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Input performance limitations of feedback control

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***Abstract*— This paper deals with input performance limitations of feedback control. The achievable input performance depends on the joint controllability and observability of unstable poles and is exactly quantified for single input single output (SISO) and multi input multi output (MIMO) systems with and without time delay. We also present a modification of μ-interaction measure to assess the feasibility of decentralized stabilization with independent designs of loops. The results are useful for various purposes including designs of the process, control structure and optimal controller synthesis.**

I. Introduction

In this paper, we study intrinsic limitations on input performance for linear systems under feedback control. The broad area of fundamental performance limitations has drawn a lot of interest in the past two decades [1] [2]. However, the focus has largely been on obtaining bounds on sensitivity and complementary sensitivity functions. Havre and Skogestad derived expressions for lower bound on achievable input performance for unstable systems [3] and extended their results to get exact expressions for rational systems with single unstable pole [4]. Chen *et. al.* [5] have studied the optimal regulation problem with input usage penalized for rational unstable systems driven by input disturbances in the $\mathcal{H}_2$ control framework. These results can be related to the present problem by appropriate choice of weights. In this paper, we focus on unstable systems driven by output disturbances and characterize the minimal input requirement for stabilization. Clearly, the achievable input performance is zero for stable systems. These results generalize the previous results of Havre and Skogestad [4] to systems with multiple unstable poles and time delay.

The μ-interaction measure (μ-IM) [6] is a useful method to assess the feasibility of decentralized stabilization through independent designs of loops. The requirement that the individual loops be designed based on the block diagonal elements, which should have the same RHP poles as the system, limits the applicability of μ-IM to open loop stable systems. We show that this difficulty can be overcome with a minor modification and present bounds on achievable input performance, when the performance of individual loops is maximized. The results presented here are useful for different purposes including: (a) process design considering achievable control performance (b) controlled and manipulated variable selection for stabilization (c) optimal controller synthesis problem formulation.

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The notation used in this paper is fairly standard. Given a matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{A}'$ is its transpose and $\mathbf{A}^*$ is its conjugate transpose. $\mathbf{A}_i$ and $\mathbf{A}_i'$ denote the i^{th} column and the i^{th} row of the matrix respectively. A matrix made of elements $a_{11}\cdots a_{1n}\cdots a_{mn}$ is represented as $[a_{ij}]$. The set of all rational stable systems is $\mathcal{RH}_\infty$. Let $\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$ such that $\mathbf{G}_1(s) \in \mathcal{RH}_\infty^\perp$ and $\mathbf{G}_2(s) \in \mathcal{RH}_\infty$. Then $\mathbf{G}_1(s)$ is the unstable projection of $\mathbf{G}(s)$ represented as $\mathcal{U}(\mathbf{G}(s))$. The symbol $\leftrightarrow$ represents the minimal state space realization of a transfer matrix, *e.g.* $\mathbf{G}(s) \leftrightarrow (\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$. For $\mathbf{G}(s) \in \mathcal{RH}_\infty$, $\sigma_{Hi}(\mathbf{G}(s))$, $\bar\sigma_H(\mathbf{G}(s))$ and $\underline{\sigma}_H(\mathbf{G}(s))$ are the i^{th}, maximum and minimum Hankel singular values [7] respectively. $\mu_{\mathbf{\Delta}}(\mathbf{G}(s))$ is the structured singular value [7], where $\mathbf{\Delta}$ represents the uncertainty structure. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of $\mathbf{G}(s) \in \mathcal{RH}_\infty$ are defined as

$$\begin{aligned}\|\mathbf{G}(s)\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left(\mathbf{G}(j\omega)^*\mathbf{G}(j\omega)\right)d\omega \\ \|\mathbf{G}(s)\|_\infty &= \sup_{\mathrm{Re}(s)>0} \bar\sigma(\mathbf{G}(s)) = \sup_{\omega\in\mathbb{R}} \bar\sigma(\mathbf{G}(j\omega))\end{aligned}$$

II. Problem formulation and simplification

In this section, we collect some general results from optimal control theory and show how they simplify when the input performance is maximized. These results form the basis for further development in this paper. Consider a finite-dimensional, linear time invariant (FDLTI) system in the standard form [8]:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_w\mathbf{w} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}_{21}\mathbf{w} \\ \mathbf{z} &= \mathbf{C}_z\mathbf{x} + \mathbf{D}_{12}\mathbf{u} \end{aligned} \quad (1)$$

where $\mathbf{z}$ is the exogenous output and $\mathbf{w}$ is the exogenous input. With $(\mathbf{A},\mathbf{B})$ stabilizable and $(\mathbf{A},\mathbf{C})$ detectable, let the Hamiltonian matrices $\mathbf{H}_2$ and $\mathbf{J}_2$ be defined as

$$\mathbf{H}_2 = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{B}^* \\ -\mathbf{C}_z^*\mathbf{C}_z & -\mathbf{A}^* \end{bmatrix} \qquad \mathbf{J}_2 = \begin{bmatrix} \mathbf{A}^* & -\mathbf{C}^*\mathbf{C} \\ -\mathbf{B}_w\mathbf{B}_w^* & -\mathbf{A} \end{bmatrix}$$

Let $\mathbf{X}_2$ and $\mathbf{Y}_2$ solve the corresponding Riccati equations or $\mathbf{X}_2 = \mathrm{Ric}(\mathbf{H}_2)$ and $\mathbf{Y}_2 = \mathrm{Ric}(\mathbf{J}_2)$. Let $\mathbf{T}_{zw}$ be the closed loop transfer matrix from $\mathbf{w}$ to $\mathbf{z}$. The unique controller minimizing $\|\mathbf{T}_{zw}(s)\|_2$ is given as [8]:

$$\mathbf{K}_{\mathrm{opt}}(s) = \left[\begin{array}{c|c} \mathbf{A}+\mathbf{B}\mathbf{F}_2+\mathbf{L}_2\mathbf{C} & -\mathbf{L}_2 \\ \hline \mathbf{F}_2 & \mathbf{0}\end{array}\right] \quad (2)$$

where $\mathbf{F}_2 = -\mathbf{B}^*\mathbf{X}_2$, $\mathbf{L}_2 = -\mathbf{Y}_2\mathbf{C}^*$ and optimal cost is [7],

$$I_2^2 = \inf_{\mathbf{K}(s)} \|\mathbf{T}_{zw}(s)\|_2^2 = \mathrm{tr}(\mathbf{B}_w^*\mathbf{X}_2\mathbf{B}_w) + \mathrm{tr}(\mathbf{F}_2\mathbf{Y}_2\mathbf{F}_2^*) \quad (3)$$

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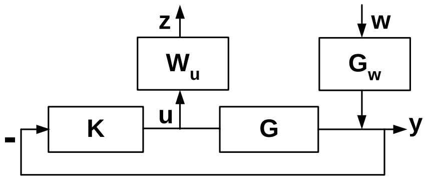

Fig. 1. Closed loop system

For the minimization of $\|\mathbf{T}_{zw}(s)\|_\infty$, similarly define

$$\begin{aligned}\mathbf{H}_\infty &= \begin{bmatrix} \mathbf{A} & \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^* - \mathbf{B}\mathbf{B}^* \\ -\mathbf{C}_z^*\mathbf{C}_z & -\mathbf{A}^* \end{bmatrix} \\ \mathbf{J}_\infty &= \begin{bmatrix} \mathbf{A}^* & \gamma^{-2}\mathbf{C}_z^*\mathbf{C}_z - \mathbf{C}^*\mathbf{C} \\ -\mathbf{B}_w\mathbf{B}_w^* & -\mathbf{A} \end{bmatrix}\end{aligned}$$

where $\gamma > 0$. If $\mathbf{X}_\infty = \mathrm{Ric}(\mathbf{H}_\infty) \geq 0$, $\mathbf{Y}_\infty = \mathrm{Ric}(\mathbf{J}_\infty) \geq 0$ and $\rho(\mathbf{X}_\infty\mathbf{Y}_\infty) < \gamma^2$, then a suboptimal controller achieving $\|\mathbf{T}_{zw}(s)\|_\infty < \gamma$ is [8]:

$$\begin{aligned}\mathbf{K}_{\mathrm{sub}}(s) &= \left[\begin{array}{c|c} \mathbf{A}_\infty & -\mathbf{Z}_\infty\mathbf{L}_\infty \\ \hline \mathbf{F}_\infty & \mathbf{0} \end{array}\right] \qquad (4) \\ \mathbf{A}_\infty &= \mathbf{A} + \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^*\mathbf{X}_\infty + \mathbf{B}\mathbf{F}_\infty + \mathbf{Z}_\infty\mathbf{L}_\infty\mathbf{C}\end{aligned}$$

where $\mathbf{F}_\infty = -\mathbf{B}^*\mathbf{X}_\infty$, $\mathbf{L}_\infty = -\mathbf{Y}_\infty\mathbf{C}^*$ and $\mathbf{Z}_\infty = (\mathbf{I} - \gamma^{-2}\rho(\mathbf{X}_\infty\mathbf{Y}_\infty))^{-1}$. The optimal cost is given as,

$$I_\infty = \inf_{\mathbf{K}(s)} \|\mathbf{T}_{zw}(s)\|_\infty = \rho^{\frac{1}{2}}(\mathbf{X}_\infty\mathbf{Y}_\infty) \qquad (5)$$

To relate these results to the problem in hand, consider the system shown in Figure 1, where all exogenous inputs have been collected in the block $\mathbf{G}_w(s)$.

***Assumption** 1:* We make the following assumptions:

1) $\mathbf{G}_w(s)$ is stable and right-invertible.
2) $\mathbf{G}(s)$ is strictly proper and has distinct poles.
3) $\mathbf{W}_u(s) = \mathbf{I}$.

In general, $\mathbf{G}_w(s)$ can share common unstable poles with $\mathbf{G}(s)$, but this case is not considered here. The right-invertibility of $\mathbf{G}_w(s)$ is necessary for the existence of a stabilizing controller. Other assumptions simplify algebraic manipulation and notation and the extension to the general case is simple. With these assumptions, the closed loop transfer matrix from disturbances to inputs is given as,

$$\mathbf{T}_{uw}(s) = \mathbf{K}(s)\left(\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\right)^{-1}\mathbf{G}_w(s) \qquad (6)$$

Let $\mathbf{G}_w(s)$ be factorized as $\mathbf{G}_w(s) = \mathbf{G}_{wm}(s)\mathbf{G}_{wa}(s)$, where $\mathbf{G}_{wm}(s)$ is minimum-phase and $\mathbf{G}_{wa}(s)$ is an all-pass factor. Define $\hat{\mathbf{G}}(s) = \mathbf{G}_{wm}^{-1}(s)\mathbf{G}(s)$ and $\hat{\mathbf{K}}(s) = \mathbf{K}(s)\mathbf{G}_{wm}(s)$, where $\hat{\mathbf{G}}(s)$ is an $n_y \times n_u$ dimensional transfer matrix. It follows from (6) that

$$\begin{aligned}\|\mathbf{T}_{uw}(s)\| &= \|\hat{\mathbf{K}}(s)(\mathbf{I} + \hat{\mathbf{G}}(s)\hat{\mathbf{K}}(s))^{-1}\mathbf{G}_{wa}(s)\| \\ &= \|\hat{\mathbf{K}}(s)(\mathbf{I} + \hat{\mathbf{G}}(s)\hat{\mathbf{K}}(s))^{-1}\|\end{aligned}$$

Then $\|\mathbf{T}_{uw}(s)\|$ is minimized by designing an optimal controller for $\hat{\mathbf{G}}(s)$. Due to the stability of $\mathbf{G}_w(s)$, the following are equivalent: (a) $\hat{\mathbf{K}}(s)$ stabilizes $\hat{\mathbf{G}}(s)$, (b) $\mathbf{K}(s)$ stabilizes $\mathbf{G}(s)$. With these observations, we can treat $\hat{\mathbf{G}}(s)$ as the system without loss of generality. These manipulations further allows us to represent the generalized plant as

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \mathbf{y} &= \hat{\mathbf{C}}\hat{\mathbf{x}} + \mathbf{w} \\ \mathbf{z} &= \mathbf{u} \qquad (7)\end{aligned}$$

where $\hat{\mathbf{G}}(s) \leftrightarrow (\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$. For the system (7), let $\hat{\mathbf{H}}_2, \hat{\mathbf{J}}_2$ and $\hat{\mathbf{H}}_\infty, \hat{\mathbf{J}}_\infty$ be the corresponding Hamiltonian matrices. By comparing (7) with (1), it follows that $\hat{\mathbf{H}}_2 = \hat{\mathbf{H}}_\infty$ and $\hat{\mathbf{J}}_2 = \hat{\mathbf{J}}_\infty$, which in turn implies that $\hat{\mathbf{X}}_2 = \hat{\mathbf{X}}_\infty = \hat{\mathbf{X}}$, $\hat{\mathbf{Y}}_2 = \hat{\mathbf{Y}}_\infty = \hat{\mathbf{Y}}$, $\hat{\mathbf{F}}_2 = \hat{\mathbf{F}}_\infty = \hat{\mathbf{F}}$ and $\hat{\mathbf{L}}_2 = \hat{\mathbf{L}}_\infty = \hat{\mathbf{L}}$.

Under Assumption 1, there exists a transformation matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}$ is diagonal and $\mathbf{T}^*\mathbf{T} = \mathbf{I}$. Rearranging and partitioning the states of the transformed system

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{T}^{-1}\hat{\mathbf{B}}\mathbf{u} = \begin{bmatrix} \mathbf{P}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{bmatrix}\tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{B}_s \\ \mathbf{B} \end{bmatrix}\mathbf{u} \\ \mathbf{y} &= \hat{\mathbf{C}}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{d} = \begin{bmatrix} \mathbf{C}_s & \mathbf{C} \end{bmatrix}\tilde{\mathbf{x}} + \mathbf{d} \qquad (8)\end{aligned}$$

where $\mathbf{P} \in \mathbb{C}^{n_p \times n_p}$ and $\mathbf{P}_s$ are diagonal matrices, which contain all the unstable and stable modes respectively. Clearly, $\mathcal{U}(\hat{\mathbf{G}}(s)) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$. Let $\tilde{\mathbf{X}} = \mathbf{T}^{-1}\hat{\mathbf{X}}\mathbf{T}$ and $\tilde{\mathbf{Y}} = \mathbf{T}^{-1}\hat{\mathbf{Y}}\mathbf{T}$ solve the corresponding Riccati equations for the transformed system (8). Then, to be non-negative definite, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ must assume the form,

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{bmatrix} \qquad \tilde{\mathbf{Y}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}$$

where $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{n_p \times n_p} > 0$. Then, it suffices to solve

$$\begin{aligned}\mathbf{X}\mathbf{P} + \mathbf{P}^*\mathbf{X} - \mathbf{X}\mathbf{B}\mathbf{B}^*\mathbf{X} &= \mathbf{0} \qquad (9) \\ \mathbf{Y}\mathbf{P}^* + \mathbf{P}\mathbf{Y} - \mathbf{Y}\mathbf{C}^*\mathbf{C}\mathbf{Y} &= \mathbf{0} \qquad (10)\end{aligned}$$

For the transformed system (8), the state feedback and the output injection matrices are given as,

$$\begin{aligned}\tilde{\mathbf{F}} &= \hat{\mathbf{F}}\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{F} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{B}^*\mathbf{X} \end{bmatrix} \qquad (11) \\ \tilde{\mathbf{L}} &= \mathbf{T}^*\hat{\mathbf{L}} = \begin{bmatrix} \mathbf{0} & \mathbf{L} \end{bmatrix}' = \begin{bmatrix} \mathbf{0} & -\mathbf{Y}\mathbf{C}^* \end{bmatrix}' \qquad (12)\end{aligned}$$

Using (3), (5) and regular algebraic manipulations,

$$\begin{aligned}I_2^2 &= \mathrm{tr}(\mathbf{F}\mathbf{Y}\mathbf{F}^*) = \mathrm{tr}(\mathbf{L}^*\mathbf{X}\mathbf{L}) \qquad (13) \\ I_\infty &= \rho^{\frac{1}{2}}(\mathbf{X}\mathbf{Y}) \qquad (14)\end{aligned}$$

The Riccati equations (9)-(10) have a special structure and are much easier to solve as compared to the general case. Consider that $\mathbf{P} = \mathrm{diag}(p_1, \cdots, p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Let the Hermitian matrix $\mathbf{M} \in \mathbb{C}^{n_p \times n_p}$ be defined as

$$[m_{ij}] = 1/(p_i + \bar{p}_j) \qquad (15)$$

***Lemma** 1:* Let $\mathbf{X}, \mathbf{Y} > 0$ solve the Riccati equations (9)-(10) and $\mathbf{M}$ be given by (15). Then

$$\begin{aligned}\mathbf{X}^{-1} &= \sum_{i=1}^{n_u} \mathrm{diag}(\mathbf{B}_i)\mathbf{M}\mathrm{diag}(\mathbf{B}_i)^* \qquad (16) \\ \mathbf{Y}^{-1} &= \sum_{j=1}^{n_y} \mathrm{diag}(\mathbf{C}_j^{'})^*\mathbf{M}\mathrm{diag}(\mathbf{C}_j^{'}) \qquad (17)\end{aligned}$$

Proof: Pre- and post multiplying (9) by $\mathbf{X}^{-1}$ gives

$$\mathbf{P}\mathbf{X}^{-1} + \mathbf{X}^{-1}\mathbf{P}^* = \mathbf{B}\mathbf{B}^* \tag{18}$$

Then [9], $\mathbf{X}^{-1} = \mathbf{M} \circ (\mathbf{B}\mathbf{B}^*)$, where $\circ$ is the Hadamard or element-wise product. Noting that $\mathbf{B}\mathbf{B}^* = \sum_{i=1}^{n_u} \mathbf{B}_i\mathbf{B}_i^*$, $\mathbf{X}^{-1} = \sum_{i=1}^{n_u} \mathbf{M} \circ (\mathbf{B}_i\mathbf{B}_i^*)$ and (16) follows. The proof of (17) follows from duality. ■

III. SISO Systems

In this section, we quantify achievable input performance of SISO systems with and without time delay.

***Lemma** 2:* For $\mathbf{M}$ defined by (15), $\mathbf{M}^{-1}$ is given as

$$[\mathbf{M}^{-1}]_{ij} = \frac{(\bar{p}_i + p_i)(p_j + \bar{p}_j)}{\bar{p}_i + p_j} \prod_{\substack{k=1\\k\neq i}}^{n_p} \frac{(\bar{p}_i + p_k)}{(\bar{p}_i - \bar{p}_k)} \prod_{\substack{k=1\\k\neq j}}^{n_p} \frac{(p_j + \bar{p}_k)}{(p_j - p_k)}$$

Lemma 2 is easily verified by evaluating $\mathbf{M}\mathbf{M}^{-1}$ or $\mathbf{M}^{-1}\mathbf{M}$. Note for $n_u = n_y = 1$, $\mathbf{B} = [b_i]$, $\mathbf{C} = [c_j]$.

***Proposition** 1:* For a rational SISO system $\mathbf{G}(s)$, let $\mathcal{U}(\mathbf{G}(s)) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$. Then

$$I_2^2 = \left[\frac{|\mathbf{q}_i|^2}{b_i c_i}\right] \mathbf{M} \left[\frac{|\mathbf{q}_i|^2}{b_i^* c_i^*}\right]' \qquad i = 1 \cdots n_p \tag{19}$$

$$I_\infty^2 = |\underline{\lambda}^{-1}(\text{diag}(\mathbf{B})\text{diag}(\mathbf{C})\mathbf{M}\text{diag}(\mathbf{B}^*)\text{diag}(\mathbf{C}^*)\mathbf{M})| \tag{20}$$

where $\mathbf{M}$ is defined by (15) and $\mathbf{q} = \mathbf{1}_{n_p}'\mathbf{M}^{-1}$.

Proof: (1) For (19), based on (11), (13) and Lemma 1, $I_2^2 = \mathbf{F}\mathbf{Y}\mathbf{F}^* = \mathbf{B}^*\mathbf{X}\mathbf{Y}\mathbf{X}\mathbf{B}$ and

$$I_2^2 = \mathbf{1}_{n_p}'\mathbf{M}^{-1}(\text{diag}(\mathbf{B})\text{diag}(\mathbf{C}))^{-1}\mathbf{M}^{-1} \\ (\text{diag}(\mathbf{B}^*)\text{diag}(\mathbf{C}^*))^{-1}\mathbf{M}^{-1}\mathbf{1}_{n_p}$$

Based on Lemma 2,

$$q_i = (p_i + \bar{p}_i) \prod_{\substack{k=1\\k\neq i}}^{n_p} \frac{(p_i + \bar{p}_k)}{(p_i - p_k)} \quad i = 1 \cdots n_p$$

Now, (19) can be obtained by noting that $\mathbf{M}^{-1} = \text{diag}(\mathbf{q}^*)\mathbf{M}'\text{diag}(\mathbf{q})$ and $\mathbf{q}_i\mathbf{q}_i^* = |\mathbf{q}_i|^2$.

(2) For (20), the result follows by using (14) and Lemma 1 and noting that for any matrix $\mathbf{A}$, $\rho(\mathbf{A}^{-1}) = |\underline{\lambda}^{-1}(\mathbf{A})|$. ■

Based on (20), for a system with real unstable poles only, $I_\infty = |\underline{\lambda}^{-1}(\text{diag}(\mathbf{B})\text{diag}(\mathbf{C})\mathbf{M})|$. Though in (19)-(20), I_2 and I_∞ depend only on the unstable poles, the stable part of the system also affects the input usage. This happens as $\mathcal{U}(\hat{\mathbf{G}}(s))$ depends on the unstable as well as stable poles of the system.

***Remark** 1:* The expression for $\mathbf{q}$ appears to suggest that in general, $I_2 \to \infty$ as $p_i \to p_j$ for some $i, j \leq n_p$, which is clearly not true. Since $b_i c_i = [\hat{\mathbf{G}}(s)(s-p_i)]_{s=p_i}$, $b_i c_i \to \infty$, as $p_i \to p_j$, which negates the effect of $\mathbf{q}$. But when the system has an RHP zero close to RHP poles, $b_i c_i$ fails to increase monotonically and stabilization can be difficult, *e.g.* if $\hat{\mathbf{G}}(s) = \frac{(s-p)}{(s-p+\epsilon)(s-p-\epsilon)}$, $I_2, I_\infty \to \infty$, as $\epsilon \to 0$.

To extend Proposition 1 to systems with a finite time delay, let $\hat{\mathbf{G}}(s)$ be expressed as,

$$\hat{\mathbf{G}}(s) = \tilde{\mathbf{G}}(s)e^{-\theta s} \tag{21}$$

where $\tilde{\mathbf{G}}$ is the delay-free part of the system. If $\mathbf{G}_w(s)$ also contains delay, it can be factored as an all-pass factor and thus $\hat{\mathbf{G}}(s)$ remains causal.

***Lemma** 3:* Consider $\mathbf{H}(s) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$. Let $\mathbf{H_1(s)} \in \mathcal{RH}_\infty$ with no zeros at p_i. Then

$$\mathcal{U}(\mathbf{H}_1(s)\mathbf{H}(s)) = \sum_{i=1}^{n_p} \frac{1}{s - p_i} \mathbf{H}_1(p_i)\mathbf{C}_i\mathbf{B}_i' \tag{22}$$

Proof: Using dyadic expansion, $\mathbf{H}(s) = \sum_{i=1}^{n_p} \frac{1}{s-p_i}\mathbf{C}_i\mathbf{B}_i'$. Let $\mathcal{U}(\mathbf{H}_1(s)\mathbf{H}(s)) \leftrightarrow (\tilde{\mathbf{P}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$. Since $\mathbf{H_1(s)}$ does not cancel RHP poles of $\mathbf{H}(s)$, $\tilde{\mathbf{P}} = \mathbf{P}$. Now, $\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i' = [\mathbf{H}_1(s)\mathbf{H}(s)(s - p_i)]_{s=p_i}$ and (22) follows. ■

Note that in Lemma 3, there is no loss of generality in assuming that all modes of $\mathbf{H}(s)$ are unstable, since $\mathcal{U}(\mathbf{H}_1(s)\mathbf{H}(s)) = \mathcal{U}(\mathbf{H}_1(s)\mathcal{U}(\mathbf{H}(s)))$.

***Proposition** 2:* For the SISO system expressed by (21), let $\mathcal{U}(\tilde{\mathbf{G}}(s)) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$ and $\mathbf{\Gamma} = \text{diag}(e^{\theta p_1} \cdots e^{\theta p_{n_p}})$. Then

$$I_2^2 = \left[\frac{\mathbf{q}_i\mathbf{q}_i^*}{\tilde{b}_i\tilde{c}_i}\right] \mathbf{\Gamma}\mathbf{M}\mathbf{\Gamma}^* \left[\frac{\mathbf{q}_i\mathbf{q}_i^*}{\tilde{b}_i^*\tilde{c}_i^*}\right]' \quad i = 1 \cdots n_p \tag{23}$$

$$I_\infty^2 = |\underline{\lambda}^{-1}(\mathbf{\Gamma}^{-1}\text{diag}(\tilde{\mathbf{B}})\text{diag}(\tilde{\mathbf{C}})\mathbf{M} \\ \mathbf{\Gamma}^{-*}\text{diag}(\tilde{\mathbf{B}}^*)\text{diag}(\tilde{\mathbf{C}}^*)\mathbf{M})| \tag{24}$$

where $\mathbf{M}$ is defined by (15) and $\mathbf{q} = \mathbf{1}_{n_p}'\mathbf{M}^{-1}$.

Proof: Let $e^{-\theta s} = f(\theta s, n) + \mathcal{O}(n)$, where $f(\theta s, n)$ is an n^{th} order rational approximation of $e^{-\theta s}$ (*e.g.* Páde approximation). For any n, if an RHP zero of $f(\theta s, n)$ cancels an RHP pole of $\tilde{\mathbf{G}}(s)$, the system is not stabilizable due to presence of hidden unstable mode. However for an FDLTI system, this situation does not occur for all $n \geq N$ for sufficiently large N, since the magnitude of RHP zeros of $f(\theta s, n)$ approaches infinity as $n \to \infty$.

(1) For (23), using (22), $b_i c_i \approx \tilde{b}_i\tilde{c}_i f(\theta p_i, n)$, $n \geq N$ and

$$I_2^2(n) \approx \left[\frac{\mathbf{q}_i\mathbf{q}_i^*}{\tilde{b}_i\tilde{c}_i f(\theta p_i, n)}\right] \mathbf{M} \left[\frac{\mathbf{q}_i\mathbf{q}_i^*}{\tilde{b}_i^*\tilde{c}_i^* f(\theta p_i, n)}\right]' \tag{25}$$

Now, $\lim_{n\to\infty} f(\theta p_i, n) = e^{-\theta p_i}$, as the exponential function is uniformly convergent in the entire complex plane. Noting that $f^{-1}(\theta p_i, n)$ appears as a bilinear term in (25), which is itself an exponential function, we conclude that $\lim_{n\to\infty} I_2^2$ exists and is given by (23).

(2) For (24), using similar arguments as before

$$I_\infty^2(n) \approx |\underline{\lambda}^{-1}(\text{diag}(f(\theta p_i, n))\text{diag}(\tilde{\mathbf{B}})\text{diag}(\tilde{\mathbf{C}})\mathbf{M} \\ \text{diag}(f(\theta p_i, n))^*\text{diag}(\tilde{\mathbf{B}}^*)\text{diag}(\tilde{\mathbf{C}}^*)\mathbf{M})|$$

The eigen values are roots of a polynomial equation, whose coefficients are functions of $f(\theta p_i, n)$. As $n \to$

∞, these coefficients and thus the roots converge. Hence, $\lim_{n\to\infty} I_\infty^2(n)$ exists and is given by (24). ■

By differentiating (23)-(24) with respect to θ, $dI_2/d\theta > 0$ and $dI_\infty/d\theta > 0$ for all θ. This shows that the input usage cannot be decreased by introducing additional lag in the system, which also follows from physical considerations.

***Corollary** 1:* Under same conditions as Proposition 2, let $\mathbf{G}_p(s) \leftrightarrow (\mathbf{P}, \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ or $(\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}\mathbf{\Gamma}^{-1})$. Then $I_2(\hat{\mathbf{G}}(s)) = I_2(\mathbf{G}_p(s))$ and $I_\infty(\hat{\mathbf{G}}(s)) = I_\infty(\mathbf{G}_p(s))$.

***Remark** 2:* Time-delay enters (23)-(24) assuming the form $e^{\theta p_i}$ and thus do not pose serious limitations on input performance for systems with slow instabilities and *vice versa*. It follows from Corollary 1 that time delay essentially reduces the controllability (or observability) of poles and the faster the instability, the less controllable (or observable) the pole is, as compared to the delay-free system.

IV. MIMO Systems

In this section, we extend the results of the last section to MIMO systems.

***Lemma** 4:* Let $\hat{\mathbf{G}}_1 = \mathcal{U}(\hat{\mathbf{G}})$ and $\mathbf{X}, \mathbf{Y} > 0$ solve the Riccati equations (9)-(10). Then,

$$\sigma_{Hi}^2(\hat{\mathbf{G}}_1(s)^*) = \lambda_i(\mathbf{X}^{-1}\mathbf{Y}^{-1}) \quad i = 1, \cdots n_p \tag{26}$$

Proof: Pre and post-multiplying (18) by $\mathbf{T}_1$ and $\mathbf{T}_1^*$ respectively, where $\mathbf{T}_1$ is a state transformation matrix,

$$\begin{aligned} \mathbf{T}_1\mathbf{P}\mathbf{X}^{-1}\mathbf{T}_1^* + \mathbf{T}_1\mathbf{X}^{-1}\mathbf{P}^*\mathbf{T}_1^* &= \mathbf{T}_1\mathbf{B}\mathbf{B}^*\mathbf{T}_1^* \\ \bar{\mathbf{P}}\bar{\mathbf{X}}^{-1} + \bar{\mathbf{X}}^{-1}\bar{\mathbf{P}}^* &= \bar{\mathbf{B}}\bar{\mathbf{B}}^* \end{aligned} \tag{27}$$

where $\bar{\mathbf{P}} = \mathbf{T}_1\mathbf{P}\mathbf{T}_1^{-1}$, $\bar{\mathbf{B}} = \mathbf{T}_1\mathbf{B}$ and $\bar{\mathbf{X}} = (\mathbf{T}_1^*)^{-1}\mathbf{X}\mathbf{T}_1^{-1}$. Similarly, by setting $\bar{\mathbf{C}} = \mathbf{C}\mathbf{T}_1^{-1}$ and $\bar{\mathbf{Y}} = \mathbf{T}_1\mathbf{Y}\mathbf{T}_1^*$,

$$\bar{\mathbf{P}}^*\bar{\mathbf{Y}}^{-1} + \bar{\mathbf{Y}}^{-1}\bar{\mathbf{P}} = \bar{\mathbf{C}}^*\bar{\mathbf{C}} \tag{28}$$

Now $\bar{\mathbf{Y}}^{-1}$ and $\bar{\mathbf{X}}^{-1}$ are the controllability and observability gramians of $\hat{\mathbf{G}}_1^*(s) \leftrightarrow (-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ and (27)-(28) are the corresponding Lyapunov equations. If $\mathbf{T}_1$ is chosen such that $(-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ is a balanced realization, then $\bar{\mathbf{X}}^{-1} = \bar{\mathbf{Y}}^{-1} = \mathrm{diag}(\sigma_{Hi}(\hat{\mathbf{G}}_1^*(s)))$ [7] and

$$\sigma_{Hi}^2(\hat{\mathbf{G}}_1^*(s)) = \lambda_i(\bar{\mathbf{X}}^{-1}\bar{\mathbf{Y}}^{-1}) = \lambda_i(\mathbf{X}^{-1}\mathbf{Y}^{-1}) \quad i = 1 \cdots n_p$$

■

***Proposition** 3:* For the rational MIMO system $\hat{\mathbf{G}}$ having n_p unstable poles, let $\hat{\mathbf{G}}_1 = \mathcal{U}(\hat{\mathbf{G}})$ and $(-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ be the balanced realization of $\hat{\mathbf{G}}_1^*$. Then

$$I_2^2 = \sum_{i=1}^{n_p} \frac{2|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|}{\sigma_{Hi}^2(\hat{\mathbf{G}}_1^*(s))} \tag{29}$$

$$I_\infty = \underline{\sigma}_H^{-1}(\hat{\mathbf{G}}_1^*(s)) \tag{30}$$

Proof: (1) For (29), based on (13),

$$I_2^2 = \mathrm{tr}(\mathbf{B}^*\mathbf{X}\mathbf{Y}\mathbf{X}\mathbf{B}) = \mathrm{tr}(\bar{\mathbf{B}}^*\bar{\mathbf{X}}\bar{\mathbf{Y}}\bar{\mathbf{X}}\bar{\mathbf{B}}) = \mathrm{tr}(\bar{\mathbf{B}}\bar{\mathbf{B}}^*\bar{\mathbf{X}}\bar{\mathbf{Y}}\bar{\mathbf{X}})$$

Define $\Sigma_H = \mathrm{diag}(\sigma_{Hi}(\hat{\mathbf{G}}_1^*(s)))$. Since $(-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ is the balanced realization of $\hat{\mathbf{G}}_1^*(s)$, using (27)

$$\begin{aligned} I_2^2 &= \mathrm{tr}\left[(-\bar{\mathbf{P}}\Sigma_H - \Sigma_H\bar{\mathbf{P}}^*)\Sigma_H^{-3}\right] \\ &= \mathrm{tr}(-\bar{\mathbf{P}}\Sigma_H^{-2}) + \mathrm{tr}(-\Sigma_H^{-2}\bar{\mathbf{P}}^*) = \sum_{i=1}^{n_p} \frac{|\bar{\mathbf{P}}_{ii} + \bar{\mathbf{P}}_{ii}^*|}{\sigma_{Hi}^2(\hat{\mathbf{G}}_1^*(s))} \end{aligned}$$

where $|\bar{\mathbf{P}}_{ii} + \bar{\mathbf{P}}_{ii}^*| = 2|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|$.

(2) For (30), based on (14) and (26)

$$I_\infty = \underline{\lambda}^{-\frac{1}{2}}(\mathbf{X}^{-1}\mathbf{Y}^{-1}) = \underline{\sigma}_H^{-1}(\hat{\mathbf{G}}_1^*(s))$$

■

The expressions (29)-(30) show that I_2 and I_∞ primarily depend on $\sigma_{Hi}(\hat{\mathbf{G}}_1^*(s))$, which is a measure of joint controllability and observability of the unstable poles. Using (16) and (17), $\sigma_{Hi}(\hat{\mathbf{G}}_1^*(s))$ is also expressed as,

$$\sigma_{Hi}(\hat{\mathbf{G}}_1^*(s)) = \lambda_i^{\frac{1}{2}}\left[((\mathbf{B}\mathbf{B}^*) \circ \mathbf{M})((\mathbf{C}^*\mathbf{C}) \circ \mathbf{M})\right] \tag{31}$$

Glover [10] studied the robust stability of systems in the presence of additive unstructured uncertainty. With this description of uncertainty, maximizing robust stability is equivalent to minimizing the $\mathcal{H}_\infty$ norm of transfer matrix from disturbances to inputs. Thus, the results of Glover [10] are also applicable to the present case. The expression for I_∞ as derived here is as an alternative proof of the same.

***Remark** 3:* In general, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a transfer matrix can be arbitrarily apart. Proposition 3 shows when input norm is minimized, I_2/I_∞ is always bounded as

$$2\frac{\underline{\sigma}_H^2(\hat{\mathbf{G}}_1^*(s))}{\bar{\sigma}_H^2(\hat{\mathbf{G}}_1^*(s))}\sum_{i=1}^{n_p}|\mathrm{Re}(\bar{\mathbf{P}}_{ii})| \le \frac{I_2^2}{I_\infty^2} \le 2\sum_{i=1}^{n_p}|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|$$

The closeness of I_2 and I_∞ partially follows from the fact that the related Riccati equations (9)-(10) for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases are the same. To extend proposition 2 to MIMO systems, we consider systems which can be expressed as

$$\hat{\mathbf{G}}(s) = \tilde{\mathbf{G}}(s) \circ \mathbf{\Theta}(s); \;\; \mathbf{\Theta}(s) = \left[e^{-\theta_{ij}s}\right] \tag{32}$$

where $\tilde{\mathbf{G}}$ is the delay-free part of the system. It is pointed that (32) does not represent the general case and in practice is satisfied only when $\mathbf{G}_w(s)$ is diagonal. The discussion is limited to the cases where $n_y \ge n_u$ and similar expressions for $n_y < n_u$ can be obtained with minor modifications.

***Lemma** 5:* Consider $\mathbf{H}(s) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{P} = \mathrm{diag}(p_1 \cdots p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Let $\mathbf{H_1(s)} \in \mathcal{RH}_\infty$ with no zeros at p_i. Then

$$\mathcal{U}(\mathbf{H}_1(s) \circ \mathbf{H}(s)) = \sum_{i=1}^{n_p} \frac{1}{s - p_i}\mathbf{H}_1(p_i) \circ (\mathbf{C}_i\mathbf{B}_i^{'}) \tag{33}$$

The proof of Lemma 5 is similar to the proof of Lemma 3 and is omitted.

***Assumption** 2:* Let $\mathcal{U}(\tilde{\mathbf{G}}(s)) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ such that $\mathbf{P} = \mathrm{diag}(p_1 \cdots p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Then the matrix $(\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i^{'}) \circ \mathbf{\Theta}(p_i)$ has full column rank for all $i = 1 \cdots n_p$.

***Proposition** 4:* For the MIMO system expressed by (32), which satisfies Assumption 2, let $\mathcal{U}(\tilde{\mathbf{G}}(s)) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ such that $\mathbf{P} = \mathrm{diag}(p_1 \cdots p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Let $\mathbf{G}_p(s) \leftrightarrow (\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p)$, where

$$\begin{aligned} \mathbf{A}_p &= \mathrm{diag}(p_1\mathbf{I}_{n_u} \cdots p_{n_p}\mathbf{I}_{n_u}); \quad \mathbf{B}_p = [\mathbf{I}_{n_u} \cdots \mathbf{I}_{n_u}]^{'} \\ \mathbf{C}_p &= \left[(\tilde{\mathbf{C}}_1\tilde{\mathbf{B}}_1^{'}) \circ \mathbf{\Theta}(p_1) \cdots (\tilde{\mathbf{C}}_{n_p}\tilde{\mathbf{B}}_{n_p}^{'}) \circ \mathbf{\Theta}(p_{n_p})\right] \end{aligned}$$

Then $I_2(\hat{\mathbf{G}}(s)) = I_2(\mathbf{G}_p(s))$, $I_\infty(\hat{\mathbf{G}}(s)) = I_\infty(\mathbf{G}_p(s))$.

Proof: Let $\Theta(s)$ be approximated by an n^{th} order rational function as before. As $n \to \infty$, using Lemma 5 and the same arguments as used in the proof of Proposition 2,

$$\mathcal{U}(\hat{\mathbf{G}}(s)) = \sum_{i=1}^{n_p} \frac{1}{s-p_i}(\tilde{\mathbf{C}}_i \tilde{\mathbf{B}}_i^{'}) \circ \mathbf{\Theta}(p_i) \quad (34)$$

Due to assumption 2, $\frac{1}{s-p_i}\mathbf{\Theta}(p_i) \circ (\mathbf{C}_i\mathbf{B}_i^{'}) \leftrightarrow (p_i\mathbf{I}_{n_u}, \mathbf{I}_{n_u}, \mathbf{\Theta}(p_i) \circ (\mathbf{C}_i\mathbf{B}_i^{'}))$. Then the result follows by considering the aggregation of these subsystems. ■

For systems not satisfying Assumption 2, the triplet $(\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p)$ is not necessarily a minimal realization. This assumption can be relaxed for generalization purposes, but this makes the expressions difficult and complex. A practical case, where Assumption 2 is always violated, occurs when the delays are associated with the sensors or actuators of the system. Systems with delay associated with sensors are handled next and the expressions for systems with delay associated with actuators can be obtained analogously.

***Corollary** 2:* Let $\hat{\mathbf{G}}(s) = \text{diag}(e^{-\theta_i s})\tilde{\mathbf{G}}(s)$ and $\mathbf{G}_p(s) \leftrightarrow (\text{diag}(p_i\mathbf{I}_{n_u}), \tilde{\mathbf{B}}, \mathbf{C}_p)$, where $\mathbf{C}_p = [\text{diag}(e^{-\theta_i p_1})\tilde{\mathbf{C}}_1 \cdots \text{diag}(e^{-\theta_i p_{n_p}})\tilde{\mathbf{C}}_{n_p}]$. Then, $I_2(\hat{\mathbf{G}}(s)) = I_2(\mathbf{G}_p(s))$ and $I_\infty(\hat{\mathbf{G}}(s)) = I_\infty(\mathbf{G}_p(s))$.

The proof of Corollary 2 follows by considering (34) and noting that $(\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i^{'}) \circ \mathbf{\Theta}(p_i) = \text{diag}(e^{-\theta_i p_i})\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i^{'}$. It is interesting to note that when $\mathbf{\Theta}(s)$ is unstructured (delays cannot be separated at inputs or outputs), stabilization of the irrational system with n_p unstable poles is equivalent to stabilizing a rational system with $n_p \times n_u$ unstable poles.

V. Decentralized Stabilization

In this section, we briefly review the available results on μ-Interaction measure (μ-IM) [6], point its limitation and suggest a modification of μ-IM to overcome the same.

Let $\hat{\mathbf{G}}(s)$ be partitioned as $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{bd}(s) + \hat{\mathbf{G}}_I(s)$ such that $\hat{\mathbf{G}}_{bd}(s)$ contains the block-diagonal elements of $\hat{\mathbf{G}}(s)$ and has same RHP poles as $\hat{\mathbf{G}}(s)$. Define $\hat{\mathbf{E}}(s) = (\hat{\mathbf{G}}(s) - \hat{\mathbf{G}}_{bd}(s))\hat{\mathbf{G}}_{bd}(s)^{-1}$ and $\hat{\mathbf{H}}_{bd}(s) = \hat{\mathbf{G}}_{bd}(s)\hat{\mathbf{K}}_{bd}(s)(\mathbf{I} + \hat{\mathbf{G}}_{bd}(s)\hat{\mathbf{K}}_{bd}(s))^{-1}$. Then the block diagonal controller $\hat{\mathbf{K}}_{bd}(s)$ stabilizing $\hat{\mathbf{G}}_{bd}(s)$ also stabilizes $\hat{\mathbf{G}}(s)$, if $\bar{\sigma}(\hat{\mathbf{H}}_{bd}(j\omega)) < \mu_{\mathbf{\Delta}}^{-1}(\hat{\mathbf{E}}(s))$ for all ω, where $\mathbf{\Delta}$ has same structure as $\mathbf{G}_{bd}(s)$ [6]. In practice, $\hat{\mathbf{G}}(s)$ and $\hat{\mathbf{G}}_{bd}(s)$ as defined above has same number of RHP poles only for open loop stable systems limiting the applicability of μ-IM.

This limitation is overcome by relaxing the requirement that $\hat{\mathbf{G}}_{bd}(s)$ contains the block diagonal elements of $\hat{\mathbf{G}}(s)$. To relate these results to the input performance, the uncertainty in $\hat{\mathbf{G}}_{bd}(s)$ is modelled as additive uncertainty as opposed to the multiplicative uncertainty form used by Grosdidier and Morari [6]. However this limits the utility of the results to the case when individual blocks of $\hat{\mathbf{G}}_{bd}(s)$ has equal number of inputs and outputs. Though $\hat{\mathbf{K}}_{bd}(s)$ designed based on $\hat{\mathbf{G}}_{bd}(s)$ is always block-diagonal, $\mathbf{K}_{bd}(s)$ is guaranteed to be block-diagonal only if $\mathbf{G}_w(s)$ is block diagonal. Note that design of $\hat{\mathbf{K}}_{bd}(s)$ based on $\hat{\mathbf{G}}_{bd}(s)$ only is equivalent to designing individual loops independently.

***Proposition** 5:* Let $\hat{\mathbf{G}}(s)$ be partitioned as $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{bd}(s) + \hat{\mathbf{G}}_I(s)$ such that $\hat{\mathbf{G}}_{bd}(s)$ is block diagonal with every block being square and has same RHP poles as $\hat{\mathbf{G}}(s)$. Define $\hat{\mathbf{S}}_{bd}(s) = (\mathbf{I} + \hat{\mathbf{G}}_{bd}(s)\hat{\mathbf{K}}_{bd}(s))^{-1}$. Then $\hat{\mathbf{K}}_{bd}(s)$ stabilizing $\hat{\mathbf{G}}_{bd}(s)$ also stabilizes $\hat{\mathbf{G}}(s)$ if

$$\|\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}_{bd}(s)\|_\infty \le \mu_{\mathbf{\Delta}}^{-1}(\hat{\mathbf{G}}_I(s)) \quad (35)$$

where $\mathbf{\Delta}$ has same structure as $\hat{\mathbf{G}}_{bd}(s)$.

Proof: Since $\hat{\mathbf{G}}_{bd}(s)$ and $\hat{\mathbf{G}}(s)$ has same RHP poles, $\hat{\mathbf{K}}_{bd}(s)$ stabilizing $\hat{\mathbf{G}}_{bd}(s)$ also stabilizes $\hat{\mathbf{G}}(s)$ if $\|\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}_{bd}(s)\|_\infty \le \|\hat{\mathbf{G}}_I(s)\|_\infty^{-1}$ [10]. Since stability is scaling invariant, closed loop system is stable if

$$\|\mathbf{D}\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}_{bd}(s)\mathbf{D}^{-1}\|_\infty \le \|\mathbf{D}\hat{\mathbf{G}}_I(s)\mathbf{D}^{-1}\|_\infty^{-1} \quad (36)$$

where $\mathbf{D}$ is a scaling matrix. Let $\mathbf{D}$ be restricted to the set $\mathcal{D} = \{\text{diag}(d_i \cdot \mathbf{I}_{m_i}), d_i \in \mathcal{R}\}$, where the dimensions of individual blocks of $\hat{\mathbf{G}}_{bd}(s)$ is $m_i \times m_i$. As $\mathbf{D}\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}_{bd}(s)\mathbf{D}^{-1} = \hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}_{bd}(s)$, the conservativeness of (36) can be reduced by choosing $\mathbf{D}$ such that the right hand side of (36) is maximized. Then the sufficient condition for the stability of closed loop system is

$$\|\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}_{bd}(s)\|_\infty \le \sup_{D\in\mathcal{D}} \|\mathbf{D}\hat{\mathbf{G}}_I(s)\mathbf{D}^{-1}\|_\infty^{-1} \le \mu_{\mathbf{\Delta}}^{-1}(\hat{\mathbf{G}}_I(s))$$

■

***Corollary** 3:* Consider that all other conditions of Proposition 5 holds, but $\hat{\mathbf{K}}_{bd}(s)$ is designed to maximize the performance of individual loops. Then the closed loop system is stable if $\underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}}_{bd}(s))^*) \ge \mu_{\mathbf{\Delta}}(\hat{\mathbf{G}}_I(s))$, where $\mathbf{\Delta}$ has same structure as $\hat{\mathbf{G}}_{bd}(s)$.

Proposition 5 provide a sufficient condition to assess if $\hat{\mathbf{K}}_{bd}(s)$, designed based on $\hat{\mathbf{G}}_{bd}(s)$, can stabilize the closed loop system. However it provides no information regarding the closed loop performance. We present bounds on input performance, when $\hat{\mathbf{K}}_{bd}(s)$ is designed to maximize the performance of individual loops in the next proposition.

***Proposition** 6:* Let all other conditions of Proposition 5 hold, but $\hat{\mathbf{K}}_{bd}(s)$ is designed to maximize the performance of individual loops. If closed loop system is stable,

$$\begin{aligned}\frac{1}{\underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}}_{bd}(s))^*) + \|\hat{\mathbf{G}}_I(s)\|_\infty} &\le \|\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}(s)\|_\infty \\ &\le \frac{1}{\max(0, \underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}}_{bd}(s))^*) - \|\hat{\mathbf{G}}_I(s)\|_\infty)} \end{aligned} \quad (37)$$

Proof: Using $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{bd}(s) + \hat{\mathbf{G}}_I(s)$, we obtain $(\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}(s))^{-1} = (\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}_{bd}(s))^{-1} + \hat{\mathbf{G}}_I(s)$. Then, using singular value inequalities [9],

$$\begin{aligned}&\underline{\sigma}((\hat{\mathbf{K}}_{bd}(j\omega)\hat{\mathbf{S}}_{bd}(j\omega))^{-1}) - \bar{\sigma}(\hat{\mathbf{G}}_I(j\omega)) \\ &\le \underline{\sigma}((\hat{\mathbf{K}}_{bd}(j\omega)\hat{\mathbf{S}}(j\omega))^{-1}) \\ &\le \underline{\sigma}((\hat{\mathbf{K}}_{bd}(j\omega)\hat{\mathbf{S}}_{bd}(j\omega))^{-1}) + \bar{\sigma}(\hat{\mathbf{G}}_I(j\omega)) \quad \forall\omega\end{aligned}$$

Now (37) is obtained by maximizing over all ω and noting that $\underline{\sigma}((\hat{\mathbf{K}}_{bd}(j\omega)\hat{\mathbf{S}}_{bd}(j\omega))^{-1}) = \underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}}_{bd}(s))^*)$. ■

Representing the individual blocks of $\hat{\mathbf{G}}_{bd}(s)$ as $[\hat{\mathbf{G}}_{bd}(s)]_{ii}$, we note that $\underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}}_{bd}(s))^*) = \min_i \underline{\sigma}_H(\mathcal{U}([\hat{\mathbf{G}}_{bd}(s)]_{ii})^*)$, $i = 1 \cdots n_p$. Then (35) is most easily satisfied by assigning the RHP poles of $\hat{\mathbf{G}}(s)$ to the blocks of $\hat{\mathbf{G}}_{bd}(s)$ such that the joint controllability and observability of each pole is maximum. This also minimizes the upper bound on $\|\hat{\mathbf{K}}_{bd}(s)\hat{\mathbf{S}}(s)\|_\infty$.

VI. Discussion and Conclusions

In this paper, we used a state space framework to obtain analytic expressions for achievable input performance for SISO and MIMO systems with and without time delay. Regarding the factors affecting achievable input performance, the following general conclusions are drawn:

1) The input performance primarily depends on the joint controllability and observability of unstable poles.
2) Time delay poses no serious limitation on the achievable input performance for a system with slow instabilities and *vice versa*.
3) The input performance of a MIMO system, where the delays cannot be separated at inputs or outputs, can be much worse as compared to a system with delays that can be factored at inputs or outputs.

The results presented here are useful for various purposes including designs of the process, control structure and controller synthesis. For optimal controller synthesis, these results can be used to assess if the input performance is overly emphasized in the mixed sensitivity objective function used to trade-off the different objectives. The utility of the results for other purposes is discussed in turn.

Consider a rational SISO system with two given unstable poles p_1, p_2 ($p_1 > p_2$) and a zero z, where z can be influenced by simple process or operating point changes. The objective is to choose z such that input usage for stabilization is minimal. Clearly as $z \to \pm\infty$, the gain of the system increases and thus the input requirement decreases monotonically. In the range $p_2 \leq z \leq p_1$, there also exists a locally optimal value of z, since a zero close to an RHP pole reduces its joint controllability and observability increasing the input requirement. This locally optimal value of z is obtained by finding the stationary points of (19) and (20),

$$\begin{aligned} z_{\mathcal{H}_2,\text{opt}} &= \frac{p_1 p_2\left(3(p_1+p_2) \pm \sqrt{5p_1^2+5p_2^2+6p_1p_2}\right)}{2(p_1^2+p_2^2+3p_1p_2)} \\ z_{\mathcal{H}_\infty,\text{sub}} &= \frac{4p_1p_2(p_1+p_2)}{p_1^2+p_2^2+6p_1p_2} \end{aligned}$$

The results presented here can also be used for controlled and manipulated variable selection for stabilization. The use of controller with minimum input usage is justified as it reduces the likelihood of input saturation and also minimizes the disturbing effect of stabilization on remaining control problem [4]. An optimal selection of variables can be done by evaluating the expression for I_2 and I_∞ for different combinations. However in the general case, the choice of norm can influence the optimal combination of variables. For example, consider the following system,

$$\hat{\mathbf{G}}(s) = \frac{1}{(s-1)(s-2)}\left[\begin{array}{cc} (0.7s-1.2) & -(2.2s+2.4) \end{array}\right]$$

where the objective is to choose one of the inputs requiring minimum usage for stabilization. Use of $\mathcal{H}_2$ and $\mathcal{H}_\infty$-norms suggests the selection of u_2 and u_1 respectively. An appropriate choice of norm can be done based on available information regarding disturbance characteristics. But noting that $\|\hat{\mathbf{K}}(s)\hat{\mathbf{S}}(s)\|_{\mathcal{L}_1}$ closely addresses the physical constraints of the system and $\|\hat{\mathbf{K}}(s)\hat{\mathbf{S}}(s)\|_\infty \leq \|\hat{\mathbf{K}}(s)\hat{\mathbf{S}}(s)\|_{\mathcal{L}_1}$ [7], use of $\mathcal{H}_\infty$-norm may be preferred. Then if for some combination of variables, $\|\hat{\mathbf{K}}(s)\hat{\mathbf{S}}(s)\|_\infty > \beta$, where β depends on physical constraints, system stabilization without input saturation using a linear feedback controller is not possible.

With a minor modification, the applicability of μ-IM is increased to unstable systems, but it can still be very conservative. Consider the following system

$$\hat{\mathbf{G}}(s) = \left[\begin{array}{cc|cc} 1 & 0 & 1 & \beta \\ 0 & 2 & \beta & 1 \\ \hline 1 & 0.1 & 0 & 0 \\ 0.1 & 1 & 0 & 0 \end{array}\right] \quad \hat{\mathbf{G}}_{bd}(s) = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

The sufficient condition for decentralized stabilization given by Corollary 3 is not satisfied, when $\beta > 0.22$, but direct controller design using Matlab® suggests that $\hat{\mathbf{K}}_{bd}(s)$ can stabilize $\hat{\mathbf{G}}(s)$ until $\beta < 0.32$. This conservativeness arises as $\mathbf{\Delta}$ is much larger than the true uncertainty in $\hat{\mathbf{G}}_{bd}(s)$. It can be reduced using frequency dependent weights [6], but the choice of such weights is non-trivial. Further note that the partition of $\hat{\mathbf{G}}(s)$ into $\hat{\mathbf{G}}_{bd}(s)$ and $\hat{\mathbf{G}}_I(s)$ is non-unique. Though some guidelines are provided, finding the optimal partition of $\hat{\mathbf{G}}(s)$ remains an issue for further research. In this paper, we assumed that the disturbance model does not share any unstable poles with the system stable. This assumption can be relaxed for generalization purposes using the results presented here and that of Havre and Skogestad [3].

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