Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Robust stabilization of discrete-time nonlinear systems by certainty equivalence output feedback with applications to model predictive control

Michael J. Messina[1], Sezai E. Tuna[1], Andrew R. Teel[1,2]
{mmessina,emre,teel}@ece.ucsb.edu
Center for Control Engineering and Computation
Department of Electrical and Computer Engineering
University of California, Santa Barbara, CA 93106

***Abstract*— We present robust stability results for discrete-time nonlinear systems using certainty equivalence output feedback, particularly those that employ a model predictive control (MPC) formulation to generate the feedback control law. To this end, we discuss nominal robustness properties of general discrete-time nonlinear systems, including those that use discontinuous control laws in the feedback loop. This is important for systems employing MPC since the method can, and sometimes necessarily does, result in discontinuous control laws. Coupling assumptions of nominal robustness with certain uniform observability or detectability assumptions (for each of which we give an observer), we assert that, in particular, MPC is robustly globally asymptotically stabilizing when used in a certainty equivalence output feedback structure. Finally, we give an example to illuminate our results.**

I. Introduction

A. Background

The presence or lack of robustness properties of model predictive control (MPC) algorithms is of key importance for their application to discrete-time systems with output feedback structure. In [19], the authors show that discrete-time MPC is stabilizing in the presence of (decaying) perturbations, such as those that can come from an observer, when the control law is assumed to be Lipschitz. In [15], more explicit results are given for stability of the interconnection of a weak detector with an MPC-stabilized closed-loop system[3] when the feedback and observer are assumed to be Lipschitz. In [9], the authors present a different framework for dealing with these robustness issues that does not make Lipschitz (or even continuity) assumptions on the feedback control law. The purpose of this paper is to show that those results are applicable to the output feedback problem. Additionally, we hope to convey the importance of general robustness properties of discrete-time (full-state or output) feedback systems that employ *discontinuous* control laws in the feedback loop. These properties are crucial to the use of nonlinear MPC (whether employed in an output feedback setting or not), as the method often results in discontinuous feedback laws, sometimes purposefully due to the fact that continuous control laws cannot stabilize some systems.

B. Problem Motivation

We consider the problem of using dynamic output feedback to stabilize the origin of a nonlinear control system

$$x^+ = f(x,u), \quad y = h(x,u), \tag{1}$$

where (1) is referred to as the *plant*, the functions f and h defining the plant are continuous, $x \in \mathbb{R}^n$ is the *state of the plant*, $u \in \mathbb{R}^m$ is the *control input*, and $y \in \mathbb{R}^p$ is the *measured output*. We are also be interested in looking at perturbations to (1) of the form $x^+ = f(x,u) + d, \quad y = h(x,u) + e$, where $d \in \mathbb{R}^n$ and $e \in \mathbb{R}^p$ are, respectively, *additive disturbance* and *measurement error*, both of which we assume to be sufficiently small. Our approach is to use certainty equivalence state feedback; i.e., a given state feedback controller is implemented using an estimate of the state. The estimate is generated by a finite-dimensional dynamical system, often called an *observer*, that is driven by the plant's control input and measured output.

We assume that we can find a state feedback control law $u = \kappa(x)$ such that the origin of the system $x^+ = f(x, \kappa(x))$ is globally asymptotically stable (GAS)[4]. Because we do not necessarily assume that $\kappa(\cdot)$ is continuous, we need to assume a bit more about the stability of this system, namely, that it is nominally robust. One of our primary motivations for studying this problem is that certain MPC algorithms produce discontinuous state feedback control laws that yield stability without nominal robustness (e.g., see [7]) and therefore have little chance of working when implemented with output feedback and in the presence of measurement error. Nevertheless, our output feedback results are not predicated on the state feedback control law coming from model predictive control.

[1]Research supported in part by AFOSR grant numbers F49620-00-1-0106 and F49620-03-1-0203 and NSF grant number ECS 9988813.

[2]Part of this research was carried out while the last author was visiting the Mittag-Leffler Institute, Sweden in March 2003, during its emphasis on Mathematical Control and Systems Theory.

[3]Corresponding results in continuous time have been given in [17] and more recently in [1], [3], [11], and [14]. See [4] for an overview.

[4]We focus on GAS for ease of exposition. Our results can be adapted to the case of local asymptotic stability (LAS).

II. Nominal robustness

A. The basic idea

In this section, we present results for a closed-loop system

$$x^+ = f(x, \kappa(x)) \tag{2}$$

for which a compact set $\mathcal{A}$ is asymptotically stable. We consider sets $\mathcal{A}$ because it allows for more generality and doesn't add significantly to the notational burden. For a given compact set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, we define $|x|_\mathcal{A} := \inf_{z \in \mathcal{A}} |x - z|$. When $\mathcal{A} = \{0\}$, $|x|_\mathcal{A} = |x|$, so the reader who is distracted by the use of a set $\mathcal{A}$ may substitute $|x|_\mathcal{A}$ by $|x|$ in all of the subsequent expressions. In everything that follows we consider global asymptotic stability of $\mathcal{A}$, but local results are similar.

We use functions of class $\mathcal{K}$, class $\mathcal{K}_\infty$ and class $\mathcal{KL}$ to characterize asymptotic stability. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. It belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \to 0} \beta(s, t) = \lim_{t \to \infty} \beta(s, t) = 0$.

Definition 0: For the system (2), the set $\mathcal{A}$ is *globally asymptotically stable with continuous Lyapunov function* if there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class-$\mathcal{K}_\infty$ functions α_1, α_2 and a continuous, positive definite function ρ such that

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \tag{3}$$

and $V(f(x, \kappa(x))) \leq V(x) - \rho(|x|_\mathcal{A})$ for all $x \in \mathbb{R}^n$. ■

Note that, since $\mathcal{A}$ is compact and V is continuous, (3) is equivalent to the statement that V is positive definite with respect to $\mathcal{A}$ and proper. We emphasize that GAS and the existence of a continuous Lyapunov function are not necessarily equivalent when $\kappa(\cdot)$ is discontinuous. This is made clear in [12], where the close relationship between nominal robustness and the existence of a continuous (in fact, smooth) Lyapunov function is illuminated.

There are various ways to characterize nominal robustness. Roughly speaking, the Lyapunov function should decrease along solutions even in the presence of sufficiently small measurement error, i.e., $\kappa(x)$ is replaced by $\kappa(x+e)$, and sufficiently small additive disturbances, i.e., $f(x, u)$ is replaced by $f(x, u) + d$, with $|e|$ and $|d|$ small. Later on we also call e and d *inner* and *outer perturbations*, respectively. It is reasonable to expect nominal robustness when a continuous Lyapunov function exists and $\kappa(\cdot)$ is locally bounded since we can use continuity of $f(\cdot, u)$, $V(\cdot)$, and $\rho(|\cdot|_\mathcal{A})$ to make the following approximations for small perturbations:

$$\begin{aligned}
&V(f(x, \kappa(x+e)) + d) - V(x) \\
&\quad \approx \quad V(f(x+e, \kappa(x+e)) + d) - V(x) \\
&\quad \approx \quad V(f(x+e, \kappa(x+e))) - V(x+e) \\
&\quad \leq \quad -\rho(|x+e|_\mathcal{A}) \approx -\rho(|x|_\mathcal{A}) \ .
\end{aligned}$$

B. Mathematical characterizations

In this and later sections we use the notation $\mathbf{w}_j^i := \left[w(i)^T \ldots w(i+j)^T\right]^T$ for $i \leq j$. We denote $\mathbf{w}_j := \mathbf{w}_j^0$, $\mathbf{w} := \mathbf{w}_\infty^0$, and use the definition $\|\mathbf{w}\| := \sup_k |w(k)|$. For the system $x^+ = f(x, \kappa(x+e)) + d$, we denote the solution k steps into the future under the influence of perturbation sequences $\mathbf{e}$ and $\mathbf{d}$ by $\phi(k, x, \mathbf{e}, \mathbf{d})$. Notice that $\phi(0, x, \mathbf{e}, \mathbf{d}) = x$. We use $\phi(k, x, \mathbf{d})$ when $\mathbf{e} = 0$. We sometimes use $x(k)$ to denote a solution when the context regarding the input is clear and we want to distinguish between different subsystems. The initial condition is then $x(0)$ in this case. Finally, we denote the unit ball with $\mathcal{B} = \{s \in \mathbb{R}^n : |s| \leq 1\}$.

We now define several concepts with the aim of making the notion of nominal robustness more precise.

Definition 1: For the system (2), the set $\mathcal{A}$ is said to be *robustly globally asymptotically stable* (RGAS) if there exist a class-$\mathcal{KL}$ function β and a continuous positive definite function ϱ such that for the system $x^+ = f(x, \kappa(x+e)) + d$ under the constraint $\max\{|e(k)|, |d(k)|\} \leq \varrho(|\phi(k, x, \mathbf{e}, \mathbf{d})|_\mathcal{A})$ for all $k \geq 0$, the solutions satisfy $|\phi(k, x, \mathbf{e}, \mathbf{d})|_\mathcal{A} \leq \beta(|x|_\mathcal{A}, k)$ for all $k \geq 0$. ■

Definition 2: For the system (2), the set $\mathcal{A}$ is said to be *semiglobally practically asymptotically stable (SPAS) in the worst case size of inner and outer perturbations* if there exists a class-$\mathcal{KL}$ function β and for each pair of strictly positive real numbers (δ, Δ) there exists a strictly positive real number ε such that for the system $x^+ = f(x, \kappa(x+e)) + d$ under the constraints $|x|_\mathcal{A} \leq \Delta$ and $\max\{\|\mathbf{e}\|, \|\mathbf{d}\|\} \leq \varepsilon$, the solutions satisfy $|\phi(k, x, \mathbf{e}, \mathbf{d})|_\mathcal{A} \leq \max\{\beta(|x|_\mathcal{A}, k), \delta\}$ for all $k \geq 0$ ■

Definition 3: For the system (2), the set $\mathcal{A}$ is said to be *semiglobally practically asymptotically stable (SPAS) in the worst case size of outer perturbations* if there exists a class-$\mathcal{K}_\infty$ function ρ and for each pair of strictly positive real numbers (δ, Δ) there exist $\varepsilon \in \mathbb{R}_{>0}$ and $\overline{k} \in \mathbb{Z}_{>0}$ such that for the system $x^+ = f(x, \kappa(x)) + d$ under the constraints $|x|_\mathcal{A} \leq \Delta$ and $\|\mathbf{d}\| \leq \varepsilon$, the solutions satisfy $|\phi(k, x, \mathbf{d})|_\mathcal{A} \leq \max\{\rho(|x|_\mathcal{A}), \delta\}$ for all $k \geq 0$ and $|\phi(k, x, \mathbf{d})|_\mathcal{A} \leq \delta$ for all $k \geq \overline{k}$. ■

Definition 4: For the system (2), the set $\mathcal{A}$ is said to be *attenuated input-to-state stable (AISS)* if there exists a continuous nonincreasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, $H(\cdot) \equiv 1$ on a neighborhood of the origin, a class-$\mathcal{KL}$ function β, and a class-$\mathcal{K}$ function γ such that the solutions of the system $x^+ = f(x, \kappa(x + H(|x|_\mathcal{A})e)) + H(|x|_\mathcal{A})d$ satisfy $|\phi(k, x, \mathbf{e}, \mathbf{d})|_\mathcal{A} \leq \max\{\beta(|x|_\mathcal{A}, k), \gamma(\|\mathbf{e}\|), \gamma(\|\mathbf{d}\|)\}$ for all $k \geq 0$. ■

Definition 5: For the system (2), the set $\mathcal{A}$ is said to be *integral input-to-state stable (IISS)* if there exist a class-$\mathcal{KL}$ function β and class-$\mathcal{K}$ functions σ_e and σ_d such that for the system $x^+ = f(x, \kappa(x+e)) + d$, the solutions satisfy $|\phi(k, x, \mathbf{e}, \mathbf{d})|_\mathcal{A} \leq \max\left\{\beta(|x|_\mathcal{A}, k), \sum_{i=0}^{k-1} \sigma_e(|e(i)|), \sum_{i=0}^{k-1} \sigma_d(|d(i)|)\right\}$ for all $k \geq 0$. ■

Proposition 1: Suppose $f(\cdot,\cdot)$ is continuous and $\kappa(\cdot)$ is locally bounded. Then the following statements are equivalent: For the system (2), the compact set $\mathcal{A}$ is

S0: globally asymptotically stable with a continuous Lyapunov function;
S1: robustly globally asymptotically stable;
S2: semiglobally practically asymptotically stable in the worst case size of inner and outer perturbations;
S3: semiglobally practically asymptotically stable in the worst case size of outer perturbations;
S4: attenuated input-to-state stable;
S5: integral input-to-state stable. ■

III. Implications of Nominal Robustness for Output Feedback

A. Certainty equivalence assumption

In this section, we make assumptions on closed-loop systems formed by implementing a certainty equivalence control law that uses an observer output in place of the state and give robust stability results. We do not presuppose that the feedback control law κ comes from an MPC formulation. In later sections, we construct observers for control systems that are uniformly completely observable (Section IV-A) or uniformly detectable (Section IV-B).

B. Dynamic output feedback structure

Our dynamic output feedback will have the general form

$$\eta^+ = g(\eta, u, y),\ \hat{x} = \Upsilon(\eta),\ u = \kappa(\hat{x}), \tag{4}$$

where $\kappa(\cdot)$ is the certainty equivalence control law coming from a state feedback algorithm. We can write the observer in the general form $\eta^+ = G(\eta, y),\ \hat{x} = \Upsilon(\eta)$ and the interconnection as

$$\begin{array}{rcl} x^+ &=& f(x, \kappa(\Upsilon(\eta))) \\ \eta^+ &=& G(\eta, h(x, \kappa(\Upsilon(\eta))))\ . \end{array} \tag{5}$$

In the presence additive disturbances, the interconnection is

$$\begin{array}{rcl} x^+ &=& f(x, \kappa(\Upsilon(\eta))) + d_1 \\ \eta^+ &=& G(\eta, h(x, \kappa(\Upsilon(\eta))) + e) + d_2\ . \end{array} \tag{6}$$

We use $X := [x^T\ \eta^T]^T$ for the composite state and $d = [d_1^T\ d_2^T]^T$ as the composite additive disturbance. In the case of observability, our closed-loop systems will satisfy the following assumption:

Assumption 1: For the closed-loop system (6) there exist an integer $r \geq 1$, class-$\mathcal{K}_\infty$ functions ρ and γ, a class-$\mathcal{KL}$ function β, and for each pair of strictly positive real numbers (δ, Δ) there exists $\varepsilon > 0$ such that if $\|\mathbf{X}\| \leq \Delta$ and $\max\{\|\mathbf{e}\|, \|\mathbf{d}\|\} \leq \varepsilon$, then, for all $k \geq 0$,

$$\begin{array}{rcl} |x(k) - \hat{x}(k)| &\leq& \max\{\rho(|X(0)|) \cdot \max\{0, r-k\}, \delta\} \\ |\eta(k)| &\leq& \max\{\beta(|X(0)|, k), \\ && \qquad \gamma(\max\{\|\mathbf{x}\|, \|\mathbf{x} - \hat{\mathbf{x}}\|\}), \delta\}. \end{array}$$

■

In the case of detectability, our closed-loop will satisfy the following assumption:

Assumption 2: For the closed-loop system (6) there exist a class-$\mathcal{K}_\infty$ function γ, class-$\mathcal{KL}$ functions β_1 and β_2, and for each pair of strictly positive real numbers (δ, Δ) there exists $\varepsilon > 0$ such that if $\|\mathbf{X}\| \leq \Delta$ and $\max\{\|\mathbf{e}\|, \|\mathbf{d}\|\} \leq \varepsilon$, then, for all $k \geq 0$,

$$\begin{array}{rcl} |x(k) - \hat{x}(k)| &\leq& \max\{\beta_1(|X(0)|, k), \delta\} \\ |\eta(k)| &\leq& \max\{\beta_2(|X(0)|, k), \\ && \qquad \gamma(\max\{\|\mathbf{x}\|, \|\mathbf{x} - \hat{\mathbf{x}}\|\}), \delta\}. \end{array}$$

■

Remark 1: The observers in the case of detectability have a slightly stronger property than this, that $\eta(k) = \hat{x}(k)$ and $|X(0)|$ can be replaced by $|x(0) - \hat{x}(0)|$, but these stronger properties aren't needed for the statements below. ■

C. Closed-loop conclusions

We now state two closed-loop robust stability results. Later, we use these results to assert robust stability for output feedback systems. Although we do not presuppose the feedback law κ comes from an MPC formulation, we do need to make the following assumption in order to state our results here.

Assumption 3: The function $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is locally bounded and the origin of $x^+ = f(x, \kappa(x))$ is RGAS. ■

Proposition 2: Under Assumptions 2 and 3, if the function σ_e from the equivalence between RGAS and IISS, for some β, and the function β_1 coming from Assumption 2 are such that $\sigma_e(\beta_1(s, \cdot))$ is summable for all s, then the origin of the closed-loop system (5) is RGAS. ■

The next result follows directly from Proposition 2 by observing that, with the definition $\beta_1(s, k) := \rho(s) \cdot \max\{0, r-k\}$, the function $\sigma_e(\beta_1(s, \cdot))$ is summable for all s since $\beta_1(s, \cdot)$ has finite support.

Corollary 1: Under Assumptions 1 and 3, the origin of the closed-loop system is RGAS. ■

IV. Observability, Observers and Dynamic Output Feedback

In this section we discuss dynamic output feedback systems satisfying certain detectability/observability properties that, when used with the given observers, satisfy either Assumption 1 or 2. We then give the corresponding results to Corollary 1 and Proposition 2, respectively.

A. Uniform complete observability

1) Notation and Definition: For the system (1) we use $\phi(k, x, \mathbf{u}_{k-1}^\ell)$ (a slight deviation from above) to denote the solution at time k starting at x produced by the input sequence $(u(\ell), \ldots, u(\ell+k-1))$. Also, we define the vector

$$\mathbf{h}_r(x, \mathbf{u}_r^\ell) := \begin{bmatrix} h(x, u(\ell)) \\ h(\phi(1, x, \mathbf{u}_0^\ell), u(\ell+1)) \\ \vdots \\ h(\phi(r, x, \mathbf{u}_{r-1}^\ell), u(\ell+r)) \end{bmatrix}.$$

The following definition is similar to concepts explored in [10] and [18] (see [5] and [22] for related concepts in continuous time):

Definition 6: The system (1) is said to be *uniformly completely observable* (UCO) if there exist an integer r and a continuous mapping $\Psi : \mathbb{R}^{p(r+1)\times m(r+1)} \to \mathbb{R}^n$ such that

$$x = \Psi(\mathbf{h}_r(x, \mathbf{u}_r), \mathbf{u}_r) \tag{7}$$

for all x and all input sequences $\mathbf{u}_r$. ■

The existence of a function satisfying (7) has been discussed in [6] (in the context of continuous time nonlinear systems with discrete measurements). For a given input sequence $\mathbf{u}_i$ and initial state x we define the output sequence $y(i) := h(\phi(i, x, \mathbf{u}_{i-1}), u(i))$. The dependence of $y(i)$ on x and $\mathbf{u}_i$ is left implicit. The following fact results from time invariance and the definition of solution.

Lemma 1: For a system that is UCO with integer r,

$$\begin{aligned}&\phi(k, x, \mathbf{u}_{k-1}) = \\ &\qquad \phi\left(r+1, \Psi\left(\mathbf{y}_r^{k-r-1}, \mathbf{u}_r^{k-r-1}\right), \mathbf{u}_r^{k-r-1}\right)\end{aligned}$$

for each $k \geq r+1$. ■

2) A deadbeat observer: We consider a system in the form of (1) and assume that it is UCO. Consider the following dynamical system

$$\begin{array}{rclcrcl} \xi_1^+ & = & \xi_2 & \qquad & \zeta_1^+ & = & \zeta_2 \\ & \vdots & & & & \vdots & \\ \xi_r^+ & = & \xi_{r+1} & & \zeta_r^+ & = & \zeta_{r+1} \\ \xi_{r+1}^+ & = & y & & \zeta_{r+1}^+ & = & u \end{array} \tag{8}$$

with dynamic output feedback controller

$$u = \kappa(\hat{x}); \; \hat{x} = \phi\left(r+1, \Psi\left(\xi, \zeta\right), \zeta\right). \tag{9}$$

This matches the form of (4) with $\eta := \left[\xi^T, \, \zeta^T\right]^T$, and we have the following result:

Lemma 2: Define:

$$\eta := \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \xi \\ \zeta \end{bmatrix},$$

$$g(\eta, u, y) :=$$

$$\begin{bmatrix} \begin{matrix} 0_{r\times 1} & \mathrm{I}_r \\ 0 & 0_{1\times r} \end{matrix} & 0_{r+1\times r+1} \\ 0_{r+1\times r+1} & \begin{matrix} 0_{r\times 1} & \mathrm{I}_r \\ 0 & 0_{1\times r} \end{matrix} \end{bmatrix} \eta + \begin{bmatrix} 0 \\ \vdots \\ y \\ 0 \\ \vdots \\ u \end{bmatrix},$$

$$\Upsilon(\eta) := \phi(r+1, \, \Psi(\eta_1, \, \eta_2), \, \eta_2) \text{ , and}$$
$$G(\eta, y) := g(\eta, \, \kappa(\Upsilon(\eta)), \, y) \text{ ,}$$

where I_i is an i-by-i identity matrix and $0_{i\times j}$ is an i-by-j zero matrix. If $h(0,0) = 0$, $\Psi(0,0) = 0$, and κ is locally bounded and satisfies $\limsup_{x\to 0} |\kappa(x)| = 0$, then the closed-loop system formed by the interconnection (6) of the dynamic output feedback controller (8) and the system (1) in the presence of measurement error and additive disturbances satisfies Assumption 1. ■

The following is an immediate consequence of Corollary 1 and Lemma 2.

Corollary 2: Suppose the system (1) is UCO, $h(0,0) = 0$, $\Psi(0,0) = 0$, $\limsup_{x\to 0} |\kappa(x)| = 0$, and Assumption 3 holds. Then the output feedback controller (8) interconnected with the system (1) is RGAS. ■

B. Uniform detectability

In this section we discuss a general notion of detectability expressed in terms of the existence of a Lyapunov function (cf. *weakly detectable* definition [24], a local version of the definition is used in [15]).

Definition 7: The system (1) is said to be *uniformly detectable* if there exist a continuous function $\Gamma : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}^n$ such that $\Gamma(0, 0, 0) = 0$, a continuous function $W : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, class-$\mathcal{K}_\infty$ functions α_1, α_2, and a continuous positive definite function α_3 such that for all $(x, \hat{x}, u) \in \mathbb{R}^{n\times n\times m}$,

$$\begin{aligned}\alpha_1(|x-\hat{x}|) \leq W(x, \hat{x}) \leq \alpha_2(|x-\hat{x}|) \\ W(x^+, \hat{x}^+) - W(x, \hat{x}) \leq -\alpha_3(|x-\hat{x}|),\end{aligned}$$

where $\hat{x}^+ := \Gamma(\hat{x}, u, h(x, u))$. ■

The corresponding dynamic output feedback controller is

$$u = \kappa(\hat{x}); \; \hat{x}^+ = \Gamma(\hat{x}, u, h(x, u)). \tag{10}$$

This controller has the structure of (4) if we choose $\eta = \hat{x}$ and $\Upsilon(\eta) = \eta$. We then have the following result.

Lemma 3: Define $\Upsilon(\eta) := \eta$, $G(\eta, y) := \Gamma(\eta, \kappa(\eta), y)$, and $\eta := \hat{x}$. If the feedback function κ is locally bounded, then the closed-loop system formed from the interconnection (6) of the dynamic output feedback controller (10) and the system (1) in the presence of measurement error and additive disturbances satisfies Assumption 2. ■

We can now state a corollary that follows from Proposition 2 and Lemma 3.

Corollary 3: Suppose the system (1) is uniformly detectable, κ is locally bounded (that is, the suppositions of Lemma 3 hold), and that Assumption 3 holds. Then if the function σ_e coming from the equivalence between RGAS (from Assumption 3) and IISS, for some β, and the function β_1 corresponding to the conclusion of Lemma 3 are such that $\sigma_e(\beta_1(s, \cdot))$ is summable for all s, then the output feedback controller (10) interconnected with the system (1) is RGAS. ■

V. Guaranteeing Assumption 3 with Model Predictive Control

In order to apply Corollary 2 or 3, we must show that Assumption 3 holds. First, we recall results (e.g., from [13]) on the existence of state feedback laws that guarantee Assumption 3 holds for systems that can be driven to the origin asymptotically using open loop controls. If the open-loop controls vanish as the trajectories approach the origin, then $\kappa(\cdot)$ can be taken to satisfy $\limsup_{x\to 0} |\kappa(x)| = 0$. The feedback laws are constructed from solutions to an appropriate infinite horizon optimal control problem. Next, we give some conditions on finite horizon model predictive control problems that generate feedback laws such that both Assumption 3 holds and $\limsup_{x\to 0} |\kappa(x)| = 0$.

A. Infinite horizon model predictive control

The following definition parallels [13, Definition 6]:

Definition 8: The system $x^+ = f(x,u)$ is said to be *asymptotically controllable to the origin with locally bounded controls* if there exist $\beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}$, $\mu \geq 0$, and for each $x \in \mathbb{R}^n$ a sequence $\mathbf{u}$ such that for each $k \geq 0$

$$\begin{aligned} |\phi(k,x,\mathbf{u})| &\leq \beta(|x|,k) \\ |u(k)| &\leq \alpha(|\phi(k,x,\mathbf{u})|) + \mu \ . \end{aligned} \tag{11}$$

The system is said to be *asymptotically controllable to the origin with vanishing controls* when $\mu = 0$. ■

The next theorem is based on [13, Theorem 2], which is established using an appropriate infinite horizon optimal control problem.

Theorem 1: If the system $x^+ = f(x,u)$ is asymptotically controllable to the origin with locally bounded controls then there exists a feedback function κ satisfying Assumption 3. Moreover, if the system is asymptotically controllable to the origin with vanishing controls, then κ can be taken to satisfy $\limsup_{x\to 0}|\kappa(x)| = 0$. ■

Proof. Using the continuity of f and α, [13, Theorem 2] is applicable and there exists a smooth, positive definite radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for each $x \in \mathbb{R}^n$,

$$\min_{u\in[\alpha(|x|)+\mu]\mathcal{B}} V(f(x,u)) \leq V(x)e^{-1} \ . \tag{12}$$

We let $\lambda \in [e^{-1}, 1)$ and then for each $x \in \mathbb{R}^n$ we let $\kappa(x) \in [\alpha(|x|) + \mu]\mathcal{B}$ satisfy $V(f(x,\kappa(x))) \leq \lambda V(x)$. With this feedback function, Assumption 3 is satisfied. Also, when $\mu = 0$ we get $\limsup_{x\to 0}|\kappa(x)| = 0$. ■

The following corollary comes from combining Theorem 1 with Corollary 2 (cf. [20], [21], and [23]).

Corollary 4: Suppose the system (1) is asymptotically controllable to the origin with vanishing controls and is UCO with $h(0,0) = 0$ and $\Psi(0,0) = 0$. Then the system (1) can be made RGAS by dynamic output feedback. ■

B. Finite horizon model predictive control

The solution to a general infinite horizon optimization problem is typically computationally intractable; consequently, finite horizon optimization algorithms, such as finite-horizon MPC, are often used instead. However, not every finite horizon optimization algorithm yields a robustly stabilizing control even if it is strengthened with so-called "stability constraints" such as terminal set constraints (see, e.g., [7] for examples of nonrobustness). For an overview of relevant MPC concepts and typically employed MPC methods, see [2] and [16] (for a more complete discussion of the methods used in this paper, see [8]). In this section we study a certain class of MPC algorithms that renders the origin of the closed-loop RGAS if the optimization horizon is chosen long enough. We assume that the optimization problem that yields the MPC feedback law is globally feasible, i.e., a solution exists for all $x \in \mathbb{R}^n$, for a long enough horizon length.

We use the *cost function*

$$\begin{aligned} J_N(x,\,\mathbf{u}_{N-1}) \;\; := \;\; & g(\phi(N,\,x,\,\mathbf{u}_{N-1})) \\ & + \sum_{k=0}^{N-1} \ell(\phi(k,\,x,\,\mathbf{u}_{N-1}),\,u(k)), \end{aligned} \tag{13}$$

constructed from the *terminal cost* $g : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ and the *stage cost* $\ell : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}_{\geq 0}$. We do not require g to be a local control Lyapunov function; unlike the typical MPC setting, our stability results do not depend on g explicitly. We consider the optimization problem

$$V_N(x) := \inf_{\mathbf{u}_{N-1}} J_N(x,\,\mathbf{u}_{N-1}) \tag{14}$$
$$\text{subject to} \begin{cases} u(k) \in \mathcal{U}, \quad k \in \{0,\,1,\,\ldots,\,N-1\} \\ \phi(N,\,x,\,\mathbf{u}_{N-1}) \in \mathcal{X}, \end{cases}$$

where $V_N(x)$ is the *value function,* $\mathcal{U} \subseteq \mathbb{R}^m$ is the *control input set* , and $\mathcal{X} \subseteq \mathbb{R}^n$ is the *terminal constraint set*. The terminal constraint set can be, for example, the origin $\mathcal{X} = \{0\}$, the whole space $\mathcal{X} = \mathbb{R}^n$ (which corresponds to the case without a terminal constraint), the interior of a sublevel set of some Lyapunov function $\mathcal{X} = \{z \in \mathbb{R}^n \,:\, V(z) \leq c\}$, or some hyperplane containing the origin.

Whenever the infimum is achieved by an admissible control input sequence, the *MPC feedback law* $\kappa_N : \mathbb{R}^n \mapsto \mathcal{U}$ is a function that returns the first element of the sequence given the current state, i.e., $\kappa_N(x) = u(0)$, where $J_N(x,\,\mathbf{u}_{N-1}) = V_N(x)$. In the following assumptions on the MPC algorithm, we consider a continuous proper indicator function $\varsigma : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, that is, there exist class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_\varsigma$, $\overline{\alpha}_\varsigma$ such that $\underline{\alpha}_\varsigma(|x|) \leq \varsigma(x) \leq \overline{\alpha}_\varsigma(|x|)$ for all $x \in \mathbb{R}^n$.

Assumption 4: The functions g and ℓ are continuous. ■

Assumption 5: For all $u \in \mathcal{U}$, $\ell(x,\,u) \geq \varsigma(x)$. ■

Assumption 6: For each x there exists an admissible control input sequence $\mathbf{u}^*_{N-1}$ such that $J_N(x,\,\mathbf{u}^*_{N-1}) = V_N(x)$ provided (14) has a solution for x. ■

Assumption 7: There exists $\overline{a} \geq 1$ such that for all $N \geq 1$, $V_N(x) \leq \overline{a}\varsigma(x)$ whenever x is feasible for $V_N(x)$. ■

Assumption 8: Either the control input set $\mathcal{U}$ is compact, or $\sup_i |u(i)| \to \infty$ implies $J_N(x,\,\mathbf{u}_{N-1}) \to \infty$. ■

Assumption 9: There exists a continuous positive definite function $\rho_\ell : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}_{\geq 0}$ such that $\ell(x,\,u) + \ell(f(x,\,u),\,v) \geq \rho_\ell(x,\,u)$ for all $v \in \mathcal{U}$. ■

Under these assumptions, we can establish the following result, a similar version of which appears in [9] along with more general results. It is also a special case of the result in [8] when there is no terminal constraint, that is, $\mathcal{X} = \mathbb{R}^n$.

Theorem 2: Consider the system $x^+ = f(x,\,u)$ under Assumptions 4-8 and further assume that there exists a horizon length $M \geq 1$ such that the optimization problem (14) has a solution for all $x \in \mathbb{R}^n$. Then, for all horizons $N > \overline{a}^2 + M - 1$, κ is locally bounded, and the origin of the MPC closed-loop $x^+ = f(x,\,\kappa_N(x))$ is RGAS. Furthermore, if, in addition to the above assumptions, Assumption 9 holds, then $\limsup_{x\to 0}|\kappa_N(x)| = 0$. ■

VI. EXAMPLE

Consider the following system:

$$x^+ = \begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} x_1 + x_2^3 \\ x_2 + u^3 \end{bmatrix} =: f(x, u), \quad y = x_1^3 =: h(x). \tag{15}$$

Note that the linearization is neither detectable nor stabilizable; however, the nonlinear system is stabilizable and UCO. The state can be reconstructed by a deadbeat observer as in (8):

$$\begin{array}{rclcrcl} \xi_1^+ & = & \xi_2 & \qquad & \zeta_1^+ & = & \zeta_2 \\ \xi_2^+ & = & y & & \zeta_2^+ & = & u \end{array} \tag{16}$$

with the continuous output map

$$\hat{x} = \phi\left(2, \Psi\left(\xi, \zeta\right), \zeta\right), \Psi\left(\xi, \zeta\right) := \left[\begin{array}{c} \xi_1^{\frac{1}{3}} \left(\xi_2^{\frac{1}{3}} - \xi_1^{\frac{1}{3}}\right)^{\frac{1}{3}} \end{array}\right] \tag{17}$$

Given any input sequence $\mathbf{v}_1$, if we let $h(x, u) = h(x)$, then $\mathbf{h}_1(x, \mathbf{v}_1) = [h(x) \quad h(f(x, v(0)))]^T$ and $x = \Psi(\mathbf{h}_1(x, \mathbf{v}_1), \mathbf{v}_1)$ for all $x \in \mathbb{R}^n$. Hence (15) is UCO.

We now formulate an MPC algorithm to stabilize (15) using the terminal equality constraint $\mathcal{X} = \{0\}$. We pick the costs $\ell(x, u) = \varsigma(x) := |x_1| + |x_2|^3$ and $g(x) = 0$. Since ℓ and g are continuous, Assumption 4 holds. Since $\ell(x, u) = \varsigma(x)$, Assumption 5 holds. The state of the system can be driven to the origin from any initial condition in two steps with the feedback law $\kappa(x) = -(x_2 + (x_1 + x_2^3)^{1/3})^{1/3}$. This resulting control input sequence is admissible, hence we have Assumption 6. Since the deadbeat control law above cannot be better than the optimal control law, we can use it to obtain the upper bound $V_N(x) \leq 3\varsigma(x)$ for all horizons $N \geq 2$. Then, Assumption 7 holds with $\overline{a} = 3$. Observe that $\sup_i |u(i)| \to \infty$ implies $J_N(x, \mathbf{u}_{N-1}) \to \infty$ since the stage cost $\ell(x, u) = \varsigma(x)$ is positive definite in x. Therefore, Assumption 8 is satisfied. Finally, since $\ell(x, u) + \ell(f(x, u), v) = |x_1| + |x_2|^3 + |x_1 + x_2^3| + |x_2 + u^3|^3 =: \rho_\ell(x, u)$ is positive definite, Assumption 9 holds. Since any initial condition can be driven to the origin in two steps, Theorem 2 can be applied with $M = 2$ and we conclude that for all $N > 3^2 + 2 - 1 = 10$, Assumption 3 holds and $\limsup_{x \to 0} |\kappa_N|(x) = 0$. Finally, since $h(0) = 0$, $\Psi(0, 0) = 0$, by Corollary 2 we conclude the origin of the interconnection of the MPC-generated controller and the observer (16-17) is RGAS for all $N > 10$.

VII. CONCLUSIONS

We have shown that discrete-time nonlinear feedback systems that employ discontinuous feedback control laws can be used in a certainty equivalence output feedback structure. We have given an MPC formulation along with two observer structures that together satisfy the conditions required to make the origin of the systems in question RGAS and have applied these ideas to an example.

REFERENCES

[1] V. Adetola and M. Guay. Nonlinear output feedback receding horizon control of sampled-data systems. In *Proc. American Cont. Conf.*, volume 6, pages 4914–4919, Denver, CO, 2003.

[2] F. Allgöwer and A. Zheng, editors. *Nonlinear Model Predictive Control*. Birkhäuser, 2000. See in particular: G. De Nicolao, L. Magni, and R. Scattolini, Stability and Robustness of Nonlinear Receding Horizon Control, pages 3–22; D. Mayne, Nonlinear Model Predictive Control: Challenges and Opportunities, pages 23–44.

[3] R. Findeisen, L. Imsland, F. Allgöwer, and B.A. Foss. Output feedback stabilization of constrained systems with nonlinear predictive control. *Int. J. Robust and Nonlinear Cont.*, 13(3-4):211–227, 2003.

[4] R. Findeisen, L. Imsland, F. Allgöwer, and B.A. Foss. State and output feedback nonlinear model predictive control: an overview. *Europ. J. Contr.*, 9(2-3):190–207, 2003.

[5] J.P. Gauthier and G. Bornard. Observability for any u(t) of a class of nonlinear systems. *IEEE Trans. Auto. Cont.*, 26(4):922–926, 1981.

[6] S.T. Glad. Observability and nonlinear deadbeat observers. In *Proc. IEEE CDC*, volume 2, pages 800–802, San Antonio, TX, 1983.

[7] G. Grimm, M.J. Messina, A.R. Teel, and S.E. Tuna. Examples of zero robustness in constrained model predictive control. In *Proc. IEEE CDC*, pages 3724–3729, Maui, HI, 2003.

[8] G. Grimm, M.J. Messina, A.R. Teel, and S.E. Tuna. Model predictive control when a local control Lyapunov function is not available. In *Proc. American Cont. Conf.*, pages 4125–4130, Denver, CO, 2003.

[9] G. Grimm, M.J. Messina, A.R. Teel, and S.E. Tuna. Nominally robust model predictive control with state constraints. In *Proc. IEEE CDC*, pages 1413–1418, Maui, HI, 2003.

[10] J.W. Grizzle and P.E. Moraal. Newton, observers and nonlinear discrete-time control. In *Proc. IEEE CDC*, volume 2, pages 760–767, Honolulu, HI, 1990.

[11] L. Imsland, R. Findeisen, F. Allgöwer, and B.A. Foss. Output feedback stabilization with nonlinear predictive control: asymptotic properties. In *Proc. American Cont. Conf.*, volume 6, pages 4908–4913, Denver, CO, 2003.

[12] C.M. Kellett and A.R. Teel. On robustness of stability and Lyapunov functions for discontinuous difference equations. In *Proc. IEEE CDC*, volume 4, pages 4282–4287, Las Vegas, NV, 2002.

[13] C.M. Kellett and A.R. Teel. Results on discrete-time control-Lyapunov functions. In *Proc. IEEE CDC*, pages 5961–5966, Maui, HI, 2003. See also C.M. Kellett, *Advances in converse and control Lyapunov functions*, PhD disseration, University of California, Santa Barbara, 2002.

[14] S.L. De Oliveira Kothare and M. Morari. Contractive model predictive control for constrained nonlinear systems. *IEEE Trans. Auto. Cont.*, 45(6):1053–1071, 2000.

[15] L. Magni, G. De Nicolao, and R. Scattolini. Output feedback receding-horizon control of discrete-time nonlinear systems. In *Preprints of the 4th IFAC NOLCOS*, volume 2, pages 422–427, 1998.

[16] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[17] H. Michalska and D.Q. Mayne. Moving horizon observers and observer-based control. *IEEE Trans. Auto. Cont.*, 40(6):995–1006, 1995.

[18] H. Nijmeijer. Observability of autonomous discrete time non-linear systems: a geometric approach. *International Journal of Control*, 36(5):867–874, 1982.

[19] P.O.M. Scokaert, J.B. Rawlings, and E.S. Meadows. Discrete-time stability with perturbations: application to model predictive control. *Automatica*, 33(3):463–470, 1997.

[20] H. Shim and A.R. Teel. Asymptotic controllability and observability imply semiglobal practical asymptotic stabilizability by sampled-data output feedback. *Automatica*, 39(3):441–454, 2003.

[21] E.D. Sontag. Conditions for abstract nonlinear regulation. *Information & Control*, 51(2):105–127, 1981.

[22] E.D. Sontag. A concept of local observability. *S&C Letters*, 5(1):41–47, 1984/5.

[23] A. Teel and L. Praly. Global stabilizability and observability imply semiglobal stabilizability by output feedback. *S&C Letters*, 22(5):313–325, 1994.

[24] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice Hall, 2nd edition, 1993.