Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

A Non-Orthogonal Projection Approach to Characterization of Almost Positive Real Systems with an Application to Adaptive Control

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Abstract: **In this paper we develop a very general projection approach to obtain and unify necessary and sufficient conditions for making a linear time-invariant system positive real using static output feedback. Such systems are called "Almost positive Real". These conditions are that the open-loop system must be weakly minimum phase, have positive definite high frequency gain and the marginally stable zero system alone must be positive real. When dynamic output feedback is used the necessary and sufficient conditions reduce to weakly minimum phase with a positive definite high frequency gain. These results yield a stability proof for Direct Model Reference Adaptive Control and Disturbance Cancellation using only almost positive real systems.**

Introduction:

There is substantial interest in positive real and strictly positive real systems for various aspects of control system design and analysis, [1]-[3]. Here we will address the problem of using
output feedback to make a system positive real. We consider the following controllable and observable linear continuous-time square plant:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx; x(0) = x_0 \in \Re^n \end{cases}$$
$$\text{where } \dim x = n, \dim u = \dim y = m \quad (1)$$
and B and C have full rank.

From [18], the transmission (or blocking) zeros of this system are the values of s where

$$rank\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} < n + m, i.e. \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \text{is singular.}$$

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The system (A, B, C) is said to be minimum phase when

$$rank\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} = n + m \quad \text{for } \mathrm{Re}(s) \geq 0 \quad (2)$$

When (2) is true only for $\mathrm{Re}(s) > 0$, (A, B, C) is weakly minimum phase; this follows a similar definition for nonlinear systems in [9]. Also A is said to be stable when all its eigenvalues lie in the open left half-plane and weakly stable when they lie in the closed left half-plane.

The high frequency gain of (1) is CB and the system is said to be relative degree one when CB is nonsingular.

We will use a static output feedback control law given by

$$u = Gy + u_r \quad (3)$$
where G is mxm

The closed-loop transfer function for (1) and (3) is

$$P_C(s) \equiv C(sI - A_C)^{-1}B \quad \text{where} \quad A_C \equiv A + BGC$$
(4)

and the open-loop version P(s) is given with G set to zero.

Definition: An m x m rational matrix transfer function P(s) is positive real (PR) when
All elements of P(s) are analytic in the open right half-plane Re(s)>0 with only simple poles on the imaginery axis (<u>the residue matrix at these poles must be positive semi-definite</u>), and
Re P(s) is positive semi-definite for Re(s)>0

$$where \ \mathrm{Re}\, P \equiv \frac{P(s) + P^T(\bar{s})}{2}$$

From the Positive Real Lemma [4], it is known that this is equivalent to

$$\exists P > 0 \ni \begin{cases} A^T P + PA = -Q \leq 0 \\ PB = C^T \end{cases} \quad (5)$$

where we have used the convention:
$W > 0$ for W symmetric and positive definite
and $W \geq 0$ for W symmetric and positive semidefinite.

There is little controversy over this classic definition of PR. However, when it comes to strict positive real, things are less clear. We use the following:
Definition: An mxm rational matrix transfer function P(s) is strictly positive real (SPR) when $\exists \varepsilon > 0 \ni P(s-\varepsilon)$ is PR. We remark that, of all the definitions of SPR, this one is equivalent to the Kalman-Yacubovic (K-Y) conditions as shown in [5]:

$$\exists P > 0 \ni \begin{cases} A^T P + PA = -Q < 0 \\ PB = C^T \end{cases} \quad (6)$$

This can be seen from the Positive Real Lemma by $\begin{cases} \text{shifting } A \text{ to } A + \varepsilon I \text{ in (5) and} \\ \text{replacing Q} \geq 0 \text{ by } Q + 2\varepsilon P > 0. \end{cases}$

Henceforth, when we speak of P(s) being SPR, we can use this equivalence of a minimal realization (A, B, C) satisfying (6).

We seek necessary and sufficient conditions on (A, B, C) for an output feedback control law (3) to make the closed-loop transfer function PR/SPR and say that "(A, B, C) is almost PR/SPR". This also called "Output feedback equivalence to a PR/SPR system." Various authors have addressed different aspects of this problem. In [6] Theo 3.1, Gao and Ioannou have shown that PR implies CB is positive definite. Additionally they have shown that SPR implies:
$a) CB > 0$
$b) \operatorname{Re} \lambda(A) < 0$ where
$\lambda(A)$ is the set of eigenvalues of A
c) $CAB + (CAB)^T < 0$
But not necessarily conversely, except for some special cases.

In [7] Theo 4.1, Weiss, Wang, and Speyer show that, if CB is positive definite and the open-loop system is minimum phase, then (3) will produce PR. Here we will show that these conditions are necessary and sufficient for (A, B, C) to be almost SPR. In [8], Gu addresses the problem of quadratic stabilization via linear control. This is related but not the same as our problem. In his Cor 3.7, Gu seems to prove a related result although it is based on a retracted or model reduced version of K-Y in Theo 2.11 of that paper. Gu's result also uses a scaled version $KP_C(s)$ of the closed-loop transfer function, where K is nonsingular. A similar result can be accomplished with a modified output feedback controller:

$$u = Gy + Hu_r \quad (7)$$

where G and H are mxm and H is nonsingular
taking H to be the inverse of K. We hope to clarify this with a clear proof that indicates the exact form of the feedback law (3) and the simple structure of the gain G.

Projections and Retractions:

We will use a non-orthogonal projection approach to obtain our results on almost PR/SPR in a unified way. An idempotent operator P on $\mathfrak{R}^n$ is a projection and, from [19] pp20-21, we have
$\mathfrak{R}^n = R(P) \oplus N(P)$ where $R(P)$ is the range of P
and $N(P)$ is the null space of P.
An orthogonal projection is one where
$N(P) = R(P)^\perp$ or $P^T = P$.

Given an m-dimensional subspace S in $\mathfrak{R}^n$ with a (possibly non-orthogonal) projection P onto it, we will define: $W \ni, \forall x \in \mathfrak{R}^n, Px = Wz$ for some $z \in \mathfrak{R}^m$. If we let the columns of W form an orthonormal basis for S, then $W^T W = I_m$ and $WW^T = P_S^\perp$ which is orthogonal projection onto S. Therefore $z = W^T Px$ is unique and we call W^T the retraction of P (or the subspace S) onto $\mathfrak{R}^m$. Thus we can retract or "pullback" a projection into a lower dimensional space.

Almost SPR Result: We now state the following:
Theorem 1: (A, B, C) is Almost SPR with control law (3) if and only if the high frequency gain CB is positive definite and the open-loop system is minimum phase. The output Feedback Gain that does this is $G \equiv -(\gamma I_m + (CB)^{-1} \overline{A}_{11})$ *with* $\gamma > 0$ *and sufficiently large (see eq (9)).*

We need three lemmae:
Lemma 1: If CB is nonsingular then $P_1 \equiv B(CB)^{-1}C$ is a (non-orthogonal) projection onto the range of B, R(B), along the null space of C,N(C) with $P_2 \equiv I - P_1$ the complementary projection, and $R^n = R(B) \oplus N(C)$.

<u>Proof of Lemma 1</u>:
Consider
$P_1^2 = (B(CB)^{-1}C)(B(CB)^{-1}C) = B(CB)^{-1}C \equiv P_1$;
hence it is a projection. Clearly,
$R(P_1) \subseteq R(B)$ and $z = Bu \in R(B)$
$\Rightarrow P_1 z = (B(CB)^{-1}C)Bu = Bu = z \in R(P_1) \therefore R(P_1) = R(B)$.
Also

$N(P_1) = N(C)$ because $N(C) \subseteq N(P_1)$
and $z \in N(P_1) \Rightarrow P_1 z = 0 \Rightarrow CP_1 z = CB(CB)^{-1}Cz = 0$ or $N(P_1) \subseteq N(C)$.

So P_2 is a projection onto $R(B)$ along $N(C)$,
but $P_2^T \neq P_2$ so it is not orthogonal projection.

We have $\Re^n = R(P_1) \oplus N(P_1)$; hence $\Re^n = R(B) \oplus N(C)$. This completes the proof of Lemma 1.

Lemma 2: If CB is nonsingular, then

$$\exists \text{nonsingular } W \equiv \begin{bmatrix} C \\ W_2^T \end{bmatrix} \ni WB = \begin{bmatrix} CB \\ 0 \end{bmatrix} \text{ and } CW^{-1} = \begin{bmatrix} I_m & 0 \end{bmatrix}$$

This coordinate transformation puts (1) into normal form:

$$\begin{cases} \dot{y} = \overline{A}_{11} y + \overline{A}_{12} z_2 + CBu \\ \dot{z}_2 = \overline{A}_{21} y + \overline{A}_{22} z_2 \end{cases} \qquad (8)$$

where the subsystem: $(\overline{A}_{22}, \overline{A}_{12}, \overline{A}_{21})$ is called the zero dynamics of (1) and

$\overline{A}_{11} \equiv CAB(CB)^{-1}; \overline{A}_{12} \equiv CAW_2;$
$\overline{A}_{21} \equiv W_2^T (I - B(CB)^{-1}B)AB(CB)^{-1};$
$\overline{A}_{22} \equiv W_2^T (I - B(CB)^{-1}B)AW_2$

<u>Proof of Lemma 2</u>:

Consider that $\begin{array}{l} y = Cx = C(B(CB)^{-1}C)x = CP_1 x \\ \text{and } P_1 x = B(CB)^{-1}Cx = B(CB)^{-1}y. \end{array}$

Also $\begin{array}{l} CP_2 = C - CB(CB)^{-1}C = 0 \\ \text{and } P_2 B = B - B(CB)^{-1}CB = 0. \end{array}$ Furthermore

$P_2 x = W_2 z_2; z_2 \in R^{n-m}$ when the n-m columns of W_2 form an orthonormal basis for $N(C)$. Then we have $W_2^T W_2 = I_{n-m}$ and the retraction: $z_2 = W_2^T P_2 x.$

Now, using $x = P_1 x + P_2 x$ from lemma 1, we have

$\dot{y} = CP_1 \dot{x} = CP_1 A(P_1 x + P_2 x) + CP_1 Bu =$
$C(B(CB)^{-1}C)AB(CB)^{-1}y + C(B(CB)^{-1}C)A(W_2 z_2)$
$+ C(B(CB)^{-1}C)Bu = \overline{A}_{11} y + \overline{A}_{12} z_2 + CBu$

And

$\dot{z}_2 = W_2^T P_2 \dot{x} = W_2^T P_2 A(P_1 x + P_2 x)$
$= W_2^T P_2 A(B(CB)^{-1}y + W_2 z_2) + W_2^T P_2 Bu$
$= W_2^T (I - B(CB)^{-1}B)AB(CB)^{-1}y$
$+ W_2^T (I - B(CB)^{-1}B)AW_2 z_2 = \overline{A}_{21} y + \overline{A}_{22} z_2.$

This yields the normal form (8).

Choose $W \equiv \begin{bmatrix} C \\ W_2^T P_2 \end{bmatrix}$. Then W has an inverse explicitly stated as $W^{-1} \equiv \begin{bmatrix} B(CB)^{-1} & W_2 \end{bmatrix}$ .This

$$WW^{-1} = \begin{bmatrix} CB(CB)^{-1} & CW_2 \\ W_2^T P_2 B(CB)^{-1} & W_2^T P_2 W_2 \end{bmatrix}$$

gives

$$= \begin{bmatrix} I_m & 0 \\ 0 & W_2^T W_2 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} = I$$

because the columns of W_2 are in $N(C)$ and P_2 projects onto $N(C)$. Furthermore

$W^{-1}W = P_1 + W_2 W_2^T P_2$
$= P_1 + P_2 = I$ because $W_2 W_2^T$ is orthogonal projection onto $N(C)$.

Also direct calculation

$$\text{yields: } \begin{cases} \overline{B} \equiv WB = \begin{bmatrix} CB \\ W_2^T P_2 B \end{bmatrix} = \begin{bmatrix} CB \\ 0 \end{bmatrix} \\ \overline{C} \equiv CW^{-1} = \begin{bmatrix} CB(CB)^{-1} & CW_2 \end{bmatrix} = \begin{bmatrix} I_m & 0 \end{bmatrix} \\ \overline{A} \equiv WAW^{-1} = \begin{bmatrix} CAB(CB)^{-1} & CAW_2 \\ W_2^T P_2 AB(CB)^{-1} & W_2^T P_2 AW_2 \end{bmatrix} \end{cases}$$

This completes the proof of Lemma 2.

Lemma 3: Assume CB is nonsingular. Define $(\overline{A} \equiv WAW^{-1}, \overline{B} \equiv WB, \overline{C} \equiv CW^{-1})$ where $\overline{A} \equiv \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}$ then the transmission zeros of (A, B, C) are the eigenvalues of $\overline{A}_{22}$.

Consequently

a) $\overline{A}_{22}$ is stable if and only if (A,B,C) is minimum phase and

b) $\overline{A}_{22}$ is weakly stable if and only if (A,B,C) is weakly minimum phase.

<u>Proof of Lemma 3</u>:

Henceforth in the interest of saving space we will omit the proofs and refer the reader to the expanded version of this paper [21].

Note that by rescaling W with the inverse of CB, this co-ordinate transformation can produce $\overline{B} = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $\overline{C} = \begin{bmatrix} (CB)^{-1} & 0 \end{bmatrix}$ which will drastically simplify the proofs in [8]. Finally, our coordinate transformation produces the so-called “normal form” and the dynamics associated with $\overline{A}_{22}$ are the “zero dynamics” [9].

In the proof of Theorem 1:
we must choose:

$$\gamma > \frac{1}{2} \lambda_{\max} (\overline{Q}_{12} \overline{Q}_{22}^{-1} \overline{Q}_{12}^T) \geq 0 \qquad (9)$$

to make $(A_C \equiv A + BGC, B, C)$ SPR.

Almost PR Results:

It is now relatively easy to state necessary conditions for output feedback to make

(A, B, C) PR as well:

Theorem 2: If the linear system (A, B, C) can be made PR by output feedback (3) then (A, B, C) is weakly minimum phase and CB is positive definite, but not conversely.

To see that the converse is false, take $A \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C \equiv [1 \quad 0]$ which is already in normal form with the weak minimum phase property and CB=1. Let $G \equiv -\gamma < 0$ which yields

$$A_C = \begin{bmatrix} -\gamma & 1 \\ 1 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 0 \\ 0 & p_{22} \end{bmatrix} > 0 \text{ when } p_{22} > 0.$$

Thus

P satisfies $PB = C^T$ but $A_C^T P + PA_C \le 0$ can only be satisfied when $p_{22} = -1$ which contradicts $P > 0$.

Note that a slight change in the above example, i.e. $A \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, will produce (A, B, C) PR via output feedback (3) with $p_{22} = 1$. Yet this system is still weakly minimum phase with CB>0.So it takes something stronger than the weakly minimum phase condition to get PR via output feedback. We now give necessary and sufficient conditions for this:

Theorem 3: (A, B, C) is almost PR with output feedback (3) if and only if CB is positive definite, the open-loop system is weakly minimum phase and the transfer function of the zero dynamics retracted onto its marginal spectral subspace:

$$P_0(s) \equiv -(CB)^{-1} \overline{A}_{12}^0 (sI - \overline{A}_{22}^0)^{-1} \overline{A}_{21}^0 \text{ is PR}$$

where $\overline{A}_{22} = \overline{U}_2 \begin{bmatrix} \overline{A}_{22}^S & 0 \\ 0 & \overline{A}_{22}^0 \end{bmatrix} \overline{V}_2$

with $\overline{U}_2 \equiv [U_1 \quad U_2]$ and $\overline{V}_2 = \overline{U}_2^{-1} \equiv \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

is the spectral decomposition of $\overline{A}_{22}$ into its stable subspace (where all eigenvalues are in the open left half-plane) and its marginal subspace (where all the eigenvalues are simple and on the imaginery axis).So $\overline{A}_{22}^0 = V_2 \overline{A}_{22} U_2; \overline{A}_{12}^0 \equiv \overline{A}_{12} U_2; \overline{A}_{21}^0 \equiv V_2 \overline{A}_{21}$ *where the zero dynamics* $(\overline{A}_{22}, \overline{A}_{12}, \overline{A}_{21})$ *are given in Lemma 2.*

Note that since CB is positive definite, scaling by it or its inverse does not change the PR property.

Consequently, $\widetilde{P}_0(s) \equiv -\overline{A}_{21}^0 (sI - \overline{A}_{22}^0)^{-1} \overline{A}_{12}^0$ PR can replace $P_0(s)$ PR in Theo. 3. Similar results using passivity were obtained recently in [20].

Dynamic Output Feedback:

Finally, one might imagine that weaker necessary and sufficient conditions might suffice in Theo.1 or Theo.3 when Dynamic Output Feedback is used, e. g.

$$\begin{cases} u = L_{11} y + L_{12} \eta + u_r \\ \dot{\eta} = L_{21} y + L_{22} \eta + \eta_r \\ \dim(\eta) = \rho \end{cases} \qquad (11)$$

The simplest way to use the dynamic controller is to cancel all the zero dynamics, i.e. choose $L_{11} \equiv G; L_{12} \equiv -(CB)^{-1} \overline{A}_{12}; L_{21} \equiv \overline{A}_{21}; L_{22} \equiv \overline{A}_{22}$ .Also $\varepsilon_2 \equiv \eta - z_2$ will satisfy $\dot{\varepsilon}_2 = \overline{A}_{22} \varepsilon_2$ which is stable when the open-loop system is minimum phase. With these dynamics cancelled, the closed-loop transfer function becomes $P_C(s) = (I_m)(sI - (\overline{A}_{11} + CBG))^{-1}(CB)$. Then taking $G \equiv -(\gamma I_m + \overline{A}_{11}), \gamma > 0$, we will have $P_C(s) = (I_m)(sI + \gamma CB)^{-1}(CB)$ which is SPR with $P \equiv (CB)^{-1} > 0$ and $Q \equiv 2\gamma I$ in the K-Y conditions (6), as long as CB is positive definite.

However we can combine (1) and (11) into the well-known Extended Output Feedback Formulation:

$$\begin{cases} \dot{\beta} = \widetilde{A}\beta + \widetilde{B}v \\ w = \widetilde{C}\beta \\ v = \widetilde{L}w + v_r \end{cases} \quad \beta \equiv \begin{bmatrix} x \\ \eta \end{bmatrix}, w \equiv \begin{bmatrix} y \\ \eta \end{bmatrix}, v_r \equiv \begin{bmatrix} u_r \\ \eta_r \end{bmatrix}$$

$$\text{where } \widetilde{A} \equiv \begin{bmatrix} A & 0 \\ 0 & 0_\rho \end{bmatrix}, \widetilde{B} \equiv \begin{bmatrix} B & 0 \\ 0 & I_\rho \end{bmatrix}, \qquad (12)$$

$$\widetilde{C} \equiv \begin{bmatrix} C & 0 \\ 0 & I_\rho \end{bmatrix}, \widetilde{L} \equiv \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

This is a minimal realization of the closed loop transfer function given by $\widetilde{P}_C(s) = \widetilde{C}(sI - (\widetilde{A} + \widetilde{B}\widetilde{L}\widetilde{C}))^{-1}\widetilde{B}$, which must be made SPR by choice of gain matrix $\widetilde{L}$. Now we can apply Theo.1 to (11) and see that necessary and sufficient conditions for this are $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ minimum phase and $\widetilde{C}\widetilde{B} > 0$. But $\widetilde{C}\widetilde{B} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} CB & 0 \\ 0 & I \end{bmatrix} > 0$ if and only if $CB > 0$. Furthermore, for Re(s) $\geq 0, n+m+2\rho = rank\begin{bmatrix} sI-\widetilde{A} & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix}$

$$= rank\begin{bmatrix} sI-A & 0 & B & 0 \\ 0 & sI_\rho & 0 & I_\rho \\ C & 0 & 0 & 0 \\ 0 & I_\rho & 0 & 0 \end{bmatrix}$$

$$= rank\begin{bmatrix} sI-A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & sI_\rho & I_\rho \\ 0 & 0 & I_\rho & 0 \end{bmatrix} =$$

$$rank\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix} + rank\begin{bmatrix} sI_\rho & I_\rho \\ I_\rho & 0 \end{bmatrix}$$

$$= 2\rho + rank\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}$$

Therefore, (A,B,C) is minimum phase if and only if $(\widetilde{A},\widetilde{B},\widetilde{C})$ is minimum phase. So the Extended Output Feedback Control Law in (11) will require the very same necessary and sufficient conditions as in Theo. 1 to produce closed-loop $\widetilde{P}_C(s)$ SPR. Consequently, static output feedback is completely adequate to the task! In [12], this same conclusion is drawn.

In the case of Almost PR, some things simplify with dynamic output feedback:

Theorem 4: (A, B, C) is Amost PR with dynamic output feedback if and only if the high frequency gain CB is positive definite and the open-loop system is weakly minimum phase.

Note that, in [9] Theo.4.7, these same conditions are shown to make nonlinear systems "passive" using dynamic output feedback. Hence, our Theo.4 is a corollary of that in [9]. However, our result uses the dynamic controller to cancel only the zero dynamics retracted onto the marginally stable subspace, whereas [9] cancels all the zero dynamics. So our approach will result in a lower order dynamic controller, in general.

Illustrative Examples: We refer the reader to [21]

Application: Adaptive Control:

In this section we will apply the above results to Direct Model Reference Adaptive Control [14]-[15] and Disturbance Rejection [16] to obtain a stronger stability result than is presented in those papers. Consider the following finite-dimensional linear time-invariant system:

$$\begin{cases} \dot{x} = Ax + Bu + \Gamma u_D \\ y = Cx \end{cases} \quad (13)$$

with a persistent disturbance of the form:

$$\begin{cases} u_D = \theta_D \phi_D \\ \dot{\phi}_D = F_D \phi_D ; F_D \text{ marginally stable} \end{cases} \quad (14)$$

where ϕ_D is a vector of known bounded basis functions which can be generated in real time, e. g. sinusoidal functions or steps generated by the known time invariant marginally stable system F_D .However θ_D is unknown.

The reference model tracking problem of [14]-[15] can be stated simply as the following:

$$e_y \equiv y - y_m \xrightarrow[t\to\infty]{} 0$$

$$\text{where} \begin{cases} y_m = C_m z_m \\ \dot{z}_m = F_m z_m ; F_m \text{ marginally stable} \end{cases} \quad (16)$$

Thus the plant (13) must track the model (16) in the presence of the disturbance (14).
This will be accomplished by the Adaptive Control Law:

$$u = G_m z_m + G_e e_y + G_D \phi_D \quad (17)$$

We introduce the ideal trajectories:

$$\begin{cases} \dot{x}_* = Ax_* + Bu_* + \Gamma u_D \\ y_* = Cx_* = y_m \end{cases} \quad (18)$$

$$\text{where } \begin{cases} x_* = S_{11}^* z_m + S_{12}^* \phi_D \\ u_* = S_{21}^* z_m + S_{22}^* \phi_D \end{cases} \quad (19)$$

The existence of gains S_{ij}^* to satisfy (18)-(19) is equivalent to the following Matching Conditions:

$$\begin{cases} S_{11}^* F_m = AS_{11}^* + BS_{21}^* \\ C_m = CS_{11}^* \end{cases} \quad (20)$$

$$\begin{cases} S_{12}^* F_D = AS_{12}^* + BS_{22}^* + \Gamma\theta_D \\ 0 = CS_{12}^* \end{cases} \quad (21)$$

We present results on the existence of solutions to these Matching Conditions next.

Theorem 5: If CB>0, then

a) (20) has unique solutions (S_{11}^*, S_{21}^*) if and only if no eigenvalue of F_m is common with a transmission zero of the open loop system (A,B,C);

b) (21) has unique solutions (S_{12}^*, S_{22}^*) *if and only if no eigenvalue of* F_D *is common with any transmission zero of the open loop system* (A,B,C).

We have the following improved adaptive stability result:

Theorem 6: Let the plant (13) with persistent disturbance (14) be controlled to track the reference model (16) by the adaptive controller (17) with adaptive gains (26). If the high frequency gain CB>0 and the open-loop plant (A,B,C) *is weakly minimum phase and the transfer function of the zero dynamics retracted to the marginal spectral subspace is positive real, then* $e_y \xrightarrow[t\to\infty]{} 0$ *and the plant output tracks the model output and rejects the persistent disturbance with bounded adaptive gains* (G_e, G_m, G_D).

Note that if (A,B,C) were almost SPR (i.e. CB>0 and (A,B,C) minimum phase) then the above result would be true. However, this was shown before in [14] and [15].

Conclusions and Remarks: Please see [21] for these.

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