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Certainty Equivalence in Constrained Linear Systems subject to Stochastic Disturbances

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***Abstract*— A sufficient condition is provided under which the optimal controller of a constrained optimization problem can be synthesized by combining an optimal state estimator with an optimal static state feedback. An application of a model predictive controller is considered that involves both input and state constraints in a system that is subject to stochastic disturbances.**

I. INTRODUCTION

In this paper we consider a linear dynamical system subject to input constraints, incomplete state observations and stochastic disturbances. The aim is to find a controller that minimizes the expected value of a cost criterion that is defined over a finite horizon.

If imperfect observations of the state are available, then a rather common approach is to substitute an estimate of the state in an optimal state feedback law. The *certainty equivalence principle* is a rigorous justification for such a substitution. If this principle applies then the design of a dynamic controller is separated in two tasks. One of *state estimation* and one of *state feedback*. Because of its recursive implementation and the minimum error variance properties, the *Kalman filter* [10] is often implemented for producing an optimal estimate of the state, which is subsequently fed into an optimal state feedback that is designed as if the full state of the system could be measured. This so called *separation principle* gives an optimal controller in some well defined problems such as LQG, LEQG ('E' stands for 'exponential' [13]). For nonlinear systems a similar separation applies under the assumption that errors introduced by the observer or the state feedback are sufficiently small.

Because of the constraints these results do not apply in general and hence it can not be taken for granted that a certainty equivalence is justified in constrained optimal control problems. We refer to [2], [12], [9], [7] for approaches of observer design in the context of control of nonlinear systems. However, these papers deal mainly with the nominal, non-stochastic case.

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In this paper we derive conditions under which separation of static state feedback design and optimal state estimation will be optimal for the problem of constrained optimizations. We formulate three constrained optimization problems in Section II and provide the main result and its proof in Section III. A discussion on implications and generalizations of the main results to include state constraints is presented in Section IV. Applications to model predictive control are discussed in Section V. A numerical example is presented in Section VI. Finally, conclusions are collected in Section VII.

II. PROBLEM FORMULATIONS

Consider the system described by the discrete time state space equations

$$\begin{cases} x(t+1) & = Ax(t) + Bu(t) + Ew(t) \\ y(t) & = C_y x(t) + \eta(t) \\ z(t) & = C_z x(t) + D_z u(t) \end{cases} \tag{1}$$

As usual, u is the control input, x is the state, y is the measured output (or observed variable), z is the to-be-controlled output (or controlled variable) and w and η are state and measurement noise, respectively. We assume a finite dimensional setting in that $u(t)$, $x(t)$, $y(t)$ and $z(t)$ are real vector valued signals of dimension m, n, d and p, respectively. The disturbance w and the measurement noise η are two mutually independent stochastic processes with $w(t) \in \mathcal{N}(0, Q_w)$ and $\eta(t) \in \mathcal{N}(0, Q_\eta)$, where $\mathcal{N}(\mu, Q)$ denotes the family of normally distributed random variables with mean μ and covariance $Q = Q^\top \geq 0$. The initial state $x(0) = x_0$ in (1) is supposed to belong to $\mathcal{N}(\bar{x}, Q_x)$. Moreover, for $t' \neq t''$, the random vectors $w(t'), w(t''), \eta(t'), \eta(t'')$ and x_0 are assumed to be independent. Hence, the state x, the measurement y and the controlled output z are *stochastic processes*.

Let $\mathbf{U} \subseteq \mathbb{R}^m$ be a closed, not necessarily bounded, convex set which contains an open neighborhood of the origin. Assume that the time evolution of the input u in (1) is constrained in the sense that for all $t \geq 0$,

$$u(t) \in \mathbf{U}. \tag{2}$$

The constrained system (1)-(2) is said to be *globally asymptotically stabilizable* if there exists a controller such that in the absence of the external input w, the equilibrium point $x = 0$ of the controlled system is globally asymptotically stable. It is well known that if $\mathbf{U}$ is bounded and if $y = x$, then global asymptotic stability can be achieved if and only

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if the poles of the open loop system lie inside or on the unit circle of the complex plane. The following assumption will be made to allow the stabilization of system (1) by means of a measurement feedback controller.

Assumption 1 All eigenvalues λ of A satisfy $|\lambda| \leq 1$ and the matrix pair $\left(C_y, A\right)$ is detectable.

This assumption is rather natural in that no measurement feedback controller will exist that stabilizes (1)-(2) if Assumption 1 is not true [3].

Let $T > 0$ denote the length of the control horizon, let $\mathbb{E}$ denote the expectation operator and let $j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ be a strictly convex function with $j(0,0) = 0$. Define the *cost criterion*

$$J(x,u) := \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{t=0}^{T} j(x(t),u(t)) \tag{3}$$

where $u(t)$ and $x(t)$ satisfy (1) with initial condition $x(0) = x$.

It is assumed that at time t the measurements $y(\tau)$, $0 \leq \tau < t$ are available for feedback and that a *controller* for (1) is a system σ that allows a state representation of the form

$$\sigma : \begin{cases} r(t+1) &= f_{\text{con}}(r(t), y(t)) \\ u(t) &= g_{\text{con}}(r(t)) \end{cases} \tag{4}$$

with $r(0) = 0$, $f_{\text{con}} : \mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R}^p$ and $g_{\text{con}} : \mathbb{R}^p \to \mathbb{R}^m$ continuous functions with $f_{\text{con}}(0,0) = 0$ and $g_{\text{con}}(0) = 0$ and where $p = \dim(r)$ is the (undecided) state dimension of the controller. Let Σ_{con} denote the set of all feedback controllers of the form (4).

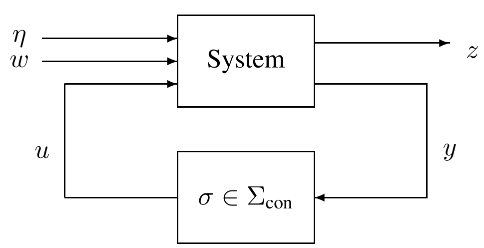

Fig. 1. Plant-controller configuration

Equations (1) and (4) define a *controlled system* with inputs w and η and output z as depicted in Figure 1. The cost invoked by connecting a controller $\sigma \in \Sigma_{\text{con}}$ to the system will, with some abuse of notation, be denoted by $J(x,\sigma)$. Due to the stochastic nature of the noise and the initial condition $x(0) = x$, also the criterion $J(x,\sigma)$ will be *stochastic* for any $\sigma \in \Sigma_{\text{con}}$. It is for this reason that we consider the *conditional expectation* with respect to the initial condition x of the plant $\mathbb{E}_x J(x,\sigma)$, to assess performance of a specific controller. This yields the following problem formulation.

Problem II.1 Consider the system (1) with initial condition $x_0 \in \mathcal{N}(\bar{x}, Q_x)$. Find an optimal controller $\tilde{\sigma} \in \Sigma_{\text{con}}$ such that (2) holds for all $t \geq 0$ and

$$\mathbb{E}_{x_0} J(x_0, \tilde{\sigma}) \leq \mathbb{E}_{x_0} J(x_0, \sigma) \tag{5}$$

for all $\sigma \in \Sigma_{\text{con}}$.

Finding an optimal, dynamic feedback controller $\tilde{\sigma} \in \Sigma_{\text{con}}$ that solves Problem II.1 is a difficult task. As mentioned in the introduction, a rather common approach is to separate the design of a controller in the two separate tasks of *state estimation* and *state feedback*. Here, we will implement the *Kalman filter* for producing an optimal (i.e., minimum error variance) estimate x^* of the state x, and subsequently feed x^* into an optimal static state feedback. We therefore define a second optimization problem as follows. Let

$$x^*(t+1) = Ax^*(t) + Bu(t) + G(t)\left(y(t) - C_y x^*(t)\right) \tag{6}$$

with $x^*(0) = \bar{x}$ denote the *Kalman filter* associated with (1), where $G(t)$, the *Kalman gain*, is given by

$$G(t) = AQ(t)C_y^\top \left(Q_\eta + C_y Q(t) C_y^\top\right)^{-1}$$

and where $Q(t)$ is the *estimation error covariance* which is a solution of the following recursive relationship

$$\begin{aligned} Q(t+1) &= AQ(t)A^\top + EQ_w E^\top \\ &\quad - AQ(t)C_y^\top \left(Q_\eta + C_y Q(t) C_y^\top\right)^{-1} C_y Q(t) A^\top \end{aligned}$$

with $Q(0) = Q_x$ and $t \geq 0$. The plant data, together with the noise assumptions allow to compute $Q(t)$ and $G(t)$ for all $t \geq 0$. Note that the Kalman filter produces an estimate x^* (optimal in a well defined sense) of the state x as function of the measurement y and the input u.

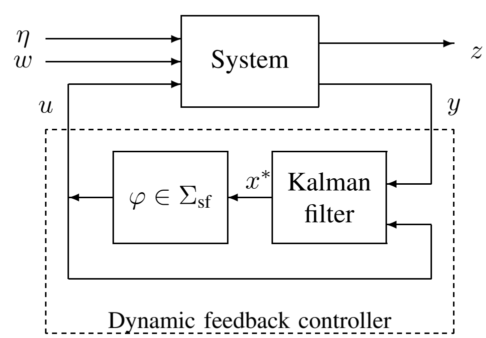

Fig. 2. Kalman filter and static state feedback

A *static state feedback* is a continuous function $\varphi : \mathbb{R}^n \to \mathbf{U}$ with the property that $\varphi(0) = 0$. Let Σ_{sf} denote the class of all such feedbacks. With $\varphi \in \Sigma_{\text{sf}}$, the control input u is now computed according to

$$u(t) = \varphi\left(x^*(t)\right) \tag{7}$$

where x^* is the state of the Kalman filter (6). With this structure we actually assemble a strictly causal dynamic controller of the form (4) as depicted in Figure 2. Let $\Sigma_{\text{kf+sf}}$ denote the set of all controllers obtained by combining the Kalman filter (6) with (7) where $\varphi \in \Sigma_{\text{sf}}$. Obviously, $\Sigma_{\text{kf+sf}} \subseteq \Sigma_{\text{con}}$ and this controller structure defines the following optimization problem.

Problem II.2 Consider the system (1) with an initial condition $x_0 \in \mathcal{N}(\bar{x}, Q_x)$. Find an optimal controller $\tilde{\sigma} \in \Sigma_{\text{kf+sf}}$ such that (2) holds for all $t \geq 0$ and

$$\mathbb{E}_{x_0} J(x_0, \tilde{\sigma}) \leq \mathbb{E}_{x_0} J(x_0, \sigma) \tag{8}$$

for all $\sigma \in \Sigma_{\text{kf+sf}}$.

We emphasize that the filter (6) in the controller structure of Problem II.2 depends only on the plant data and can be assumed fixed in the optimization over the state feedback laws. Hence, the optimization in Problem II.2 basically runs over Σ_{sf}.

To investigate the relation between Problem II.1 and Problem II.2 we define a third problem as follows. Consider the *auxiliary system*

$$x(t+1) = Ax(t) + Bu(t) + G(t)\zeta(t) \tag{9}$$

where ζ is a white noise stochastic process with zero mean and time dependent covariance matrix $Q(t)$. That is $\zeta(t) \in \mathcal{N}(0, Q(t))$ for all $t \geq 0$ and $\zeta(t')$ and $\zeta(t'')$ are independent for all $t' \neq t''$. Note that, if the initial condition $x(0)$ of (9) coincides with the initial condition $x^*(0)$ of the Kalman filter (6), then the stochastic properties (mean and variance) of the state $x(t)$ of (9) and $x^*(t)$ of (6) coincide for all time $t \geq 0$. Let (3) be the cost function for the auxiliary system (9) with $x(0) = \bar{x}$ and consider the following static state feedback problem for system (9).

Problem II.3 Consider the auxiliary system (9) with initial condition $x(0) := \bar{x}$. Find an optimal static state feedback controller $\varphi^* \in \Sigma_{\text{sf}}$ such that

$$J(\bar{x}, \varphi^*) \leq J(\bar{x}, \varphi) \tag{10}$$

for all $\varphi \in \Sigma_{\text{sf}}$.

This control structure is shown in Figure 3.

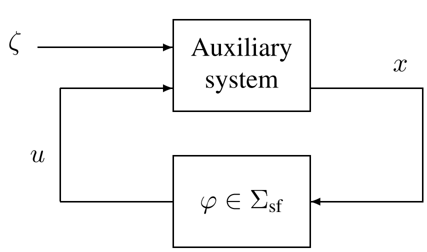

Fig. 3. The auxiliary system (9) with state feedback

III. MAIN RESULTS

The main result of this section shows that, under certain conditions, the state feedback φ^* that is optimal for the auxiliary Problem II.3 is, in combination with the Kalman filter (6), an optimal solution for Problem II.2 and Problem II.1. That is, the configuration of Figure 2 for Problem II.2 will be optimal for Problem II.1 by solving Problem II.3. The result is as follows.

Theorem III.1 *Suppose that the function j is a non-negative quadratic form in its arguments. Let $\varphi^* \in \Sigma_{sf}$ be an optimal controller that solves Problem II.3. Then the controller given by* (6) *and*

$$u(t) = \varphi^* \left(x^*(t) \right) \tag{11}$$

will be an optimal solution to both Problem II.2 and Problem II.1.

Proof: A non-negative quadratic function j allows a representation of the form $j(x, u) = \|C_z x + D_z u\|^2$ for suitable matrices C_z and D_z. To distinguish between the states of (1) and (9), let x^a denote the state variable of the auxiliary system (9). Similarly, let $J^a(x, u)$ denote the cost (3) associated with the auxiliary system subject to input u and initial condition $x^a(0) = \bar{x}$. Then

$$\begin{aligned} \mathbb{E}\big\{ j(x(t), u(t)) \big| y(0), \cdots, y(t-1), u(0), \cdots, u(t-1) \big\} \\ = \mathbb{E} j(x^a(t), u(t)) \\ + \operatorname{Trace} \mathbb{E}\big\{ C_z (x(t) - x^a(t))(x(t) - x^a(t))^\top C_z^\top \big\}. \end{aligned}$$

Note that

$$\mathbb{E}\big\{ C_z (x(t) - x^a(t))(x(t) - x^a(t))^\top C_z^\top \big\} = C_z Q(t) C_z^\top$$

where $Q(t)$ is the estimation error covariance matrix.

Next, we can rewrite the cost criterion (3) in terms of the state $x^a(t)$ instead of $x(t)$ as

$$\begin{aligned} J(x, u) = \lim_{T \to \infty} \mathbb{E} \frac{1}{T} \sum_{t=0}^{T} j(x^a(t), u(t)) \\ + \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{\top} \operatorname{Trace} C_z Q(t) C_z^\top \end{aligned} \tag{12}$$

where $x^a(0) = \bar{x}$. In turn, (12) can be rewritten as

$$J(x, u) = J^a(x, u) + c \tag{13}$$

where c is a constant given by

$$c = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \operatorname{Trace} C_z Q(t) C_z^\top$$

Since c does not depend on u, (13) implies that the input u that minimizes J^a also minimizes J and vice versa. This gives the result. ∎

IV. GENERALIZATION TO INCLUDE STATE CONSTRAINTS

The separation result of Theorem III.1 is well known for unconstrained optimization problems (LQG control), but is new for constrained optimizations as formulated in Problem II.1. The result facilitates the synthesis of controllers that solve Problem II.1 in that it amounts to solving two separate optimization problems: the optimal state estimator defined by the Kalman filter (6) and the optimal static state feedback defined in Problem II.3. The latter problem (i.e., Problem II.3) has been solved in [3] where explicit algorithms were given to compute optimal state feedbacks $\varphi^* \in \Sigma_{\text{sf}}$.

We will next discuss to what extent Theorem III.1 can be generalized to problems that include both input and state constraints. For this, let $\mathbf{X} \subseteq \mathbb{R}^n$ be a closed, not necessarily bounded, convex set which contains an open neighborhood of the origin and suppose that time evolutions of (1) are constrained in the sense that for all $t \geq 0$, $u(t) \in \mathbf{U}$ and $x(t) \in \mathbf{X}$. The approach in [6] amounts to incorporating the state constraint into the cost criterion by a soft *exponential* weighting on constraint violations. That is, the function j in cost (3) is decomposed according to

$$j\big(x, u\big) := g\big(x, u\big) + h(x), \qquad x \in \mathbb{R}^n \quad u \in \mathbf{U} \tag{14}$$

where $h : \mathbb{R}^n \to \mathbb{R}_+$ is a convex function that vanishes on $\mathbf{X}$ and where g and h are chosen so that j is in the class of functions that have "polynomial-exponential growth". This choice implies that j is not quadratic, but grows exponentially with $\|x\|$ i.e.,

$$j\big(x, u\big) \sim e^{\|x\|_R^2} \qquad \text{as} \quad \|x\| \to \infty.$$

Here, $\|x\|_R^2 = \langle x, Rx\rangle$ where $R > 0$ denotes the exponential growth of j. Hence, Theorem III.1 does not apply in this case, but with x the state of (1) and x^a the state of (9), the expectation of the exponential cost function satisfies

$$\begin{aligned}\mathbb{E}\, e^{\|x\|_R^2} &= \mathbb{E}\, e^{\|(x-x^a)+x^a\|_R^2} \\ &= \mathbb{E}\, e^{(x-x^a)^\top R(x-x^a)} e^{\|x^a\|_R^2} e^{2(x-x^a)^\top Rx^a}.\end{aligned}$$

Since the probability distribution of $x - x^a$ is known, this shows that $\mathbb{E}\, e^{\|x\|_R^2}$ is basically a function of x^a only. This suggests that also in this case some kind of separation will be possible. However, details for a generalization of Theorem III.1 in this direction are still under investigation.

V. MODEL PREDICTIVE CONTROL BY MEASUREMENT FEEDBACK FOR STOCHASTIC SYSTEMS

Within the model predictive framework, a receding horizon controller is obtained by solving an optimization problem at each time instant t. The usual formulation of this problem amounts to finding a control input u that minimizes a cost criterion defined on an interval $I_t := \{t + k | k \in T\}$ where T denotes the control horizon $T := \{0, \cdots, N\}$ and $N > 0$. Only the control at time t, $u(t)$, is fed into the system, after which the optimization is repeated at the next time instant. Because (1) is time-invariant, all variables involved in the optimization can be viewed, without loss of generality, as functions of $k \in T$, rather than the current time t. To cope with the noise acting on the system, we will not optimize over time trajectories $u(\cdot)$, but over time-varying feedback laws of the form

$$u(k) = \pi_k(\hat{x}_N(k))$$

where $k \in T$ and $\hat{x}_N : T \to \mathbb{R}^n$ is a *state sequence* of length N and π_k is a continuous feedback mapping. Formally, let Π denote the set of feedback laws where $\pi \in \Pi$ is a vector $(\pi_k)_{k=0}^N$ of continuous mappings $\pi_k : \mathbb{R}^n \to \mathbf{U}$. With $\pi \in \Pi$, the dynamics of the controlled plant on the control interval is described by

$$\begin{aligned} x_N(k+1) &= Ax_N(k) + B\,\pi_k(\hat{x}_N(k)) + Ew(k) \\ y_N(k) &= C_y x_N(k) + \eta(k) \end{aligned} \tag{15}$$

where $k \in T$ and the disturbances w and η are mutually independent Gaussian white noise processes with $w(k) \in \mathcal{N}(0, Q_w)$ and $\eta(k) \in \mathcal{N}(0, Q_\eta)$. The initial condition for the recursion (15) is given by $x_N(0) \in \mathcal{N}(x_t^*, Q_t)$ where x_t^* and Q_t are the state estimate and the error covariance matrix obtained by the Kalman filter (6) at time t i.e., $x_t^* = x^*(t)$ and $Q_t = Q(t)$. The *state sequence* $(\hat{x}_N(k))_{k=0}^{N+1}$ is generated as follows

$$\begin{aligned}\hat{x}_N(k+1) &= A\hat{x}_N(k) + B\pi_k\big(\hat{x}_N(k)\big) + \\ &\quad + G(k)\big(y_N(k) - C_y\hat{x}_N(k)\big), \quad k = 0, 1, \cdots, N \end{aligned} \tag{16}$$

where $\hat{x}_N(0) = x_t^*$ and

$$G(k) := AP(k)C_y^\top \left(Q_\eta + C_y P(k) C_y^\top\right)^{-1}$$

The matrix $P(k)$ is the covariance matrix of the estimation error and a solution of the Riccati equation

$$\begin{aligned} P(k+1) &= AP(k)A^\top - AP(k)C_y^\top \\ &\times \big(Q_\eta + C_y P(k) C_y^\top\big)^{-1} C_y P(k) A^\top + EQ_w E^\top \end{aligned} \tag{17}$$

with $P(0) = Q_t$. Note that the difference between the Kalman filter (6) and the *sequence* (16) lies in the fact that the innovation in the Kalman filter depends on the measurement y which is not available in (16). Define

$$\omega(k) := y_N(k) - C_y\hat{x}_N(k), \tag{18}$$

which is a normally distributed random vector with zero mean and covariance given by

$$\mathbb{E}\left(\omega(k)\omega(k)^\top\right) = C_y P(k) C_y^\top + Q_\eta \tag{19}$$

while $\omega(t')$ and $\omega(t'')$ are independent for all $t' \neq t''$. With the decomposition (14), the cost criterion is given by

$$\begin{aligned} J(x, \pi) := \mathbb{E}\Big\{ \sum_{k\in T} &\big\{g\big(C_z\hat{x}_N(k) + D_z\pi_k(\hat{x}_N(k))\big) \\ &+ h(\hat{x}_N(k))\big\} + \|\hat{x}_N(N+1)\|_Q^2 \Big\} \end{aligned} \tag{20}$$

where $\hat{x}_N(0) = x$ and $\pi \in \Pi$ is the control. Here, $\|x\|_Q^2 := \langle x, Qx \rangle$ with $Q > 0$ is an end point penalty. The optimization problem to be solved is stated next.

Problem V.1 Find, for given initial condition $x \in \mathbb{R}^n$, a vector of optimal feedback mappings $\pi^* \in \Pi$ such that $J(x, \pi^*) \leq J(x, \pi)$ for all $\pi \in \Pi$ and determine the optimal cost $V(x) := \inf_{\pi \in \Pi} J(x, \pi)$.

An analytical computation of the 'cost-to-go' associated with the criterion function (20) is generally a difficult task. We therefore replace (20) by its *empirical mean*

$$\hat{J}(x,\pi) := \hat{\mathbb{E}}\Big\{ \sum_{k \in T} \big\{ g\big(C_z \hat{x}_N(k) + D_z \pi_k(\hat{x}_N(k))\big) + h(\hat{x}_N(k)) \big\} + \|\hat{x}_N(N+1)\|_Q^2 \Big\} \quad (21)$$

where $\hat{\mathbb{E}} f := \frac{1}{m} \sum_{j=1}^m f(\theta_j)$ and $\theta_1, \ldots, \theta_m \in \Theta$ are m independent identically distributed samples drawn according to the probability measure on Θ.

Problem V.1 is replaced by the optimal control problem in which the empirical cost (21) is minimized instead of (20). In a receding horizon implementation of an optimal feedback $\pi^* \in \Pi$, only the first element of π^* is significant. The model predictive controller in our setting is thus given by $u(t) = \pi_0^*(x^*(t)) \qquad t \in \mathbb{Z}_+$. The algorithm to compute such a feedback is based on the following result from [4].

Theorem V.2 *Consider Problem V.1 in which the empirical cost* (21) *is minimized instead of* (20). *Suppose Assumption 1 holds. The empirical optimal cost* $\hat{V}(x) := \inf_\pi \hat{J}(x, \pi)$ *is given by* $\hat{V}(x) = \hat{V}_0(x)$ *where* $\hat{V}_s(x)$ *and the vector of feedback mappings* π *can be obtained recursively from*

$$\hat{V}_s(x) := \inf_{u \in \mathbf{U}} \big\{ g(C_z x + D_z u) + h(x) + \hat{\mathbb{E}}_\nu \hat{V}_{s+1}\left(Ax + Bu + K(s)\omega(s)\right) \big\} \quad (22)$$

with $\hat{V}_{N+1}(x) := \|x\|_Q^2$, *and* s *ranging from* N *to* 0.

To compute the empirical mean (22), a number of realizations of the innovation process (18) is needed. The samples are chosen randomly, according to the distribution of the innovation process ω. For the details of this algorithm, we refer to [4] and [6].

VI. NUMERICAL EXAMPLES

In this section we present an example in which we consider a "double integrator" of the form:

$$\begin{cases} x(t+1) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w(t) \\ y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t) + \eta(t) \\ z(t) &= \begin{pmatrix} 0 & 0 \\ 0.7 & 0 \\ 0 & 0.7 \end{pmatrix} x(t) + \begin{pmatrix} 0.33 \\ 0 \\ 0 \end{pmatrix} u(t) \end{cases} \quad (23)$$

Physically, this can be a system that describes the 1-dimensional motion of a unit mass under influence of a force. The force is the input to the system and the position of the mass is measured.

Here, $w(k) \in \mathcal{N}(0, 0.4)$ and $\eta(k) \in \mathcal{N}(0, 0.2)$ are mutually independent. The input and state are constrained as:

$$\mathbf{U} = [-0.5, 0.5], \qquad \mathbf{X} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_2 \geq 0 \right\}$$

and it is assumed that $x(0) = (0, \ \ 10)^\top$ is the initial state. The task is to steer the state from the initial state to the origin while respecting the state constraints.

The design of the Kalman filter for this system is straightforward. The remaining task is to approximate the static state feedback (7). For this, we design the model predictive controller based on Theorem V.2. Specifically, we use (16) with asymptotic gain

$$K := \lim_{k \to \infty} G(k) = \begin{pmatrix} 0.5857 \\ 1.4142 \end{pmatrix}.$$

The innovation process ω defined in (18) then satisfies $\omega(k) \in \mathcal{N}(0, 1.1657)$ which is sampled to determine the empirical cost (21) with $g(z) = \|z\|^2$ and constraint violation cost

$$h(x) = \begin{cases} 0 & \text{if } x_2 \geq 0 \\ e^{4.5 x_2^2} - 1 & \text{if } x_2 < 0 \end{cases}$$

With these specifications, the controller minimizes the expectation of a quadratic cost when the state is away from the constraint $x_2 > 0$. When the state is near or on the boundary of the constraint the exponential constraint violation cost h dominates and the main objective of the controller is to avoid a constraint violation. Conditions derived in [6] prove that this optimization problem is, in fact, solvable.

Let $N = 10$ and let ω be sampled according to its distribution. We take 10 samples at the first time instant and 5 samples at the second time instant in the control horizon. In this way, we obtain 50 samples of the innovation process over the control horizon. To assess the performance of the stochastic predictive controller we perform 100 simulations. Each of them is performed with a different realization of the disturbance w and the measurement noise η. The resulting measurement trajectories y are plotted in Figure 4.

To compare the performance of the stochastic predictive controller we also implement a standard predictive controller which is designed under the assumption that the innovation process ω takes its mean value over the control horizon i.e., we assume that $\omega(k) = 0$ for all $k \in T$. The controller is then obtained by minimizing the same criterion function, but now in an open-loop optimization. The results of 100 simulations are plotted in Figure 5.

The two controllers show very different performance. The standard MPC controller is not able to realistically predict a constraint violation. On the other hand, the stochastic predictive controller computes an optimal map from the

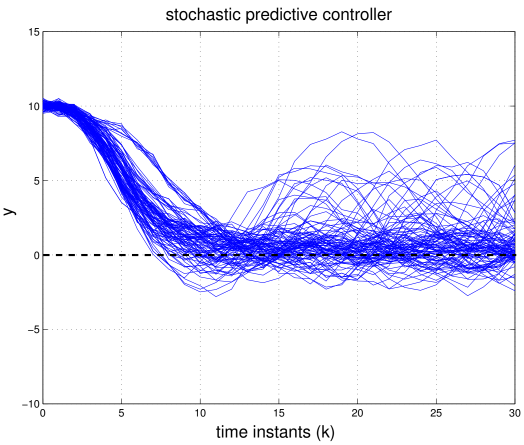

Fig. 4. Double integrator controlled by **stochastic predictive controller**

state to the input for a number of points in the state space. Points are determined with the stochastic sampling of the innovation process and therefore there is a large probability that the optimal map for the predicted states is computed in the region in which the estimated state of the system will be. This leads to the more realistic "prediction" and the control strategy that respects the state constraints better.

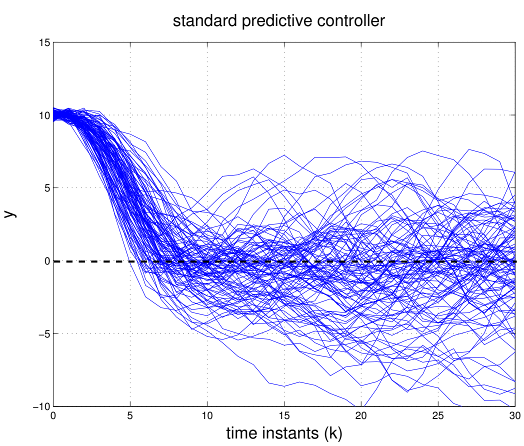

Fig. 5. "Double integrator" controlled by **standard predictive controller**

In Figure 6 we also plot the mean and variance of the obtained trajectories. The mean of the standard MPC scheme converges to a point that is in the region $x_2 < 0$, i.e., it violates the state constraint. The mean response of the system controlled by the stochastic predictive controller converges to a point in the region $x_2 \geq 0$. This point is larger than the set point.

VII. CONCLUSION

In this paper we considered optimal control of linear, constrained stochastic systems via measurement feedback. It has been shown that if the cost criterion is a quadratic function, then the optimal dynamic controllers can be decomposed in a Kalman filter and a static state feedback that

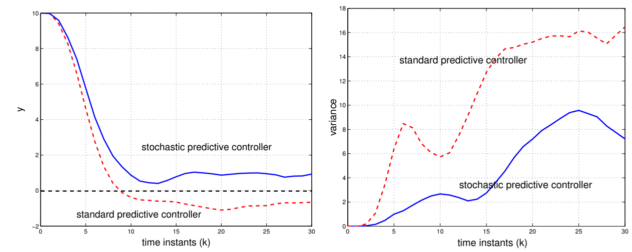

Fig. 6. Mean and the variance of the trajectories from figures 5 and 4

is derived from the solution of a constrained optimization problem of an auxiliary system in which the state is assumed to be measured. We have shown how such a controller can be designed within the model predictive control framework. To make prediction in the model predictive controller as realistic as possible, the estimation structure of the Kalman filter has been exploited in the prediction. A difficulty is that there is no measurement available over the control horizon. It has been shown that the innovation process of the Kalman filter can substitute such a measurement by sampling the innovation process according to its stochastic properties. By doing so, a cost criterion with an empirical mean has been minimized so as to synthesize a dynamic feedback control law in a receding horizon setting. An example shows that this stochastic predictive controller outperforms a standard MPC controller even for relatively small number of samples.

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