \frac{1}{2}(-x^\times + \sqrt{1 - \|x\|^2} I_3) \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \tag{31}$$

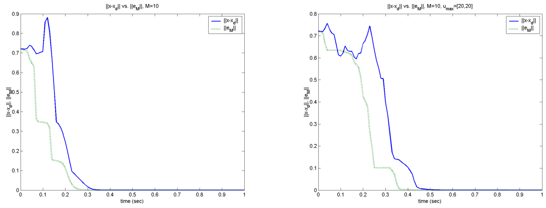
where $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ is the vector quaternion, $x^\times$ is the 3×3 matrix representation of the vector cross product, and (u_1, u_2) are the two angular velocities considered as the control variables. The equation of motion ensures that $\|x(t)\| \leq 1$ if $\|x(0)\| \leq 1$. We discretize the equation using finite difference with $t_s = 0.01$sec. The initial state is chosen to be $x_0 = (0.1, 0.1, 0.707)$. The initial control vector, $\underline{u}_{0,M}$, is chosen to be all zeros. The look ahead horizon is $M = 10$.

The predicted state error vs. the actual state error plots for the unconstrained input and input constrained at $[20, 20]$ are shown in Figure 5. In both cases, the predicted error converges monotonically, and the actual error converges asymptotically. The convergence in the constrained input case takes longer due to the input constraint.

D. A Nonlinear Example from [9]

In [9], the following single input discrete time example is used to demonstrate the lack of robustness of conventional NMPC schemes when a terminal constraint (the origin) is used together with a short horizon ($M = 2$):

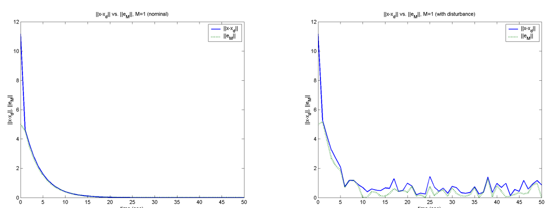
$$x_{1_{k+1}} = x_{1_k}(1 - u_k), \quad x_{2_{k+1}} = \sqrt{x_{1_k}^2 + x_{2_k}^2}\, u_k. \tag{32}$$



unconstrained input-constrained

Fig. 5. Satellite Example: Convergence of predicted State vs. actual state, $M = 10$, no input constraint vs. $u_{\max} = [20, 20]$

Note that in contrast to the earlier nonlinear examples, this system does not correspond to the finite difference discretization of a continuous time driftless system. The initial condition is arbitrarily chosen to be $x_0 = (5, 10)$. Since this example was used to show the lack of robustness when the prediction horizon is small, we set $M = 1$; other prediction horizons produced similar results. In addition to the nominal case, we also consider the case when a random state noise is added (which would cause any NMPC with terminal constraint and $M = 2$ to be unstable). The predicted state vs. actual state plots for the nominal and perturbed cases are shown in Figure 6. Though we have not proven robustness of our method, it can be seen at least for this example that the system remains stable in the perturbed case.



nominal disturbed

Fig. 6. Example from [9]: Convergence of predicted State vs. actual state, nominal vs. disturbed, $M = 1$

IV. Conclusions

This paper presents a gradient-based nonlinear feedback stabilization strategy for discrete time nonlinear systems. The approach is similar to NMPC but the control vector is updated to reduce predicted state error rather than minimizing an optimization index. We show that under the full rank assumption of a gradient matrix and the boundedness of the predicted control sequence refinement by the predicted state error, the closed loop system is globally asymptotically stable. These assumptions are reasonable for the finite difference approximation of driftless nonlinear systems motivated by systems subject to nonholonomic constraints or non-integrable conservation laws. The efficacy of the algorithm is demonstrated by several nonlinear examples.

Acknowledgment

The authors gratefully acknowledge the careful and helpful comments from the anonymous reviewers, and helpful discussion with Professors Fernando Lizarralde, Liu Hsu, Murat Arcak, Andy Teel, and Dr. Dan Popa. This material is based upon work supported in part by the National Science Foundation under Grants CMS-9813099 and CMS-0301827. This work is supported in part by the Center for Automation Technologies under a block grant from the New York State Office of Science, Technology, and Academic Research.

References

[1] J. Wen, "Control of nonholonomic systems," in *Control Handbook*, W. Levine, Ed. CRC Press, 1995, pp. 1359–1368.

[2] R. Brockett, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*, edited by R.W. Brockett, R.S. Millman and J.J. Sussmann. Birkhauser, 1983, vol. 27, pp. 181–208.

[3] H. Sussmann, "A continuation method for nonholonomic path-finding problem," in *Proc. 32nd IEEE Conference on Decision and Control*, San Antonio, TX, Dec. 1993, pp. 2718–2723.

[4] A. Divelbiss and J. Wen, "Trajectory tracking control of a car–trailer system," *IEEE Transaction on Control System Technology*, vol. 5, no. 3, pp. 269–278, May 1997.

[5] E. D. Sontag, "Control of systems without drift via generic loops," *IEEE Trans. on Automatic Control*, vol. 40, no. 7, pp. 1210–1219, July 1995.

[6] F. Lizarralde, J. T. Wen, and L. Hsu, "A new model predictive control strategy for affine nonlinear control systems," in *Proc. 1999 American Control Conference*, San Diego, CA, June 1999, pp. 4263–4267.

[7] B. Kouvaritakis, M. Cannon, and J. Rossiter, "nonlinear model based predictive control," *International Journal of Control*, vol. 72, no. 10, pp. 919–928, 1999.

[8] S. L. Oliveira and M. Morari, "Contractive model predictive control for constrained nonlinear systems," *IEEE Trans. on Automatic Control*, vol. 45, no. 6, pp. 1053–1071, June 2000.

[9] G. Grimm, M. Messina, A. Teel, and S. Tuna, "Examples of zero robustness in constrained model predictive control," in *Proc. of 42nd Conference on Decision and Control*, Maui, HI, Dec. 2003.

[10] J. Wen and S. Jung, "Gradient-based model predictive control design with application to driftless nonlinear systems," Center for Automation Technologies, Rensselaer Polytechnic Institute, Troy, NY (also available at http://www.cat.rpi.edu/˜wen/papers/nmpc.pdf), CAT Report, June 2003.

[11] G. Golub and C. V. Loan, *Matrix Computations*, 3rd ed. Johns Hopkins University Press, 1996.

[12] R. Choquet and J. Erhel, "Newton-GMRES algorithm applied to compressible flow," *International Journal on Numerical Methods for Fluids*, pp. 177–190, 1996.

[13] H. Sussmann and Y. Chitour, "A continuation method for non-holonomic path finding problem," IMA Workshop on Robotics, Jan. 1993.

[14] D. Popa and J. Wen, "Singularity computation for iterative control of nonlinear affine systems," *Asian Journal of Control*, vol. 2, no. 2, pp. 57–75, June 2000.

[15] K. Tchon, "On kinematic singularities of nonholonomic robotic systems," in *13th CISM-IFToMM*, Zakopane, Poland, July 2000.

[16] E. Polak, *Optimization: Algorithms and Consistent Approximations*. New York: Springer-Verlag, 1997.