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State feedback gain scheduling for linear systems with time-varying parameters

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***Abstract*— This paper addresses the problem of parameter dependent state feedback control (i.e. gain scheduling) for linear systems with parameters that are assumed to be measurable in real time and are allowed to vary in a compact polytopic set with bounded variation rates. A new sufficient condition given in terms of linear matrix inequalities permits to determine the controller gain as a function of the time-varying parameters and of a set of constant matrices. The closed-loop stability is assured by means of a parameter dependent Lyapunov function. The condition proposed encompasses the well known quadratic stabilizability condition and allows to impose structural constraints such as decentralization to the feedback gains. Numerical examples illustrate the efficiency of the technique.**

I. INTRODUCTION

Gain scheduling control has motivated several studies in the recent years, as can be inferred from the survey papers [1] and [2]. This technique is appealing to deal with systems subject to parametric variations, which include linear systems with time-varying parameters and nonlinear systems modeled as linear parameter-varying systems. The classical approach is to design several controllers for chosen linear models of a parameterized family of models and then, based on the measurement of the time-varying parameters, to schedule the controller gain using some interpolation method [3]. Although the results can be improved by the refinement of the grid on the parametric space, at the price of increasing the computational effort, this procedure cannot guarantee stability and performance for the controlled system except in some special cases as for slow varying parameters [4], [5].

Regarding to linear systems affected by arbitrarily time-varying parameters in polytopic domains, it is well known that the quadratic stabilizability condition [6] is an important tool to cope with the design of a robust state feedback gain for the system. This condition is appealing by its numerical simplicity, since it uses a common Lyapunov matrix to determine a fixed gain that stabilizes the closed-loop system, and can be formulated in terms of linear matrix inequalities (LMIs) [7]. Other classes of Lyapunov functions have been investigated as, for instance, piecewise quadratic Lyapunov functions, yielding less conservative results than quadratic stability for both analysis and control design at the price of a considerably higher numerical complexity [8]. However, quadratic or piecewise quadratic functions can lead to conservative results for systems with bounded time derivatives on the parameter variations. As an alternative, parameter dependent Lyapunov functions seem to be a useful tool since they can incorporate the rates of parameter variation into the analysis and synthesis problems.

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Actually, parameter dependent Lyapunov functions have been recently used to derive important conditions for the robust stability of linear time-invariant systems with polytopic uncertainty. In [9], a simple LMI feasibility test defined at the vertices of the uncertainty domain provides a set of Lyapunov matrices whose convex combination yields a parameter dependent Lyapunov function used to assess the robust stability of the uncertain linear time-invariant system, encompassing the analysis results provided by quadratic stability. Improved LMI conditions appeared in [10] and, more recently, in [11], encompassing the previous results. In the context of linear time-varying systems, the robust stability of uncertain linear systems has been investigated by means of parameter dependent Lyapunov functions in many papers [12], [13], [14], [15], some of them also discussing possible extensions to cope with control design for certain classes of uncertain systems. In most of cases, restrictive assumptions on the structure of the uncertainty are made. In [14], LMI sufficient conditions for the robust stability and performance of continuous time-varying systems with affine parameter dependence are presented. The results are still conservative, since the multiconvexity of the time derivative of the parameter dependent Lyapunov function is imposed, implying that the analysis at the vertices of the uncertainty polytope (usually only necessary) provides a conclusive evaluation about the overall stability. The multiconvexity has also been used in [16] as a tool to compute robust stability and performance of uncertain systems and to simplify, in some cases, the gridding procedure to compute a linear parameter varying feedback control.

The extension of analysis conditions based on parameter dependent Lyapunov functions to cope with control design problems often results in gain scheduling strategies. In [17], an approach which combines methods of analysis of polytopic systems with conventional constant scaling techniques to solve the robustness analysis problem for uncertain systems admitting a linear fractional transformation representation is extended to deal with gain scheduling controllers. The design of gain scheduling for polytopic systems with bounded time derivatives on the parameters has been addressed by means of LMIs in [18], [19], [20]. The conditions must be solved upon a grid on the parameter space, which results in testing a finite number of LMIs. In a sense, the results using finite gridding points are unreliable and the

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numerical complexity of the tests grows rapidly. In [19], LMI conditions use a combination of parameter dependent Lyapunov functions and the so called S-procedure to reduce the conservatism in the design problem, but the problem still has to be addressed by means of a grid in the parameter space. Although parameter gridding can be avoided in some cases, the use of conservative assumptions for the set of uncertainties is required. For instance, when the plant and the controller admit a linear fractional transformation, the existence of a stabilizing control can be determined through the feasibility of a finite set of LMIs [21], [22], [23].

This paper is concerned with the use of a parameter dependent Lyapunov function to derive a state feedback gain scheduling control for linear continuous time-varying system with uncertain parameters belonging to a polytope and satisfying known bounds on their time derivatives. There are no restrictive assumptions on the structure of the uncertainties. It is assumed that the vector of uncertain parameters can be measured on-line. A finite set of LMIs is defined at the vertices of the uncertainty domain in such a way that, if a feasible solution exists, a parameter dependent Lyapunov function constructed as the convex combination of a set of Lyapunov matrices assures the robust stability of the uncertain time-varying system by means of a parameter dependent stabilizing gain. The results encompass the quadratic stabilizability condition in the sense that feasible solutions always exist for quadratically stabilizable uncertain systems. For uncertain domains and time derivatives bounds known *a priori*, the conditions provide a simple convex LMI stabilizability test that can be performed in polynomial time by specialized algorithms [24], yielding a parameter dependent gain. There is no need of gridding on the parameter space. Using line searches and LMI feasibility tests, the maximum bounds on the uncertain parameter time derivatives for which there exists a stabilizing parameter dependent feedback gain can be estimated. Moreover, structural constraints to the feedback gains such as decentralization or output feedback can be easily imposed. Examples illustrate the method proposed.

II. STATE FEEDBACK GAIN SCHEDULING DESIGN

Consider the linear continuous time-varying system

$$\dot{x}(t) = A(\alpha(t))x(t) + B(\alpha(t))u(t) \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A(\alpha(t)) \in \mathbb{R}^{n\times n}$, and $B(\alpha(t)) \in \mathbb{R}^{n\times m}$. Suppose that matrices $A(\alpha(t))$ and $B(\alpha(t))$ belong to the polytope $\mathcal{D}$ given by

$$\mathcal{D} = \Big\{(A,B)(\alpha(t)) \ : \ (A,B)(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t)(A,B)_j, \\ \sum_{j=1}^{N} \alpha_j(t) = 1,\ \alpha_j(t) \geq 0,\ j = 1,\ldots,N\Big\} \tag{2}$$

with bounds on the time derivatives of the uncertain parameters given by

$$|\ \dot{\alpha}_j(t)\ | \leq \rho_j\ ,\ j = 1,\ldots,N-1 \tag{3}$$

Notice that the constraint $\sum_{j=1}^{N} \alpha_j(t) = 1$ implies, without loss of generality, $\dot{\alpha}_N(t) = \sum_{j=1}^{N-1} \dot{\alpha}_j(t)$ and the bound on this parameter can be expressed by $|\ \dot{\alpha}_N(t)\ | \leq \sum_{j=1}^{N-1} \rho_j$.

The well known quadratic stabilizability condition [6] assures that, if there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$ and a matrix $Z \in \mathbb{R}^{m\times n}$ such that

$$A_jW + WA_j' + B_jZ + Z'B_j' < 0 \quad , \quad j = 1,\ldots,N \tag{4}$$

then the stability of the system (1)-(2) is guaranteed (independently of the bounds (3)) by the state feedback control

$$u(t) = Kx(t) \quad , \quad K = ZW^{-1} \tag{5}$$

This result has been largely used as a starting point to compute guaranteed costs, robust controllers and filters (see [7] and references therein). It is important to stress that the feasibility of condition (4) guarantees that system (1)-(2) is stabilizable by means of a fixed gain K given by (5), for any arbitrary $\dot{\alpha}(t)$. However, many times this condition leads to conservative results, since the stabilizability of the entire polytope is based on a fixed Lyapunov matrix.

Less conservative evaluations are provided by Theorem 1, which presents a sufficient condition with finite number of LMIs to design a parameter dependent state feedback controller (i.e. a gain scheduling controller) for the system (1)-(2), taking into account the bounds on the parameter time derivatives, given by (3). Neither a gridding on the parameter space nor assumptions on the structure of the uncertainties are needed.

Theorem 1 If there exist symmetric positive definite matrices $W_j \in \mathbb{R}^{n\times n}$ and matrices $Z_j \in \mathbb{R}^{m\times n}$, with $j = 1,\ldots,N$, for given bounds $\rho_i \geq 0$, $i = 1,\ldots,N-1$, such that[1]

$$A_jW_j + W_jA_j' + B_jZ_j + Z_j'B_j' + \sum_{i=1}^{N-1} \pm\rho_i(W_i - W_N) < 0 \\ j = 1,\ldots,N \tag{6}$$

$$A_jW_k + W_kA_j' + A_kW_j + W_jA_k' + B_jZ_k + Z_k'B_j' + B_kZ_j + Z_j'B_k' \\ + 2\sum_{i=1}^{N-1} \pm\rho_i(W_i - W_N) < 0 \qquad \begin{array}{l} j = 1,\ldots,N-1 \\ k = j+1,\ldots,N \end{array} \tag{7}$$

then the parameter dependent state feedback control law $u(t) = K(\alpha(t))x(t)$ with

$$K(\alpha(t)) = Z(\alpha(t))W(\alpha(t))^{-1} \tag{8}$$

and

$$Z(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t)Z_j;\ W(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t)W_j; \\ \sum_{j=1}^{N} \alpha_j(t) = 1;\ \alpha_j(t) \geq 0,\ j = 1,\ldots,N \tag{9}$$

[1]The LMIs must be implemented with all the combinations $\pm$.

assures the closed-loop stability of the uncertain system (1)-(2) under the bounds (3) by means of the positive definite parameter dependent Lyapunov matrix $P(\alpha(t)) = W(\alpha(t))^{-1}$, with $W(\alpha(t))$ given by (9).

Proof Consider the parameter dependent Lyapunov function $v(x(t)) = x(t)'P(\alpha(t))x(t)$ with $P(\alpha(t)) = W(\alpha(t))^{-1}$ given by (9). Clearly, $W(\alpha(t)) > 0$, implying that $v(x(t))$ is positive for all $x(t) \neq 0$. Its time derivative is given by $\dot{v}(x(t)) \triangleq x(t)'Q(\alpha(t))x(t)$ with

$$Q(\alpha(t)) = \big(A(\alpha(t)) + B(\alpha(t))K(\alpha(t))\big)'P(\alpha(t)) + P(\alpha(t))\big(A(\alpha(t)) + B(\alpha(t))K(\alpha(t))\big) + \dot{P}(\alpha(t)) \quad (10)$$

Multiplying $Q(\alpha(t))$ at right and at left by $P(\alpha(t))^{-1}$ and making the change of variables

$$W(\alpha(t)) = P(\alpha(t))^{-1} \quad ; \quad Z(\alpha(t)) = K(\alpha(t))W(\alpha(t)) \quad (11)$$

one has $R(\alpha(t)) \triangleq W(\alpha(t))Q(\alpha(t))W(\alpha(t))$ with

$$R(\alpha(t)) = A(\alpha(t))W(\alpha(t)) + W(\alpha(t))A(\alpha(t))' + B(\alpha(t))Z(\alpha(t)) + Z(\alpha(t))'B(\alpha(t))' + W(\alpha(t))\dot{P}(\alpha(t))W(\alpha(t)) \quad (12)$$

From (11), $P(\alpha(t))W(\alpha(t)) = \mathbf{I}$, implying that $\dot{P}(\alpha(t)) = -W(\alpha(t))^{-1}\dot{W}(\alpha(t))W(\alpha(t))^{-1}$ which leads to

$$R(\alpha(t)) = A(\alpha(t))W(\alpha(t)) + W(\alpha(t))A(\alpha(t))' + B(\alpha(t))Z(\alpha(t)) + Z(\alpha(t))'B(\alpha(t))' - \dot{W}(\alpha(t)) \quad (13)$$

Using (2) and (9), one has

$$R(\alpha(t)) = \sum_{j=1}^{N} \alpha_j^2(t)\big(A_jW_j + W_jA_j' + B_jZ_j + Z_j'B_j'\big) + \sum_{j=1}^{N-1}\sum_{k=j+1}^{N} \alpha_j(t)\alpha_k(t)\big(A_jW_k + W_kA_j' + A_kW_j + W_jA_k' + B_jZ_k + Z_k'B_j' + B_kZ_j + Z_j'B_k'\big) - \sum_{j=1}^{N}\dot{\alpha}_j(t)W_j \quad (14)$$

Since

$$\Big(\sum_{j=1}^{N}\alpha_j(t)\Big)^2 = \sum_{j=1}^{N}\alpha_j^2(t) + 2\sum_{j=1}^{N-1}\sum_{k=j+1}^{N}\alpha_j(t)\alpha_k(t) = 1 \quad (15)$$

it is possible to rewrite (14) as

$$R(\alpha(t)) = \sum_{j=1}^{N} \alpha_j^2(t)\Big(A_jW_j + W_jA_j' + B_jZ_j + Z_j'B_j' - \sum_{j=1}^{N}\dot{\alpha}_j(t)W_j\Big) + \sum_{j=1}^{N-1}\sum_{k=j+1}^{N} \alpha_j(t)\alpha_k(t)\Big(A_jW_k + W_kA_j' + A_kW_j + W_jA_k' + B_jZ_k + Z_k'B_j' + B_kZ_j + Z_j'B_k' - 2\sum_{j=1}^{N}\dot{\alpha}_j(t)W_j\Big) \quad (16)$$

Recalling that $\dot{\alpha}_N(t) = -\sum_{i=1}^{N-1}\dot{\alpha}_i(t)$ and $\sum_{j=1}^{N}\dot{\alpha}_j(t)W_j = \sum_{i=1}^{N-1}\dot{\alpha}_i(t)(W_i - W_N)$ which, replaced in (16), results in

$$R(\alpha(t)) = \sum_{j=1}^{N} \alpha_j^2(t)\Big(A_jW_j + W_jA_j' + B_jZ_j + Z_j'B_j' - \sum_{i=1}^{N-1}\dot{\alpha}_i(t)(W_i - W_N)\Big) + \sum_{j=1}^{N-1}\sum_{k=j+1}^{N} \alpha_j(t)\alpha_k(t)\Big(A_jW_k + W_kA_j' + A_kW_j + W_jA_k' + B_jZ_k + Z_k'B_j' + B_kZ_j + Z_j'B_k' - 2\sum_{i=1}^{N-1}\dot{\alpha}_i(t)(W_i - W_N)\Big) \quad (17)$$

Taking into account (3), conditions (6)-(7) are sufficient to guarantee that $R(\alpha(t))$ is negative definite for all $\alpha_j(t) \geq 0$, $\sum_{j=1}^{N}\alpha_j(t) = 1$. As a consequence, $Q(\alpha(t)) = W(\alpha(t))^{-1}R(\alpha(t))W(\alpha(t))^{-1} < 0$ and thus $\dot{v}(x(t)) < 0$. □

Theorem 1 deserves some remarks. First of all, it provides an easy way to determine a gain scheduling for the system (1)-(2) when the bounds on the time derivatives of the parameters ρ_i, $i = 1,\ldots,N-1$ are known *a priori*. Of course, the number of LMIs of Theorem 1, given by $N + N2^{N-1} + N(N-1)2^{N-2}$ (including $P_j > 0$, $j = 1\ldots N$), increases rapidly with N (the number of vertices of the system), but efficient polynomial time algorithms [24] can be used to solve the problem. When the bounds on the time derivatives are not known *a priori*, Theorem 1 allows to determine a parameter dependent stabilizing gain $K(\alpha(t))$ and the bounds ρ_i, $i = 1,\ldots,N-1$ for which this gain is valid. This can be done by means of solving a convex problem with a line search procedure.

A second remark is that Theorem 1 encompasses the quadratic stabilizability condition in the sense that if the system (1)-(2) is quadratically stabilizable with fixed W and Z, satisfying (4), then Theorem 1 will be feasible for $W_1 = W_2 = \ldots = W_N = W$ and $Z_1 = Z_2 = \ldots = Z_N = Z$ for some ρ_i, $i = 1,\ldots,N-1$. Moreover, when the system is quadratically stabilizable, the matrices W_j and Z_j of Theorem 1 tend to fixed matrices W and Z as $\rho_i \to \infty$, $i = 1,\ldots,N-1$.

Theorem 1 can be seen as an extension for synthesis of the recently published conditions for robust stability of linear time-varying systems in polytopic domains [25]. A numerical comparison illustrated that, in the case of the analysis, the conditions from [25] leaded to less conservative results than the multiconvexity from [14].

Finally, note that although $\alpha(t)$ may not exactly represent the actual time-varying vector of parameters $p(t)$ of a physical system, a linear relationship between $\alpha(t)$ and $p(t)$ (as well as between $\dot{\alpha}(t)$ and $\dot{p}(t)$) can be readily established whenever there is an affine dependence on the uncertainty.

III. STRUCTURALLY CONSTRAINED CONTROL

Sometimes the control design problem need to take into account structural constraints, such as decentralization (when interconnected systems must be controlled by means of local information only) or output feedback (which occurs very frequently, since in general only a linear combination of the states is available for feedback).

Following ideas similar to the ones used in the context of quadratically stabilizing robust control [26], structural constraints can be easily imposed to the matrix variables, yielding sufficient conditions for the existence of structured constrained robust as well as parameter dependent feedback gains. A special structure on matrices W_j, Z_j of Theorem 1 can provide decentralized or output feedback gain scheduling, as shown in corollaries 2 and 3.

Corollary 2 A decentralized parameter dependent state feedback control law can be obtained from the previous results by imposing to matrices W_j and Z_j in Theorem 1 a block diagonal (*bl. d.*) structure

$$W_{jD} = bl.\ d.\{W_j^1,\ldots,W_j^M\};\ Z_{jD} = bl.\ d.\{Z_j^1,\ldots,Z_j^M\} \quad (18)$$

with M being the number of subsystems. If a feasible solution exists, the gain scheduled control (8) is such that

$$K_D(\alpha(t)) = bl.\ d.\{K(\alpha(t))^1,\ldots,K(\alpha(t))^M\} \quad (19)$$

Suppose now that only a subset of the states of the system (1) is available for feedback; in other words, $y(t)\in\mathbb{R}^p$ is an output given by[2] $y(t) = \left[\begin{array}{c|c} \mathbf{I}_p & \mathbf{0}\end{array}\right] x(t)$. In this case, the output feedback gain scheduled control problem can be formulated as the search of a parameter dependent state feedback gain $K(\alpha(t))\in\mathbb{R}^{m\times n}$ with the following structure

$$K(\alpha(t)) = \left[\begin{array}{c|c} K_O(\alpha(t)) & 0\end{array}\right] \quad (20)$$

with $K_O(\alpha(t))\in\mathbb{R}^{m\times p}$. Following the ideas presented in [27], [26], a sufficient condition for the existence of such gain is easily obtained as follows.

Corollary 3 A parameter dependent output feedback control gain can be obtained from the previous results by imposing to matrices W_j and Z_j in Theorem 1 the structure constraints

$$Z_{jO} = \left[\begin{array}{c|c} Z_{jO}^1 & \mathbf{0}\end{array}\right]\ ;\quad W_{jO} = \left[\begin{array}{c|c} W_{jO}^{11} & \mathbf{0} \\ \hline \mathbf{0} & W_{jO}^{22}\end{array}\right]$$

with $Z_{jO}^1\in\mathbb{R}^{m\times p}$, $W_{jO}^{11}\in\mathbb{R}^{p\times p}$ and $W_{jO}^{22}\in\mathbb{R}^{(n-p)\times(n-p)}$. If a feasible solution exists, the control gain given by (8) is such that (20) holds.

IV. EXAMPLES

Some examples are presented in this section to illustrate the usefulness of the proposed conditions to address the problems of analysis and synthesis for the class of time-varying systems under investigation.

[2]There always exists a similarity transformation that allows the output of a linear system to be written in this way.

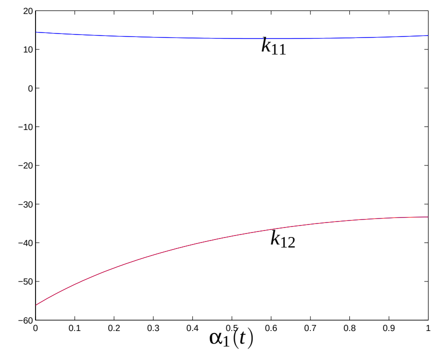

Fig. 1. Nonlinear behavior of the entries of the gain scheduled state feedback control (24) as a function of $\alpha_1(t)$ for system (21) with $\rho_1 = 1$.

As a first example, consider the system (1)-(2) with $N = 2$ vertices given by

$$A_1 = \begin{bmatrix} 0.2 & -0.8 \\ 0.3 & -1.3\end{bmatrix},\ B_1 = \begin{bmatrix} 0.4 \\ 0.8\end{bmatrix}\ ;$$
$$A_2 = \begin{bmatrix} 0.0 & -0.3 \\ 0.5 & 0.0\end{bmatrix},\ B_2 = \begin{bmatrix} 0.3 \\ 0.1\end{bmatrix} \quad (21)$$

The uncertain parameters $\alpha_1(t)$, $\alpha_2(t)$ are such that $\alpha_2(t) = 1-\alpha_1(t)$ and $|\,\dot\alpha_1(t)\,| = |\,\dot\alpha_2(t)\,| \le \rho_1$. Despite the simplicity, this system is not quadratically stabilizable, that is, (4) fails to provide a fixed gain K to stabilize the entire polytope. On the other hand, Theorem 1 allows to determine a parameter dependent state feedback stabilizing gain $K(\alpha_1(t))$ that guarantees the closed-loop system stability for bounds on the time derivative of its parameters.

Conditions (6)-(7) of Theorem 1 provide feasible solutions until $\rho_1 = 10000$, that is, stabilizing the uncertain time-varying system for $-10000 \le \dot\alpha_1(t) \le 10000$.

To illustrate the efficiency of the analytical gain calculated by Theorem 1 when compared to a standard gain scheduling controller obtained from a linear interpolation, consider $\alpha_1(t) = 0.5+0.5\sin(2t)$, such that $|\,\dot\alpha_1(t)\,| \le 1$. For this bound (i.e. $\rho_1 = 1$), Theorem 1 yields the solution

$$W_1 = \begin{bmatrix} 52.3249 & 21.6632 \\ 21.6632 & 10.2517\end{bmatrix}; Z_1' = \begin{bmatrix} -11.7960 \\ -47.6387\end{bmatrix} \quad (22)$$

$$W_2 = \begin{bmatrix} 44.3201 & 12.2531 \\ 12.2531 & 4.8696\end{bmatrix}; Z_2' = \begin{bmatrix} -47.2555 \\ -96.3004\end{bmatrix} \quad (23)$$

$$K(\alpha_1(t)) = \Big(\alpha_1(t)Z_1 + (1-\alpha_1(t))Z_2\Big) \Big(\alpha_1(t)W_1 + (1-\alpha_1(t))W_2\Big)^{-1} \quad (24)$$

This gain $K(\alpha_1(t))$ has entries that are nonlinear functions of $\alpha_1(t)$, as shown in Figure 1.

If a pole location strategy is used to compute one stabilizing state feedback control gain for each vertex, for instance placing the closed-loop poles at $(-10,-20)$ for vertex #1 and at $(-30,-40)$ for vertex #2, one has

$$K_1 = \begin{bmatrix} 2014.65 & -1043.45\end{bmatrix},\ K_2 = \begin{bmatrix} 2280.94 & -7542.81\end{bmatrix}$$

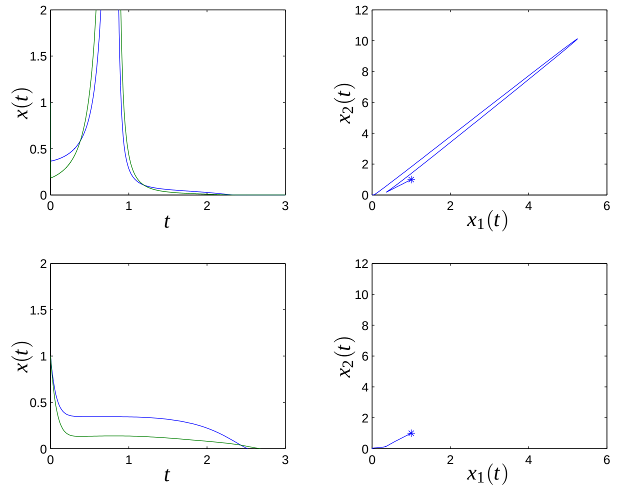

Fig. 2. Trajectories and phase portraits of the closed-loop system for the standard gain scheduling with linear interpolation, given by (25) (top), and for the nonlinear analytical gain scheduling strategy proposed in Theorem 1, given by (24) (bottom), for system (21) with $\rho_1 = 1$.

resulting in the linear interpolated gain

$$K_I(\alpha_1(t)) = \alpha_1(t)K_1 + (1-\alpha_1(t))K_2 \qquad (25)$$

A time simulation has been performed for both gain scheduling strategies, i.e., the one proposed by Theorem 1, which provides the nonlinear analytical gain given by (24), and the standard linear interpolation resulting in K_I given by (25). The results are shown in Figure 2, for the initial condition $x(0) = [1\ 1]'$, marked with $*$ in the phase portraits. Although the system with the gain K_I does not become unstable, its performance is clearly worst than the one of the closed-loop system with the parameter dependent gain from Theorem 1.

It is interesting to note that, if higher rates of parameter variation are allowed, the LMI conditions of Theorem 1 tend to force the solution $W_1 = W_2 = \ldots = W_N = W$, indicating that the closed-loop stability for fast parameter variation requires constant Lyapunov matrices. In this case, the scheduled state feedback control will have an almost linear behavior with respect to $\alpha_1(t)$, as shown in Figure 3 for the same example, with $\rho_1 = 100$. This means Theorem 1 provides a useful characterization for systems non quadratically stabilizable through constant feedback gains, but that admit a parameter dependent state feedback control, such that the closed-loop system tends to be quadratically stable. As the closed-loop quadratic stability is attained, the scheduled gain tends to a linear parameter varying state feedback control.

The second example (randomly generated) illustrates how additional information about stabilizability domains can be obtained from Theorem 1. Consider the system (1)-(2) with

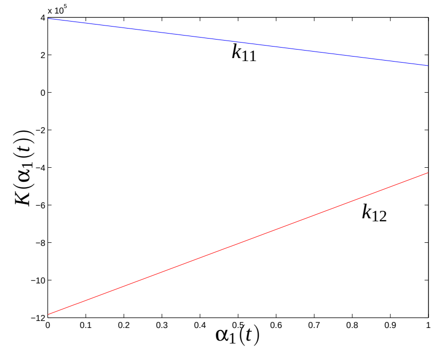

Fig. 3. Parameter dependent gain entries obtained from Theorem 1 for system (21) with $\rho_1 = 100$.

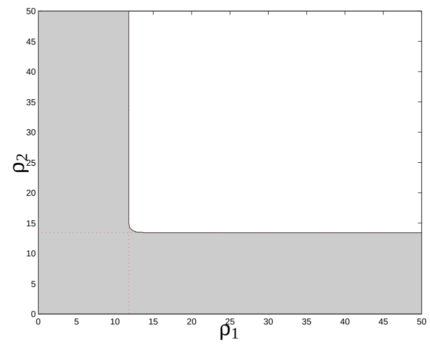

Fig. 4. Stabilizability domains for the uncertain system given by (26)-(28): bounds on the parameter time derivatives.

$N = 3$ vertices given by

$$A_1 = \begin{bmatrix} 0.4615 & 0.0041 & 0.0952 \\ 0.7684 & -0.0633 & 0.9556 \\ 0.1943 & 0.6249 & 0.2818 \end{bmatrix}, B_1 = \begin{bmatrix} 0.7652 \\ 0.7489 \\ 0.9525 \end{bmatrix} \qquad (26)$$

$$A_2 = \begin{bmatrix} -0.0207 & 0.3513 & 0.4003 \\ 0.8288 & 0.2590 & 0.9077 \\ 0.6369 & 0.1098 & -0.0047 \end{bmatrix}, B_2 = \begin{bmatrix} 0.9274 \\ 0.4693 \\ 0.3157 \end{bmatrix} \qquad (27)$$

$$A_3 = \begin{bmatrix} 0.4992 & 0.3527 & 0.1125 \\ 0.3698 & 0.7311 & 0.3518 \\ 0.2291 & 0.7293 & 0.3262 \end{bmatrix}, B_3 = \begin{bmatrix} 0.3907 \\ 0.1346 \\ 0.1496 \end{bmatrix} \qquad (28)$$

This system is not quadratically stabilizable. However, Theorem 1 can provide stabilizing gains for the system with bounds on its parameters time derivatives. Figure 4 shows the feasibility region of conditions (6)-(7) as a function of ρ_1 and ρ_2 (remembering that $|\dot{\alpha}_3(t)| \leq \rho_1 + \rho_2$). For instance, if the parameter $\alpha_1(t)$ is time-invariant, that is, $\rho_1 = 0$, then the system (1)-(2) with vertices (26)-(28) is stabilized by means of a parameter dependent gain obtained through the conditions of Theorem 1 for arbitrarily high values of ρ_2. Actually, this behavior occurs until $\rho_1 = 11.8$. The same considerations are valid for $0 \leq \rho_2 \leq 13.4$. In other words, this non-quadratically stabilizable system can be stabilized by a parameter dependent state feedback gain $K(\alpha(t))$ and the closed-loop system will admit arbitrarily high rates of change in some of its parameters.

The third example addresses the problem of structural constrained control. Consider the system (1)-(2) with vertices given by

$$A_1 = \begin{bmatrix} 0.4886 & 0.1740 \\ 0.8502 & -0.0006 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.5841 \\ 0.0812 \end{bmatrix} \;\; ;$$
$$A_2 = \begin{bmatrix} 0.7794 & 0.8577 \\ 0.5147 & -0.0392 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.2725 \\ 0.7584 \end{bmatrix} \quad (29)$$

and $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$. The aim here is to obtain an output feedback control gain using the results of Corollary 3. For this system, the quadratic stabilizability condition fails to provide a fixed gain with the structure $K = [k_{11} \quad 0]$ when a block diagonal constraint as in (18) is imposed to Z and W in the LMIs (4), but Corollary 3 yields a parameter dependent stabilizing gain in the form $K(\alpha_1(t)) = [k_{11}(\alpha_1(t)) \quad 0]$. For instance, choosing $\rho_1 = 10000$ one has $Z_1 = \begin{bmatrix} -69.5612 & 0 \end{bmatrix}$, $Z_2 = \begin{bmatrix} -45.4018 & 0 \end{bmatrix}$ and

$$W_1 \simeq W_2 = \begin{bmatrix} 0.6066 & 0 \\ 0 & 40.4422 \end{bmatrix}$$

implying that, in this case, the gain (8) is affine in $\alpha_1(t)$: $K(\alpha_1(t)) = \begin{bmatrix} (-39.8250\alpha_1(t) - 74.8416) & 0 \end{bmatrix}$. Note that this linear parameter varying output feedback gain assures to the closed-loop system robust stability for very high parameter variation rates, thanks to the almost constant closed-loop Lyapunov matrix.

V. CONCLUSION

A new sufficient condition for the synthesis of gain scheduling controllers applied to linear systems with time-varying polytopic uncertainty is provided in terms of LMIs. A parameter dependent Lyapunov function, which allows to determine a parameter dependent stabilizing gain, is obtained from the feasibility of a set of LMIs defined at the vertices of the polytope, assuring robust stabilizability for all uncertain parameters whose time derivatives satisfy some given bounds. The maximum rate of variation of the uncertain parameters can be estimated by means of line search procedures and LMI feasibility tests. Additional constraints can be easily incorporated, permitting to address the problems of decentralized control or output feedback gains. The time-varying gain scheduling is given as a function of the time-varying uncertain parameters, supposed to be measurable, and of a set of fixed matrices that can be calculated and stored *a priori*, allowing the implementation of real time control.

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REFERENCES

[1] D. J. Leith and W. E. Leithead, "Survey of gain-scheduling analysis and design," *Int. J. Contr.*, vol. 73, no. 11, pp. 1001–1025, July 2000.

[2] W. J. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, October 2000.

[3] D. J. Stilwell and W. J. Rugh, "Interpolation of observer state feedback controllers for gain scheduling," *IEEE Trans. Automat. Contr.*, vol. 44, no. 6, pp. 1225–1229, June 1999.

[4] J. S. Shamma and M. Athans, "Guaranteed properties of gain scheduled control for linear parameter-varying plants," *Automatica*, vol. 27, no. 3, pp. 559–564, 1991.

[5] ——, "Gain scheduling: potential hazards and possible remedies," *IEEE Control Syst. Mag.*, vol. 12, no. 3, pp. 101–107, June 1992.

[6] J. Bernussou, P. L. D. Peres, and J. C. Geromel, "A linear programming oriented procedure for quadratic stabilization of uncertain systems," *Syst. Contr. Lett.*, vol. 13, no. 1, pp. 65–72, July 1989.

[7] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1994.

[8] M. Johansson, *Piecewise Linear Control Systems — A Computational Approach*, ser. Lecture Notes in Control and Information Science. Heidelberg, Germany: Springer-Verlag, 2003, vol. 284.

[9] J. C. Geromel, M. C. de Oliveira, and L. Hsu, "LMI characterization of structural and robust stability," *Lin. Alg. Appl.*, vol. 285, no. 1–3, pp. 69–80, December 1998.

[10] D. C. W. Ramos and P. L. D. Peres, "An LMI condition for the robust stability of uncertain continuous-time linear systems," *IEEE Trans. Automat. Contr.*, vol. 47, no. 4, pp. 675–678, April 2002.

[11] V. J. S. Leite and P. L. D. Peres, "An improved LMI condition for robust $\mathcal{D}$-stability of uncertain polytopic systems," *IEEE Trans. Automat. Contr.*, vol. 48, no. 3, pp. 500–504, March 2003.

[12] F. Amato, M. Corless, M. Mattei, and R. Setola, "A multivariable stability margin in the presence of time-varying, bounded rate gains," *Int. J. Robust Nonlinear Contr.*, vol. 7, pp. 127–143, 1997.

[13] E. Feron, P. Apkarian, and P. Gahinet, "Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions," *IEEE Trans. Automat. Contr.*, vol. 41, no. 7, pp. 1041–1046, July 1996.

[14] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Trans. Automat. Contr.*, vol. 41, no. 3, pp. 436–442, March 1996.

[15] W. M. Haddad and D. S. Bernstein, "Parameter-dependent Lyapunov functions and the Popov Criterion in robust analysis and synthesis," *IEEE Trans. Automat. Contr.*, vol. 40, no. 3, pp. 536–543, 1995.

[16] P. Apkarian and H. D. Tuan, "Parametrized LMIs in control theory," *SIAM J. Control Optim.*, vol. 38, no. 4, pp. 1241–1264, May 2000.

[17] F. Wang and V. Balakrishnan, "Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems," *IEEE Trans. Automat. Contr.*, vol. 47, no. 5, pp. 720–734, May 2002.

[18] P. Apkarian and R. J. Adams, "Advanced gain-scheduling techniques for uncertain systems," *IEEE Trans. Trans. Contr. Syst. Technol.*, vol. 6, no. 1, pp. 21–32, 1998.

[19] F. Wu, "A generalized LPV system analysis and control synthesis framework," *Int. J. Contr.*, vol. 74, no. 7, pp. 745–759, May 2001.

[20] F. Wu, X. H. Yang, A. Packard, and G. Becker, "Induced $\mathcal{L}_2$-norm control for LPV systems with bounded parameter variation rates," *Int. J. Robust Nonlinear Contr.*, vol. 6, pp. 983–998, 1996.

[21] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled $\mathcal{H}_\infty$ controllers," *IEEE Trans. Automat. Contr.*, vol. 40, no. 5, pp. 853–864, May 1995.

[22] A. Packard, "Gain scheduling via linear fractional transformations," *Syst. Contr. Lett.*, vol. 22, no. 2, pp. 79–92, February 1994.

[23] G. Scorletti and L. E. Ghaoui, "Improved LMI conditions for gain scheduling and related control problems," *Int. J. Robust Nonlinear Contr.*, vol. 8, no. 10, pp. 845–877, August 1998.

[24] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox for use with Matlab*. The Math Works Inc.: User's Guide, 1995.

[25] V. F. Montagner and P. L. D. Peres, "A new LMI condition for the robust stability of linear time-varying systems," in *Proc. 42nd IEEE Conf. Decision Contr.*, Maui, HI, USA, December 2003, pp. 6133–6138.

[26] P. L. D. Peres and J. C. Geromel, "An alternate numerical solution to the linear quadratic problem," *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 198–202, January 1994.

[27] V. J. S. Leite, V. F. Montagner, and P. L. D. Peres, "Robust pole location by parameter dependent state feedback control," in *Proc. 41st IEEE Conf. Decision Contr.*, Las Vegas, December 2002, pp. 1864–1869.