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Controllability of Piecewise Linear Descriptor Systems

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***Abstract*— The controllability of piecewise linear descriptor systems is considered in this paper. Necessary and sufficient geometric criteria for C-controllability and R-controllability of such systems are established, respectively. These conditions can be easily transformed into algebraic form. Furthermore, the intrinsic relationship between our results and the existing results are also discussed. Then a novel necessary and sufficient criterion for C-controllability of linear time-invariant descriptor systems is derived as a byproduct.**

I. Introduction

The problems of controllability and observability of descriptor systems have been well studied[1-13]. There are several definitions of controllability. For a linear time-invariant descriptor system, the system is called *completely controllable (C-controllable)*[1], if it can be driven to any terminal state from any admissible initial state; the system is called *R-controllable*[1], if it can be driven to any terminal state in the reachable set from any admissible initial state; the system is called *impulse controllable (I-controllable)*[9], if for every initial condition there exist a smooth(impulse-free) control $u(t)$ and a smooth state trajectory $x(t)$ solution; and the system is called *strongly controllable (S-controllable)*[2], if it is both R-controllable and I-controllable. [10] investigated C-controllability of descriptor systems with single time-delay in control, and necessary and sufficient conditions were established. Then [11] extended the results in [10] to multiple time-delays case, and necessary and sufficient criteria for R-controllability and I-controllability were derived as well. [12] and [13] studied the issues of controllability and observability for analytically solvable linear time-varying singular systems, but the model considered in [12] and [13] was assumed to be in the standard canonical form.

Despite these important results on controllability analysis of time-invariant or time-varying descriptor systems, very few papers consider piecewise linear descriptor systems. In this paper, we aim to derive necessary and sufficient criteria for controllability of piecewise linear descriptor systems. For a piecewise linear descriptor system, a distinct feature is that the trajectory of the system is discontinuous and jumps at the discontinuous point. We investigate C-controllability and R-controllability of such systems, and necessary and sufficient geometric conditions are established. Then, the algebraic criteria are obtained as well. Furthermore, the intrinsic relationship between our results and the existing results for linear time-invariant descriptor systems and piecewise linear systems are also addressed. A novel necessary and sufficient criterion for C-controllability of linear time-invariant descriptor systems is derived as a byproduct.

This paper is organized as follows. Section II formulates the problem and presents some preliminary results. C-controllability and R-controllability are investigated in Section III. The relationship between our results and the existing ones are discussed in Section IV. Section V presents two illustrating examples. Finally, Section VI concludes the whole paper.

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II. Preliminaries

Consider the piecewise linear descrptor system given by

$$E_i\dot{x}(t) = A_ix(t) + B_iu(t), t \in [t_{i-1}, t_i), i = 1, \cdots, k, \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \mathbb{R}^p$ is the input vector; $x(t^+) := \lim_{h\to 0^+} x(t+h)$, $x(t^-) := \lim_{h\to 0^+} x(t-h)$, $x(t^-) = x(t)$ implies that the solution of system (1) is left continuous, E_i, A_i, B_i are the known $n\times n$, $n\times n$ and $n\times p$ constant matrices, for $i = 1, 2, \cdots, k$, E_i is a singular matrix and $\det(sE_i - A_i) \not\equiv 0$, and the discontinuous points $t_1 < t_2 < \cdots < t_{k-1}$, where $t_0 < t_1$ and $t_{k-1} < t_k = t_f < \infty$.

The system is said to be *regular* if each triple (E_i, A_i, B_i) is regular as a time-invariant descriptor system, for $i = 1, 2, \cdots, k$. In this paper, we assume that system (1) is regular.

Since each triple (E_i, A_i, B_i) is regular, there exist nonsingular matrices P_i and Q_i such that

$$Q_iE_iP_i = \begin{bmatrix} I_{n_i} & 0 \\ 0 & N_i \end{bmatrix}, Q_iA_iP_i = \begin{bmatrix} G_i & 0 \\ 0 & I_{n-n_i} \end{bmatrix}, Q_iB_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \end{bmatrix} \tag{2}$$

where $N_i \in \mathbb{R}^{(n-n_i)\times p}$ is nilpotent, $B_{i,1} \in \mathbb{R}^{n_i\times p}$, $B_{i,2} \in \mathbb{R}^{(n-n_i)\times p}$ and $0 < n_i < n$. The matrix P_i and its inverse are decomposed as

$$P_i = [P_{i,1},\ P_{i,2}], \quad P_i^{-1} = \begin{bmatrix} P_{i,1}^{-1} \\ P_{i,2}^{-1} \end{bmatrix} \tag{3}$$

where $P_{i,1} \in \mathbb{R}^{n\times n_i}$, $P_{i,2} \in \mathbb{R}^{n\times(n-n_i)}$, $P_{i,1}^{-1} \in \mathbb{R}^{n_i\times n}$ and $P_{i,2}^{-1} \in \mathbb{R}^{(n-n_i)\times n}$. Moreover, denote $h_i = t_i - t_{i-1}$, $H_i = P_{i,1}\exp(G_ih_i)P_{i,1}^{-1}$, $i = 1, \cdots, k$.

In the rest of the paper, denote $\mathbb{U}$ the set of functions with piecewise differentiable $n-1$ times. As usual, we assume that all the control input $u(t) \in \mathbb{U}$. Given an input function $u(t) \in \mathbb{U}$, $u^{(i)}(t)$ denotes the i^{th} derivative of $u(t)$,

$i=0,1,\cdots,n-1$. Let $\prod_{i=1}^{n} A_i$ be the matrices product $A_1\cdots A_n$ and $\prod_{i=n}^{1} A_i$ be the matrices product $A_n\cdots A_1$.

Now we consider the general solution of system (1).

Lemma 1: For any $t\in(t_{k-1},t_k]$, given the initial state x_0 and an input $u\in\mathbb{U}$, the general solution of system (1) is given as follows:

(a) if $k=1$,

$$\begin{aligned} x(t) &= P_{1,1}\exp[G_1(t-t_0)]P_{1,1}^{-1}x(t_0)\\ &+P_{1,1}\int_{t_0}^{t} e^{G_1(t-s)}B_{1,1}u(s)ds\\ &-P_{1,2}\sum_{j=1}^{n-n_1}(N_1)^{j-1}B_{1,2}u^{(j-1)}(t), \end{aligned} \tag{4}$$

(b) if $k=2,3,\cdots$,

$$\begin{aligned} &x(t)=P_{k,1}\exp[G_1(t-t_{k-1})]P_{k,1}^{-1}\Bigg\{\prod_{j=k-1}^{1}H_jx(t_0)\\ &+\sum_{m=1}^{k-2}\Big[\prod_{j=k-1}^{m+1}H_j\big(P_{m,1}\int_{t_{m-1}}^{t_m}e^{G_m(t_m-s)}B_{m,1}u(s)ds\\ &-P_{m,2}\sum_{j=1}^{n-n_m}(N_m)^{j-1}B_{m,2}u^{(j-1)}(t_m)\big)\Big]\\ &+P_{k-1,1}\int_{t_{k-2}}^{t_{k-1}}\exp[G_{k-1}(t_{k-1}-s)]B_{k-1,1}u(s)ds\\ &-P_{k-1,2}\sum_{j=1}^{n-n_{k-1}}(N_{k-1})^{j-1}B_{k-1,2}u^{(j-1)}(t)\Bigg\}\\ &+P_{k,1}\int_{t_{k-1}}^{t}\exp[G_k(t-s)]B_{k,1}u(s)ds\\ &-P_{k,2}\sum_{j=1}^{n-n_k}(N_k)^{j-1}B_{k,2}u^{(j-1)}(t) \end{aligned} \tag{5}$$

Proof: See Appendix A. ■

Remark 1: By Lemma 1, we know that the solution of system (1) is discontinuous at t_i, $i=0,1,\cdots,k$. At the discontinuous point t_i, the state jumps from $x(t_i^-)$ to $x(t_i^+)$. One part of $x(t_i^+)$ is inherited from $x(t_i^-)$, and the other part is corresponding to the control input $u(t_i)$.

Now, we'll give some mathematical preliminaries as the basic tools in the following discussion.

Given matrices $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times p}$, denote $\mathcal{I}m(B)$ the *range* of B, i.e., $\mathcal{I}m(B)=\{y|y=Bx,x\in\mathbb{R}^p\}$, and denote $\langle A|B\rangle$ the *minimal invariant subspace*[15] of A on $\mathcal{I}m(B)$, i.e., $\langle A|B\rangle=\sum_{i=1}^{n}A^{i-1}\mathcal{I}m(B)$.

The following lemma is a generalization of Theorem 7.8.1 in [14], which is the starting point for deriving the controllability criteria.

Lemma 2: [17] Given matrices $G\in\mathbb{R}^{n_1\times n_1}$, $B_1\in\mathbb{R}^{n_1\times p}$, $N\in\mathbb{R}^{n_2\times n_2}$, $B_2\in\mathbb{R}^{n_2\times p}$, $P_1\in\mathbb{R}^{n\times n_1}$ and $P_2\in\mathbb{R}^{n\times n_2}$, where $n_1+n_2=n$, for any $0\le t_0<t_f<+\infty$, we have

$$\begin{aligned} &\{x|x=P_1\int_{t_0}^{t_f}e^{G(t_f-s)}B_1u(s)ds\\ &-P_2\sum_{j=1}^{n_2}(N)^{j-1}B_2u^{(j-1)}(t_f),u\in\mathbb{U}\}\\ &=[P_1,P_2]\langle\left[\begin{smallmatrix}G&0\\0&N\end{smallmatrix}\right]|\left[\begin{smallmatrix}B_1\\B_2\end{smallmatrix}\right]\rangle \end{aligned} \tag{6}$$

Lemma 3: Given matrices $A,Q\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times p}$, we have

$$\langle QA-I_n|QB\rangle=Q\langle AQ|B\rangle \tag{7}$$

Proof: Since

$$\begin{aligned} \langle QA|QB\rangle &=\sum_{i=1}^{n}(QA)^{i-1}\mathcal{I}m(QB)=\sum_{i=1}^{n}(QA)^{i-1}Q\mathcal{I}m(B)\\ &=Q\sum_{i=1}^{n}(AQ)^{i-1}\mathcal{I}m(B)=Q\langle AQ|B\rangle \end{aligned}$$

We only need to verify that $\langle A-I_n|B\rangle=\langle A|B\rangle$. In fact, it is easy to see that $\langle A-I_n|B\rangle,\langle A+I_n|B\rangle\subseteq\langle A|B\rangle$. Then we have $\langle A|B\rangle=\langle A-I_n+I_n|B\rangle\subseteq\langle A-I_n|B\rangle$. Hence, we have $\langle A-I_n|B\rangle=\langle A|B\rangle$. ■

In the following, we'll discuss the controllability of system (1) at time instant t_f. If $k=1$, then the system is reduced to a linear time-invariant descriptor system, for which many controllability definitions and criteria have been established[1][4][7]. Thus, in the remaining part of the paper, we concentrate on the case when $k=2,3,\cdots$.

III. Controllability

First, we discuss the reachability of system (1). For system (1), a state x_f is called *reachable* from initial state $x_0\in\mathbb{R}^n$ at time instant $t_f(t_0<t_f)$, if there exists an input $u(t)\in\mathbb{U}$ such that the system is driven from $x(t_0)=x_0$ to $x(t_f)=x_f$. Let $R_{[t_0,t_f]}(x_0)$ be the set of reachable states from x_0. The reachable set of the system is $\mathcal{R}_{[t_0,t_f]}=\bigcup_{\forall x_0}R_{[t_0,t_f]}(x_0)$.

Theorem 1: For system (1), the reachable set from state x_0 in $[t_0,t_f]$ is given by

$$\begin{aligned} &\mathcal{R}_{[t_0,t_f]}(x_0)=\mathcal{I}n(x_0)+\\ &\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}H_jP_mQ_m\langle(A_m+E_m)P_mQ_m|B_m\rangle\Big)\\ &+P_kQ_k\langle(A_k+E_k)P_kQ_k|B_k\rangle \end{aligned} \tag{8}$$

where $\mathcal{I}n(x_0)=\prod_{j=k}^{1}H_jx_0$.

Proof: First, we consider $\mathcal{R}_{[t_0,t_f]}(0)$. By Lemma 1, let $x(t_0)=0$, we have

$$\begin{aligned} x(t_f)&=\sum_{m=1}^{k-1}\Big[\prod_{j=k}^{m+1}H_j\Big(P_{m,1}\int_{t_{m-1}}^{t_m}e^{G_m(t_m-s)}B_{m,1}u(s)ds\\ &-P_{m,2}\sum_{j=1}^{n-n_m}(N_m)^{j-1}B_{m,2}u^{(j-1)}(t_m)\Big)\Big]\\ &+P_{k,1}\int_{t_{k-1}}^{t_f}\exp[G_k(t_f-s)]B_{k,1}u(s)ds\\ &-P_{k,2}\sum_{j=1}^{n-n_k}(N_k)^{j-1}B_{k,2}u^{(j-1)}(t_f) \end{aligned} \tag{9}$$

It follows that

$$
\begin{aligned}
&\mathcal{R}[t_0,t_f]\\
&=\{x|x=\sum_{m=1}^{k-1}\Big[\prod_{j=k}^{m+1}H_j\Big(P_{m,1}\int_{t_{m-1}}^{t_m}e^{G_m(t_m-s)}B_{m,1}u(s)ds\\
&-P_{m,2}\sum_{j=1}^{n-n_m}(N_m)^{j-1}B_{m,2}u^{(j-1)}(t_m)\Big)\Big]\\
&+P_{k,1}\int_{t_{k-1}}^{t_f}\exp[G_k(t_f-s)]B_{k,1}u(s)ds\\
&-P_{k,2}\sum_{j=1}^{n-n_k}(N_k)^{j-1}B_{k,2}u^{(j-1)}(t_f),u\in\mathbb{U}\}\\
&=\sum_{m=1}^{k-1}\Big[\prod_{j=k}^{m+1}H_j\{x|x=P_{m,1}\int_{t_{m-1}}^{t_m}e^{G_m(t_m-s)}B_{m,1}u(s)ds\\
&-P_{m,2}\sum_{j=1}^{n-n_m}(N_m)^{j-1}B_{m,2}u^{(j-1)}(t_m),u\in\mathbb{U}\}\Big]\\
&+\{x|x=P_{k,1}\int_{t_{k-1}}^{t_f}\exp[G_k(t_f-s)]B_{k,1}u(s)ds\\
&-P_{k,2}\sum_{j=1}^{n-n_k}(N_k)^{j-1}B_{k,2}u^{(j-1)}(t_f),u\in\mathbb{U}\}
\end{aligned}
$$

By Lemma 2, we get

$$
\begin{aligned}
\mathcal{R}[t_0,t_f]=&\sum_{m=1}^{k-1}\left(\prod_{j=k}^{m+1}H_jP_m\langle\begin{bmatrix} G_m & 0\\ 0 & N_m\end{bmatrix}|\begin{bmatrix} B_{m,1}\\ B_{m,2}\end{bmatrix}\rangle\right)\\
&+P_k\langle\begin{bmatrix} G_k & 0\\ 0 & N_k\end{bmatrix}|\begin{bmatrix} B_{k,1}\\ B_{k,2}\end{bmatrix}\rangle
\end{aligned}
$$

For $m=1,\cdots,k$, by Lemma 3,

$$
\begin{aligned}
&\langle\begin{bmatrix} G_m & 0\\ 0 & N_m\end{bmatrix}|\begin{bmatrix} B_{m,1}\\ B_{m,2}\end{bmatrix}\rangle\\
&=\langle Q_m(A_m+E_m)P_m-I_n|Q_mB_m\rangle\\
&=Q_m\langle(A_m+E_m)P_mQ_m|B_m\rangle
\end{aligned}
$$

It is easy to verify that (8) holds for zero state. For non-zero state x_0, the proof is similar and thus omitted. ■

Definition 1 (C-controllability): System (1) is said to be completely controllable (C-controllability) in $[t_0,t_f]$ ($t_0<t_f$), if for any state $x_0,x_f\in\mathbb{R}^n$, there exists an input $u(t)\in\mathbb{U}$ such that the system is driven from $x(t_0)=x_0$ to $x(t_f)=x_f$.

Corollary 1: System (1) is C-controllable in $[t_0,t_f]$ if and only if

$$
\begin{aligned}
&\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}H_jP_mQ_m\langle(A_m+E_m)P_mQ_m|B_m\rangle\Big)\\
&+P_kQ_k\langle(A_k+E_k)P_kQ_k|B_k\rangle=\mathbb{R}^n
\end{aligned}
\tag{10}
$$

Proof: System is C-controllable if and only if, for any $x_0,x_f\in\mathbb{R}^n$, equation (5) has a solution $u(t)\in\mathbb{U}$. By Theorem 1, this is equivalent to

$$
\begin{aligned}
&x_f-\mathcal{I}n(x_0)\\
&\in\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}H_jP_mQ_m\langle(A_m+E_m)P_mQ_m|B_m\rangle\Big)\\
&+P_kQ_k\langle(A_k+E_k)P_kQ_k|B_k\rangle
\end{aligned}
$$

for any $x_0,x_f\in\mathbb{R}^n$. Thus, this is equivalent to (10). ■

Definition 2 (R-controllability): System (1) is said to be controllable in the set of reachable states (R-controllable) in $[t_0,t_f]$ ($t_0<t_f$), if for any initial state $x_0\in\mathbb{R}^n$ and any terminal state $x_f\in\mathcal{R}_{[t_0,t_f]}$, there exists an input $u(t)\in\mathbb{U}$ such that the system is driven from $x(t_0)=x_0$ to $x(t_f)=x_f$.

Corollary 2: System (1) is R-controllable in $[t_0,t_f]$ if and only if

$$
\begin{aligned}
&\mathcal{I}m\left(\prod_{j=k}^{1}H_j\right)\\
&\subseteq\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}H_jP_mQ_m\langle(A_m+E_m)P_mQ_m|B_m\rangle\Big)\\
&+P_kQ_k\langle(A_k+E_k)P_kQ_k|B_k\rangle
\end{aligned}
\tag{11}
$$

Proof: By Theorem 1, it is easy to see that

$$
\begin{aligned}
&\mathcal{R}_{[t_0,t_f]}=\mathcal{I}m\left(\prod_{j=k}^{1}H_j\right)\\
&+\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}H_jP_mQ_m\langle(A_m+E_m)P_mQ_m|B_m\rangle\Big)\\
&+P_kQ_k\langle(A_k+E_k)P_kQ_k|B_k\rangle
\end{aligned}
\tag{12}
$$

Then the system is R-controllable if and only if, for any $x_0\in\mathbb{R}^n$ and any $x_f\in\mathcal{R}_{[t_0,t_f]}$, equation (5) has a solution $u(t)\in\mathbb{U}$. This is equivalent to

$$
\begin{aligned}
&x_f-\mathcal{I}n(x_0)\\
&in\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}H_jP_mQ_m\langle(A_m+E_m)P_mQ_m|B_m\rangle\Big)\\
&+P_kQ_k\langle(A_k+E_k)P_kQ_k|B_k\rangle
\end{aligned}
\tag{13}
$$

for any $x_0\in\mathbb{R}^n$ and any $x_f\in\mathcal{R}_{[t_0,t_f]}$. This is also equivalent to

$$
\begin{aligned}
\mathcal{R}_{[t_0,t_f]}\subseteq&\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}H_jP_mQ_m\langle(A_m+E_m)P_mQ_m|B_m\rangle\Big)\\
&+P_kQ_k\langle(A_k+E_k)P_kQ_k|B_k\rangle
\end{aligned}
\tag{14}
$$

Obviously, this is equivalent to (11). ■

IV. Relations Between Our Results and the Existing Results

In Section III, necessary and sufficient conditions for controllability of piecewise linear descriptor systems are derived. Since piecewise linear descriptor systems are extensions of linear time-invariant descriptor systems and piecewise linear systems, our results generalize the existent results on controllability of linear time-invariant descriptor systems and piecewise linear systems.

A. Extension from linear time-invariant descriptor systems

For system (1), if $(E_i,A_i,B_i)=(E,A,B)$, $i=1,\cdots,k$, then the system is reduced to a linear time-invariant descriptor system. We'll show that criteria (10) and (11) are reduced to the traditional ones. In fact, in this case we can assume that $(P_i,Q_i)=(P,Q)$, $(G_i,N_i)=(G,N)$, $i=1,\cdots,k$, where $G\in\mathbb{R}^{n_1\times n_1}$, $P=[P_1,P_2]$, $P^{-1}=\begin{bmatrix} P_1^{-1}\\ P_2^{-1}\end{bmatrix}$ and $B=\begin{bmatrix} B_1\\ B_2\end{bmatrix}$. We will discuss them respectively:

(a) C-controllability. Criterion (10) is simplified as

$$
\begin{aligned}
&\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}H_jPQ\langle(A+E)PQ|B\rangle\Big)\\
&+PQ\langle(A+E)PQ|B\rangle=\mathbb{R}^n
\end{aligned}
\tag{15}
$$

First, it is easy to see that $PQ\langle (A+E)PQ|B\rangle = P_1\langle G|B_1\rangle + P_2\langle N|B_2\rangle$. Next, for any $t\in\mathbb{R}$, we have

$$\begin{aligned} & P_1\exp(Gt)P_1^{-1}(P_1\langle G|B_1\rangle + P_2\langle N|B_2\rangle) \\ &= P_1\exp(Gt)P_1^{-1}P_1\langle G|B_1\rangle + P_1\exp(Gt)P_1^{-1}P_2\langle N|B_2\rangle \\ &= P_1\exp(Gt)\langle G|B_1\rangle \subset P_1\langle G|B_1\rangle. \end{aligned}$$

Thus, we know that the left part of the equation (15) is just $P_1\langle G|B_1\rangle + P_2\langle N|B_2\rangle$. Since P is nonsingular, (15) is equivalent to

$$\langle G|B_1\rangle \oplus \langle N|B_2\rangle = \mathbb{R}^n. \tag{16}$$

It is obvious that (16) is just the traditional criterion for C-controllability of linear time-invariant descriptor systems. Thus, the existent result is a special case of our result. Moreover, we get a new criterion for C-controllability of linear constant descriptor systems as follows.

Corollary 3: A linear time-invariant descriptor system (E,A,B) is C-controllable if and only if one of the following condition holds:

$$\langle (A+E)PQ|B\rangle = \mathbb{R}^n, \tag{17}$$

$$\text{rank}([B, (A+E)PQB, \cdots, ((A+E)PQ)^{n-1}B]) = n, \tag{18}$$

$$\text{rank}([(A+E)PQ - I_n s, B]) = n, \forall s. \tag{19}$$

(b) R-controllability. Criterion (11) is simplified as

$$\begin{aligned} &\mathcal{I}m(\prod_{j=k}^{1} H_j) \subseteq \sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1} H_j PQ\langle (A+E)PQ|B\rangle\Big) \\ &+PQ\langle (A+E)PQ|B\rangle \end{aligned} \tag{20}$$

Then, (20) is equivalent to

$$P_1\exp(G\sum_{j=1}^{k} h_j)\mathcal{I}m(P_1^{-1}) \subseteq P_1\langle G|B_1\rangle + P_2\langle N|B_2\rangle \tag{21}$$

Moreover, since $\mathcal{I}m(P_1)\bigcap\mathcal{I}m(P_2) = 0$, (21) is equivalent to

$$P_1\exp(G\sum_{j=1}^{k} h_j)\mathcal{I}m(P_1^{-1}) \subseteq P_1\langle G|B_1\rangle \tag{22}$$

Since P_1, P_1^{-1} and $\exp(G\sum_{j=1}^{k} h_j)$ are all full rank, (22) is equivalent to $\mathbb{R}^{n_1} \subseteq \langle G|B_1\rangle$. Obviously, this is also equivalent to $\langle G|B_1\rangle = \mathbb{R}^{n_1}$. This is just the traditional criterion for R-controllability of linear time-invariant descriptor systems.

B. Extension from piecewise linear systems

For system (1), if $E_1,\cdots,E_k$ are nonsingular, then the system is reduced to a piecewise linear system. We'll show that the criteria (10) and (11) are also reduced to the traditional ones. In fact, in this case we can assume that $(P_i,Q_i) = (I_n, E_i^{-1})$, $i=1,\cdots,k$, then (10) is rewritten as

$$\begin{aligned} &\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}\exp(A_j h_j)E_j^{-1}\langle A_m E_m^{-1}|B_m\rangle\Big) \\ &+E_m^{-1}\langle A_k E_k^{-1}|B_k\rangle = \mathbb{R}^n \end{aligned} \tag{23}$$

In particular, if $E_i = I_n$, $i=1,\cdots,k$, then (23) is just

$$\sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}\exp(A_j h_j)\langle A_m|B_m\rangle\Big) + \langle A_k|B_k\rangle = \mathbb{R}^n. \tag{24}$$

It is easy to see that (24) is just the traditional criterion for controllability of piecewise linear systems[16].

As to criterion (11), we have

$$\begin{aligned} &\mathcal{I}m(\textstyle\prod_{j=K}^{1}\exp(A_j h_j)) \\ &\subseteq \sum_{m=1}^{k-1}\Big(\prod_{j=k}^{m+1}\exp(A_j h_j)\langle A_m|B_m\rangle\Big) + \langle A_k|B_k\rangle \end{aligned} \tag{25}$$

Obviously, (25) is also equivalent to (24). Thus, we show that C-controllability and R-controllability are both reduced to the general controllability of piecewise linear systems.

V. Illustrating Examples

In this section, we give two numerical examples to illustrate how to utilize our criteria.

Example 1: Consider a 6-dimensional linear piecewise constant impulsive system with

$$E_1 = \left[\begin{array}{c|ccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$A_1 = \left[\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], B_1 = \left[\begin{array}{c} -1 \\ \hline 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$E_2 = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$A_2 = \left[\begin{array}{cccc|cc} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], B_2 = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ \hline 1 \\ 0 \end{array}\right]$$

where $t_0 = 0$, $t_1 = 1$ and $t_2 = t_f = 2$.

Now, we try to use our criteria to study the controllability of the system in Example 1. By simple calculation, we get $P_1 = P_2 = Q_1 = Q_2 = I_6$, $n_1 = 1$ and $n_2 = 4$. Moreover, it is easy to verify that $H_2\langle A_1+E_1|B_1\rangle + \langle A_2+E_2|B_2\rangle = \mathbb{R}^6$. By Corollary 1, the system is C-controllable. In fact, we take the control input as

$$u(t) = \begin{cases} c_1, & t\in(0,1); \\ c_2(t-1)+c_3, & t=1; \\ c_4, & t\in(1,2) \\ c_5(t-2)+c_6, & t=2. \end{cases} \tag{26}$$

Then we have $x(2) = H_2H_1x(0) + \Phi \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix}$, where $\Phi = \begin{bmatrix} -e^2 & 0 & 0 & 0 & 0 & 0 \\ e-e^2 & -e & 0 & 0 & 0 & 0 \\ 0 & 0 & -e & e-1 & 0 & 0 \\ 0 & 0 & -e & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$. It is easy to verify that the matrix Φ is nonsingular. This shows that the system is C-controllable indeed.

Example 2: Consider a 6-dimensional linear piecewise constant impulsive system with

$$E_1 = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$A_1 = \left[\begin{array}{cccc|cc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], B_1 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \end{array}\right]$$

$$E_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$A_2 = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], B_2 = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ 1 \\ 0 \end{array}\right]$$

where $t_0 = 0$, $t_1 = 1$ and $t_2 = t_f = 2$.

By simple calculation, we get $P_1 = P_2 = Q_1 = Q_2 = I_6$, $n_1 = 1$ and $n_2 = 2$. Moreover, it is easy to verify that $H_2\langle A_1 + E_1|B_1\rangle + \langle A_2 + E_2|B_2\rangle \subsetneq \mathbb{R}^6$, but $\mathcal{I}m(H_2H_1) \subseteq H_2\langle A_1 + E_1|B_1\rangle + \langle A_2 + E_2|B_2\rangle$. By Corollaries 1 and 3, the system is R-controllable, but not C-controllable. In fact, we take the control input as

$$u(t) = \begin{cases} c_1, & t \in (0, 0.5]; \\ c_2, & t \in (0.5, 1]; \\ c_3, & t \in (1, 2); \\ c_4, & t = 2. \end{cases} \tag{27}$$

Then we have $x(2) = H_2H_1x(0) + \Phi[c_1, c_2, c_3, c_4]^T$, where $\Phi = \begin{bmatrix} 2.0037 & 1.1867 & 0 & 0 \\ 2.8956 & 1.5344 & 0 & 0 \\ 1.7663 & 1.1875 & 1.7183 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Meanwhile, by simple calculation, we get $\mathcal{R}_{[0,2]} = \mathcal{I}m(\begin{bmatrix} 1.7839 & 3.7982 & 0 & 0 \\ 2.0143 & 5.8126 & 0 & 0 \\ 2.0143 & 3.0943 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix})$. It is easy to verify that $\mathcal{I}m(\Phi) = \mathcal{R}_{[0,2]}$. This shows that the system is R-controllable indeed. As to C-controllability, it is obvious to see that the 6^{th} variable of the state remains zero all the time. It can not be affected by any input. It is easy to see that the system is not C-controllable.

VI. Conclusion

This paper has dealt with the controllability of piecewise linear descriptor systems. Necessary and sufficient geometric criteria for C-controllability and R-controllability have been established, respectively. These criteria are easily transformed into the algebraic forms. Furthermore, the relationship between our results and the existing results in the literature have also been discussed. A novel necessary and sufficient criterion for C-controllability of linear time-invariant descriptor systems has been derived as a byproduct.

Appendix A

Proof: [Proof of Lemma 1] For $i = 1, 2, \cdots, k$, let $z_i(t) = P_i^{-1}x(t)$, $t \in [t_{i-1}, t_i)$, we decompose $z_i(t)$ as $z_i(t) = \begin{bmatrix} z_{i,1}(t) \\ z_{i,2}(t) \end{bmatrix}$, where $z_{i,1}(t) \in \mathbb{R}^{n_i}$ and $z_{i,2}(t) \in \mathbb{R}^{n-n_i}$, then we get

$$\begin{cases} \dot{z}_{i,1}(t) = G_iz_{i,1}(t) + B_{i,1}u(t), \\ N_i\dot{z}_{i,2}(t) = z_{i,2}(t) + B_{i,2}u(t), \\ z_i(t_{i-1}) = P_ix(t_{i-1}), \quad t \in [t_{i-1}, t_i). \end{cases} \tag{28}$$

The solution of (28) is, for $t \in (t_{i-1}, t_i)$

$$z_i(t) = \begin{bmatrix} z_{i,1}(t) \\ z_{i,2}(t) \end{bmatrix} = \begin{bmatrix} e^{G_i(t-t_{i-1})}z_{i,1}(t_{i-1}) + \int_{t_{i-1}}^t e^{G_i(t-s)}B_{i,1}u(s)ds, \\ -\sum_{j=1}^{n-n_i}(N_i)^{j-1}B_{i,2}u^{(j-1)}(t), \end{bmatrix} \tag{29}$$

Then, for $t \in (t_{i-1}, t_i)$, we get

$$\begin{aligned} x(t) &= P_iz_i(t) = P_{i,1}z_{i,1}(t) + P_{i,2}z_{i,2}(t) \\ &= P_{i,1}e^{G_i(t-t_{i-1})}z_{i,1}(t_{i-1}) + P_{i,1}\int_{t_{i-1}}^t e^{G_i(t-s)}B_{i,1}uds \\ &\quad -P_{i,2}\sum_{j=1}^{n-n_i}(N_i)^{j-1}B_{i,2}u^{(j-1)}(t) \\ &= P_{i,1}e^{G_i(t-t_{i-1})}P_{i,1}^{-1}x(t_{i-1}) + P_{i,1}\int_{t_{i-1}}^t e^{G_i(t-s)}B_{i,1}uds \\ &\quad -P_{i,2}\sum_{j=1}^{n-n_i}(N_i)^{j-1}B_{i,2}u^{(j-1)}(t) \end{aligned}$$

Since $z_{1,1}(t_0) = P_{1,1}^{-1}x(t_0)$, for $t \in (t_0, t_1)$, we have

$$\begin{aligned} x(t) &= P_{1,1}e^{G_1(t-t_0)}P_{1,1}^{-1}x(t_0) + P_{1,1}\int_{t_0}^t e^{G_1(t-s)}B_{1,1}uds \\ &\quad -P_{1,2}\sum_{j=1}^{n-n_1}(N_1)^{j-1}B_{1,2}u^{(j-1)}(t). \end{aligned}$$

Since $x(t_1) = x(t_1^-)$, the above equation also holds for $t = t_1$. Thus, we know that (a) holds.

For $k = 2, 3, \cdots$, we use mathematical induction.

(i) For $k = 2$, $t \in (t_1, t_2)$,

$$
\begin{aligned}
&x(t) = P_{2,1}e^{G_2(t-t_1)}P_{2,1}^{-1}x(t_1) \\
&+P_{2,1}\int_{t_1}^{t} e^{G_2(t-s)}B_{2,1}u(s)ds \\
&-P_{2,2}\sum_{j=1}^{n-n_2}(N_2)^{j-1}B_{2,2}u^{(j-1)}(t) \\
&= P_{2,1}\exp[G_2(t-t_1)]P_{2,1}^{-1}\Big(P_{1,1}\exp(G_1h_1)P_{2-1,1}^{-1}x(t_0) \\
&+P_{1,1}\int_{t_0}^{t_{2-1}}\exp[G_1(t_1-s)]B_{1,1}u(s)ds \\
&-P_{1,2}\sum_{j=1}^{n-n_1}(N_1)^{j-1}B_{1,2}u^{(j-1)}(t_1)\Big) \\
&+P_{2,1}\int_{t_1}^{t}\exp[G_2(t-s)]B_{2,1}u(s)ds \\
&-P_{2,2}\sum_{j=1}^{n-n_2}(N_2)^{j-1}B_{2,2}u^{(j-1)}(t)
\end{aligned}
$$

Since $x(t_2) = x(t_2^-)$, the above equation also holds for $t = t_2$. Then we know that (5) holds for $k = 2$.

(ii) Suppose that (5) holds for $2, 3, \cdots, k-1$, we'll prove that (5) holds for k. For $t \in (t_{k-1}, t_k)$,

$$
\begin{aligned}
&x(t) = P_{k,1}e^{G_k(t-t_{k-1})}P_{k,1}^{-1}x(t_{k-1}) \\
&+P_{k,1}\int_{t_{k-1}}^{t} e^{G_k(t-s)}B_{k,1}uds \\
&-P_{k,2}\textstyle\sum_{j=1}^{n-n_k}(N_k)^{j-1}B_{k,2}u^{(j-1)}(t) \\
&= P_{k,1}e^{G_k(t-t_{k-1})}P_{k,1}^{-1}\Big(P_{k-1,1}e^{G_{k-1}h_{k-1}}P_{k-1,1}^{-1}x(t_{k-2}) \\
&+P_{k-1,1}\int_{t_{k-2}}^{t_{k-1}}\exp[G_{k-1}(t_{k-1}-s)]B_{k-1,1}uds \\
&-P_{k-1,2}\sum_{j=1}^{n_{k-1}}(N_{k-1})^{j}B_{k-1,2}u^{(j-1)}(t_{k-1})\Big) \\
&+P_{k,1}\int_{t_{k-1}}^{t} e^{G_k(t-s)}B_{k,1}uds \\
&-P_{k,2}\sum_{j=1}^{n-n_k}(N_k)^{j-1}B_{k,2}u^{(j-1)}(t) \\
&= \cdots \\
&= P_{k,1}\exp[G_k(t-t_{k-1})]P_{k,1}^{-1} \\
&\Bigg\{\textstyle\prod_{j=k-1}^{1}P_{j,1}\exp(G_jh_j)P_{j,1}^{-1}x(t_0) \\
&+\sum_{m=1}^{k-2}\Bigg[\prod_{j=k-1}^{m+1}P_{j,1}e^{G_jh_j}P_{j,1}^{-1} \\
&\Big(P_{m,1}\int_{t_{m-1}}^{t_m} e^{G_m(t_m-s)}B_{m,1}u(s)ds \\
&-P_{m,2}\sum_{j=1}^{n-n_m}(N_m)^{j-1}B_{m,2}u^{(j-1)}(t_m)\Big) \\
&+P_{k-1,1}\int_{t_{k-2}}^{t_{k-1}}\exp[G_{k-1}(t_{k-1}-s)]B_{k-1,1}u(s)ds \\
&-P_{k-1,2}\sum_{j=1}^{n-n_{k-1}}(N_{k-1})^{j-1}B_{k-1,2}u^{(j-1)}(t_{k-1})\Bigg]\Bigg\} \\
&+P_{k,1}\int_{t_{k-1}}^{t}\exp[G_k(t-s)]B_{k,1}u(s)ds \\
&-P_{k,2}\sum_{j=1}^{n-n_k}(N_k)^{j-1}B_{k,2}u^{(j-1)}(t)
\end{aligned}
$$

Since $x(t_k) = x(t_k^-)$, the above equation also holds for $t = t_k$. Thus, we know that (5) holds for k. ■

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