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H^∞ Collocated Control of Structural Systems: An Analytical Bound Approach

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***Abstract*— The paper examines the H^∞ norm analysis and output feedback control synthesis problems for structural systems with collocated sensors and actuators. Using a particular solution of the Bounded Real Lemma for an open loop collocated structural system we obtain an explicit expression to compute an upper bound on the H^∞ norm of such systems. Then, for the corresponding output feedback H^∞ control synthesis problem we obtain an explicit parametrization of the output feedback control gains that achieve the proposed H^∞ norm bound. These results have obvious computational advantages for large scale systems where standard H^∞ analysis and control design methods are computationally intractable. Computational examples demonstrate the advantages of the proposed results.**

Keywords: H^∞ control, Control of structural systems, Linear Matrix Inequalities.

I. Introduction

The control of structural systems with collocated sensors and actuators has been shown to provide great advantages from a stability, passivity, robustness and an implementation viewpoint. For example, collocated control can easily be achieved in a space structure when an attitude rate sensor is placed at the same point as a torque actuator [3][13]. Collocation of sensors and actuators leads to symmetric transfer functions. Several other classes of engineering systems, such as circuit systems, chemical reactors and power networks, can be modelled as systems with symmetric transfer functions. Stabilization, robustness, model reduction and control of such systems has been examined recently [2][14][15][20].

State space H^∞ control based on the standard Riccati equation approach or the recent linear matrix inequality (LMI) formulation is now a well developed control synthesis tool. The optimal static state feedback and full-order dynamic output feedback H^∞ control synthesis problems can be solved using iterations on the corresponding Riccati solutions or via the computational solution of a convex LMI optimization problem [7][8][12]. On the other hand, the static output feedback and the fixed-order dynamic output feedback H^∞ control synthesis problems are difficult computational problems since they require the solution of (nonconvex) bilinear matrix inequalities (BMIs) or LMIs with coupling rank constraints [9][11][17].

In this work, we examine the H^∞ control analysis and the symmetric output feedback H^∞ control synthesis problems for systems with symmetric transfer functions. The objective of the paper is to show that, by exploiting the particular structure of these systems, explicit bounds for the H^∞ control problems can be obtained. To this end, a particular solution of the Bounded Real Lemma is proposed and an explicit expression for an upper bound of the H^∞ norm of such a symmetric system is obtained that requires only the computation of the maximum eigenvalue of a symmetric matrix. Subsequently, we derive an explicit parametrization for the output feedback H^∞ control gains that guarantee this bound. The proofs of the results are purely algebraic based on simple matrix algebra tools. This work generalizes the results of [19] that consider systems with state space symmetry, which is a special case of transfer function symmetry. However, in [19] the corresponding algebraic results are exact although in the present paper the results provide a conservative bound on the H^∞ norm.

The notation to be used in this paper is as follows: Given a real matrix N, the orthogonal complement $N^\perp$ is defined as the (possibly non-unique) matrix with maximum row rank that satisfies $N^\perp N = 0$ and $N^\perp N^{\perp T} > 0$. Hence, $N^\perp$ can be computed from the singular value decomposition of N as follows: $N^\perp = TU_2^T$ where T is an arbitrary nonsingular matrix and U_2 is defined from the singular value decomposition of N

$$N = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

The standard notation $>$ ($<$) is used to denote the positive (negative) definite ordering of symmetric matrices. The ith eigenvalue of a real symmetric matrix N will be denoted by $\lambda_i(N)$ where the ordering of the eigenvalues is defined as $\lambda_{\max}(N) = \lambda_1(N) \geq \lambda_2(N) \geq \ldots \geq \lambda_n(N)$. The maximum singular value of a (not necessarily square) matrix N will be denoted by $\sigma_{\max}(N)$, which is also its spectral norm $||N||$. N^+ will denote the Moore-Penrose generalized inverse of a matrix N.

II. The Collocated H^∞ Control Analysis Problem

Consider the following vector second-order representation of a structural system with collocated sensors and

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actuators

$$\begin{aligned} M\ddot{q} + D\dot{q} + Kq &= Fu \\ y &= F^T\dot{q} \end{aligned} \quad (1)$$

where $q(t) \in \mathbb{R}^n$ is the generalized coordinate vector, $u(t) \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^k$ is the measured output vector. The matrices M, D and K are symmetric positive definite matrices that represent the structural system mass, damping and stiffness distribution, respectively. The system has a state-space realization as follows

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad (2)$$

with

$$\begin{aligned} A &= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}, \ C = \begin{bmatrix} 0 & F^T \end{bmatrix} \end{aligned} \quad (3)$$

where $x = [q^T \quad \dot{q}^T]^T$. Notice that the transfer function $G(s)$ of the system (2)-(3)

$$G(s) = sF^T(Ms^2 + Ds + K)^{-1}F$$

is symmetric, i.e., $G(s) = G^T(s)$. The system (2)-(3) is an *externally symmetric* state-space realization, that is, there exists a nonsingular matrix T such that

$$A^TT = TA, \ C^T = TB \quad (4)$$

This class of systems is more general than the class of *internally* or *state space symmetric* systems that satisfy the symmetry conditions (4) with a positive definite transformation matrix T [20]. Obviously, state-space symmetry implies external symmetry, but the converse is not true, that is, there exist symmetric transfer matrices for which there is no internally symmetric realization. An analytical solution of the H^∞ control problem for internally symmetric systems has been presented in [19].

Recall that the H^∞ norm of the system (2) is given by

$$||G||_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{G(j\omega)\}$$

where $G(s) = C(sI - A)^{-1}B$ is the transfer function of the system and $\sigma_{\max}$ denotes the maximum singular value of a matrix. It is well known that for a stable LTI system, its H^∞ norm can be approximated iteratively, for example using a bisection method [5]. The next result shows that for a vector second-order realization (2)-(3), an upped bound of its H^∞ norm can be computed using a simple explicit formula.

Theorem 1: Consider the vector second-order system realization (2)-(3). The system has an H^∞ norm γ that satisfies

$$\gamma < \bar{\gamma} = \lambda_{\max}(F^TD^{-1}F) \quad (5)$$

To prove this result recall the following Bounded Real Lemma (BRL) characterization of the H^∞ norm of a system.

Lemma 2: [6] A stable system (2) has an H^∞ norm less than or equal to γ if and only if there exists a matrix $P > 0$ satisfying

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} \leq 0 \quad (6)$$

Also, we will need the following Schur complement formula [1].

Lemma 3: The block matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix},$$

where S_{11} and S_{22} are symmetric ,is positive definite if and only if

$$S_{11} > 0 \ \text{ and } S_{22} - S_{12}^TS_{11}^{-1}S_{12} > 0$$

or

$$S_{22} > 0 \ \text{ and } S_{11} - S_{12}S_{22}^{-1}S_{12}^T > 0$$

These conditions can be easily modified to test negative definiteness and semidefiniteness of a matrix [4]. Now, Theorem 1 follows from the symmetric system BRL condition and the following Finsler's Lemma [17].

Lemma 4: (Finsler's Lemma) Consider matrices Γ and Q such that Γ has full column rank and Q=Q^T.Then the following statements are equivalent:

(i) There exists a scalar μ such that

$$\mu\Gamma\Gamma^T - Q > 0 \quad (7)$$

(ii) The following condition holds

$$\Gamma^\perp Q\Gamma^{\perp T} < 0 \quad (8)$$

If the above statements hold, then all scalars μ satisfying (7) are given by

$$\mu > \mu_{\min} \triangleq \lambda_{\max}[\Gamma^+(Q - Q\Gamma^{\perp \mathbf{T}}(\Gamma^\perp Q\Gamma^{\perp T})^{-1}\Gamma^\perp Q)\Gamma^{+T}]. \quad (9)$$

Proof: For the Theorem 1. The result follows from the Bounded Real Lemma 2 by utilizing the following Lyapunov matrix

$$P = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}. \quad (10)$$

Using (10), the Bounded Real Lemma 2 provides

$$\begin{bmatrix} -2D & F & F \\ F^T & -\gamma I & 0 \\ F^T & 0 & -\gamma I \end{bmatrix} \leq 0$$

Application of the Schur complement formula in Lemma 3 results in the following condition

$$\begin{aligned} & -2D - \begin{bmatrix} F & F \end{bmatrix} \begin{bmatrix} -\gamma I & 0 \\ 0 & -\gamma I \end{bmatrix}^{-1} \begin{bmatrix} F^T \\ F^T \end{bmatrix} \\ = & \ -D + \frac{1}{\gamma}FF^T \leq 0. \end{aligned}$$

Which using Schur complement formula again results in

$$\gamma \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} - \begin{bmatrix} -D & F \\ F^T & 0 \end{bmatrix} \geq 0$$

Then, application of Finsler's Lemma (4) provides the bound (5). ■

III. THE H^∞ CONTROL SYNTHESIS PROBLEM

Now consider the following controlled vector second-order system

$$\begin{aligned} M\ddot{q} + D\dot{q} + Kq &= F(u+w) \\ z &= F^T\dot{q} \\ y &= F^T\dot{q} \end{aligned} \tag{11}$$

where $q(t) \in \mathbb{R}^n$ is the generalized coordinate vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $w(t) \in \mathbb{R}^m$ is the external and disturbance input, $y(t) \in \mathbb{R}^k$ is the measured output vector and $z(t) \in \mathbb{R}^k$ is the performance output vector. The collocated H^∞ control synthesis problem is to design a symmetric static feedback gain $G = G^T$ such that the output feedback control law

$$u = -Gy \tag{12}$$

renders the closed-loop system stable with an H^∞ norm less than a given scalar $\gamma > 0$.

The closed-loop system of the plant (11) and the controller (12) is

$$M\ddot{q} + (D + FGF^T)\dot{q} + Kq = Fu \tag{13}$$

$$z = F^T\dot{q} \tag{14}$$

The following result provides an explicit expression for the output feedback gains that guarantee a closed-loop H^∞ norm less than a given bound γ. For simplicity, we assume that the input matrix F has full column rank.

Theorem 5: Consider the vector second-order system (11). For any $\gamma > 0$ there exists a symmetric output feedback control law (12) to provide a closed-loop H^∞ norm less than γ.

1) If F is square and invertible then G can be selected as

$$G \geq \frac{1}{\gamma} I - F^{-1} D F^{-1T} \tag{15}$$

2) If FF^T is singular then G can be selected as

$$\begin{aligned} G \geq\ & F^+[DF^{\perp T}(F^{\perp}DF^{\perp T})^{-1} \\ & F^{\perp}D - D + \frac{1}{\gamma}FF^T]F^{+T} \end{aligned} \tag{16}$$

This result follows from the BRL condition (2) and the following Generalized Finsler's Lemma [17].

Lemma 6: (Generalized Finsler's Lemma) Consider matrices M and Q such that M has full column rank and $Q = Q^T$. Then the following statements are equivalent:

(i) There exists a symmetric matrix X such that

$$MXM^T - Q > 0 \tag{17}$$

(ii) The following condition holds

$$M^{\perp}QM^{\perp T} < 0 \tag{18}$$

If the above statements hold, then all matrices X satisfying (17) are given by

$$X > M^+ \left[Q - QM^{\perp \mathbf{T}}(M^{\perp}QM^{\perp T})^{-1}M^{\perp}Q\right] M^{+T}. \tag{19}$$

Proof: For the Theorem 5. Applying the Bounded Real Lemma 2 to the closed loop system (13)-(14) using the Lyapunov matrix (10) results in

$$\begin{bmatrix} -2(D + FGF^T) & F & F \\ F^T & -\gamma I & 0 \\ F^T & 0 & -\gamma I \end{bmatrix} \leq 0.$$

Then using Schur complement formula (Lemma 3) we obtain the following condition for the control gain G

$$FGF^T + D - \frac{1}{\gamma}FF^T \geq 0$$

Applying the Generalized Finsler's Lemma 6 and simplifying the corresponding expressions provides the required control gains in Theorem 5 that guarantee the desired closed-loop H^∞ gain. ■

IV. NUMERICAL EXAMPLES

Consider the following structural system that consists of three masses interconnected with springs and dampers with the following structural parameters: $m_i = 1$, $d_i = \delta$ and $k_i = 1$ for $i = 1, 2, 3$.

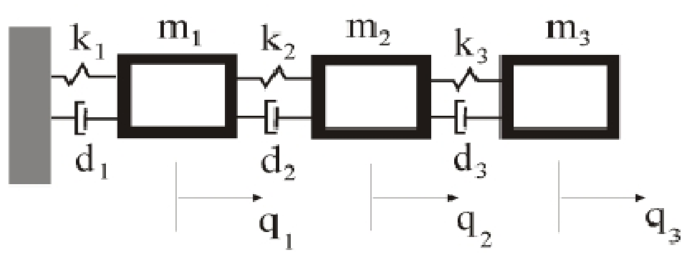

Fig. 1: Spring-mass-damper system

The corresponding structural matrices of the system are as follows:

$$M = I_{3\times3},\ K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 2\delta & -\delta & 0 \\ -\delta & 2\delta & -\delta \\ 0 & -\delta & \delta \end{bmatrix}$$

We seek to examine the values of the H^∞ norm bound (5) compared to the exact H^∞ norm value when the damping parameter δ of the system varies. The following figure (Fig. 2) shows the relative error between the H^∞ norm bound (5) and the exact H^∞ norm .

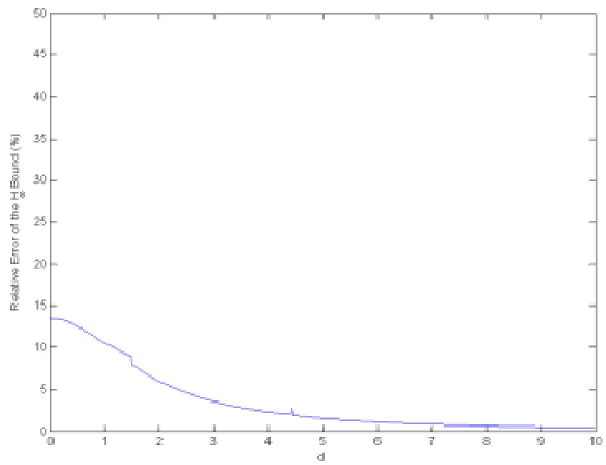

Fig. 2 Relative error of the proposed H$^\infty$ norm bound

Now consider a control design problem for the same structural system as above with $di = 1$. We assume an input matrix

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The open loop system has an H^∞ norm equal to 2.3681. We seek to find a symmetric output feedback gain matrix G such that the H^∞ norm of the closed-loop system is less than $\gamma = 0.5$. Theorem 5 provides a parametrization of such gains as follows

$$G > \bar{G} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Notice that $\bar{G}$ results in a closed-loop H^∞ norm equal to $0.4918 < 0.5$. For a desired $\gamma = 0.2$ Theorem 5 results in

$$G > \bar{G} = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$$

and $\bar{G}$ results in a closed-loop H^∞ norm equal to $0.1988 < 0.2$.

The real benefit of the proposed bounds is evident in the analysis and control of very large scale symmetric systems, such as large scale structures and power networks, where standard H^∞ analysis and design tools are computationally prohibitive. To demonstrate this point consider the finite element structural model for the assembly phase 8A-OBS of the International Space Station with collocated control and Rayleigh damping [18]. This model is in the form (1) with 360 degrees of freedom, that is, the corresponding state space model (2)-(3) has 720 states. Computation of an H^∞ control design via standard Riccati equation or LMI methods is computationally intractable. In fact it takes 3282.8 sec to calculate the exact H^∞ norm of the system which equals to 82.747. However, the proposed bound (5) provides an open-loop H^∞ norm bound of the system equal to $\bar{\gamma} = 83.102$ which takes only 0.501 sec to calculate. In addition, a symmetric static output feedback gain to reduce this bound to, say, $\bar{\gamma} = 5$ is easily computed using the results of Theorem 5. It takes only 0.671 sec to compute this control gain. The exact H^∞ norm of the closed-loop system is 4.9988 and it takes 1762.22 sec to compute it using standard methods. The result in Theorem 1 provides a closed-loop H^∞ norm bound $\bar{\gamma} = 5$ in 0.17 sec. The above computaions have been performed in a 1.33GHz Athlon PC and the corresponding results and computational times for different values of the desired closed-loop H^∞ norm bound $\bar{\gamma} = 5$, 1, 0.5 and 0.1 of the system are shown in Table 1. Fig. 3 shows the open-loop maximum singular value (sigma) plot of the above system and Fig. 4 - 7 show the corresponding closed-loop singular value plots for $\bar{\gamma} = 5$, 1, 0.5, and 0.1 using the feedback gain formula in Theorem 5. It can be easily observed from these Fig.s that the closed-loop system satisfies the desired bounds.

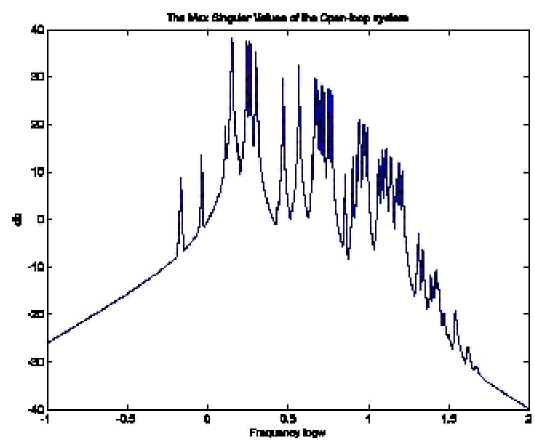

Fig. 3: Maximum singular value plot of the open-loop system

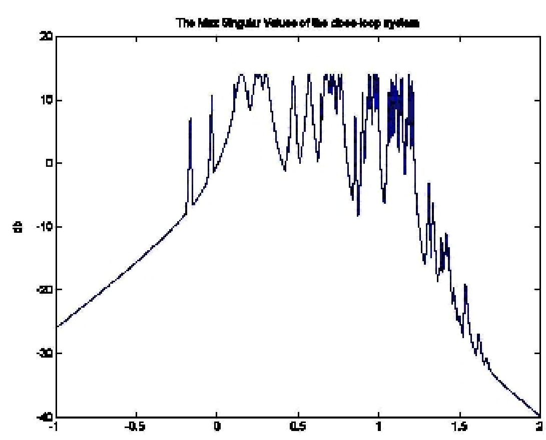

Fig. 4: Maximum singular value of the closed-loop system for $\bar{\gamma}$=5

Desired close-loop H^∞ norm bound $\bar{\gamma}$	Exact H^∞ norm of the closed-loop system	Time to calculate the feedback gain using Theorem 5(sec)	Time to calculate the exact H^∞ norm (sec)	Time to calculate the H^∞ bound (5) (sec)
5	4.9988	0.671	1762.220	0.1700
1	0.99997	0.681	1963.553	0.2099
0.5	0.499999	0.671	1961.891	0.1800
0.1	0.099999999	0.661	1926.120	0.1910

TABLE I

RESULTS FOR DIFFERENT VALUES OF THE DESIRED H^∞ NORM BOUND $\bar{\gamma}$

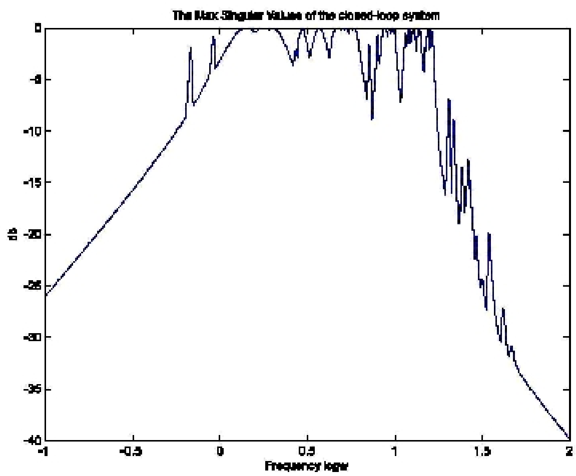

Fig. 5: Maximum singular value plot of the closed-loop system for $\bar{\gamma}$=1

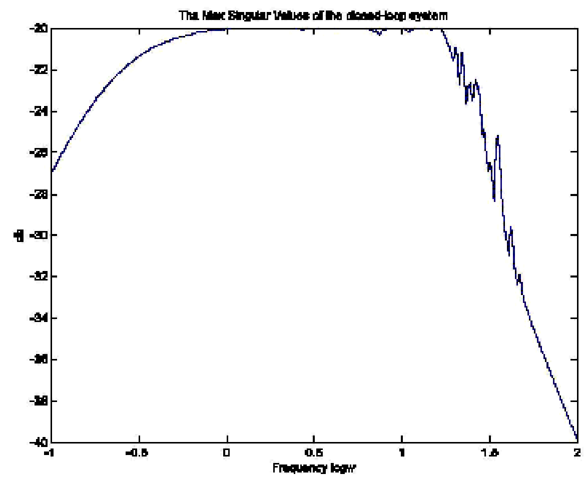

Fig. 7: Maximum singular value plot of the closed-loop system for $\bar{\gamma}$=0.1

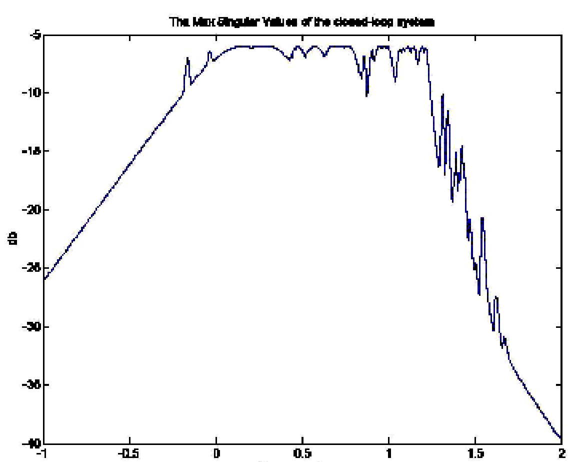

Fig. 6: Maximum singular value plot of the closed-loop system for $\bar{\gamma}$=0.5

V. CONCLUSIONS

We have obtained a simple explicit expression for an H^∞ norm bound of structural systems with collocated sensors and actuators. In addition, an explicit parametrization of symmetric output feedback gains that lead to a desired closed-loop H^∞ norm bound has been derived. The results provides easily computable guidelines for H^∞ analysis and control of collocated structural systems and are particularly useful for very large scale systems where standard H^∞ analysis and design methods are compuationally intractable. The results are applicable to any system with a symmetric transfer function.

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