Stability Test Based on Eigenvalue Loci for Bimodal Piecewise Linear Systems

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***Abstract*— In this paper, we consider convergence and stability analysis for a class of bimodal piecewise linear systems. We first discuss some properties of trajectories of bimodal piecewise linear systems and derive a necessary condition and a sufficient condition for the stability. The conditions are given in terms of the eigenvalue loci and the detectability of subsystems. In addition, we provide two necessary and sufficient conditions for the planar bimodal piecewise linear systems to be stable. These two conditions are given in terms of eigenvalue loci of subsystems and coefficients of characteristic polynomials, respectively. Furthermore, we discuss a stabilizing controller design based on the derived sufficient condition.**

I. Introduction

Hybrid control has been paid much attention in the area of control system design, because we have many practical control applications which contain both continuous-time dynamical systems and logical or switching elements. There have been a lot of mathematical models proposed to represent behaviors of hybrid control systems. One of the typical models is the piecewise linear system (PLS); The system consists of some pairs of linear time-invariant dynamics and a cell which is a piece of a partition of the state space, and the state evolves along the dynamics corresponding to the cell in which the state exists. The class of PLSs is one of the fundamental classes of hybrid dynamical systems, because the continuous dynamics is linear in each cell and the discrete dynamics is the simplest one. Therefore, a study on PLSs is important as a first step to establish hybrid control theory.

In spite of recent progress in hybrid control theory, there still remain fundamental issues to be clarified. The most fundamental issue is stability. Recently, many results based on Lyapunov functions have been obtained on stability of several classes of hybrid dynamical systems (see [2], [3], [8] and the references therein), where we need to show the existence of a Lyapunov function which guarantees the stability. On the other hand, no converse theorem[1] has been derived due to its hybrid nature. In other words, no necessary and sufficient stability condition based on Lyapunov functions has been derived for any class of hybrid dynamical systems.

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[1]For smooth dynamical systems, there are theorems such that the given conditions are necessary for stability. Such theorems are usually called converse theorems [6].

In fact, we can not completely check the stability even for the class of PLSs. In general, direct applications of the Lyapunov methods to the class of PLSs lead to not any necessary and sufficient conditions but only sufficient conditions for the stability. In addition, we must restrict the available class of Lyapunov functions within a class of piecewise quadratic functions to give a systematic way of finding the Lyapunov functions [9], [10]. This also causes the conservativeness of the stability conditions. Therefore, we need a new approach to get a less conservative stability condition or hopefully to derive a necessary and sufficient condition for the stability.

To this end, we here try to investigate the stability problem for bimodal PLSs (BPLSs) from a different perspective. Instead of Lyapunov methods, we focus on eigenvalue loci of subsystems to investigate the stability. We first discuss several properties of trajectories of BPLSs. In particular, a necessary condition and a sufficient condition for stability are derived. The conditions are given in terms of the eigenvalue loci and the detectability of subsystems. Fortunately, we can derive two necessary and sufficient conditions for the planar BPLSs. The first one is characterized by the eigenvalue loci of subsystems. The second one consists of a condition for each subsystems and a coupling condition, and they are given in terms of coefficients of characteristic polynomials of subsystems.

This paper is organized as follows. Section II describes basic setup for representing a class of BPLSs. Section III is devoted to some preliminaries on analysis of eigenvalues. In Sections IV and V, a necessary condition and a sufficient condition for stability are derived, respectively. Section VI is devoted to stability analysis of the planar BPLS. We give a necessary and sufficient condition for the planar BPLS to be stable in terms of eigenvalue loci of the subsystems. Moreover, another stability condition is provided in terms of coefficients of characteristic polynomials in Section VII. In Section VIII, we propose a method of designing a stabilizing controller based on the derived sufficient condition for stability.

In this paper, we will use the following notation. $\mathcal{Z}$, $\mathcal{R}$, and $\mathcal{R}_+$ represent the set of integers, the set of real numbers, and the set of positive real numbers, respectively. The symbol $\mathcal{R}^{n\times m}$ denotes the set of all $n \times m$ real matrices. If $x_i > 0$ and $x_j = 0$ $(j = 1, 2, \ldots, i-1)$ for some i, we denote it by $x \succ 0$. Furthermore, if $x = 0$ or $x \succ 0$, we denote it by $x \succeq 0$. Also $x \prec 0$ and $x \preceq 0$ mean that $-x \succ 0$ and $-x \succeq 0$, respectively.

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II. Bimodal piecewise linear system

We consider a class of bimodal piecewise linear systems (BPLSs) represented by

$$\dot{x} = \begin{cases} A_1 x, & \text{if } y \geq 0, \\ A_2 x, & \text{if } y \leq 0, \end{cases} \tag{1}$$

$$y = Cx, \tag{2}$$

where $A_1, A_2 \in \mathcal{R}^{n\times n}$, $C \in \mathcal{R}^{1\times n}$, and C is not zero.

We here assume that the BPLS (1)–(2) is well-posed [4], i.e. the BPLS has a unique solution for each initial state. The solution from a given initial state x_0 is denoted by $x(t, x_0)$ where the initial time is always set 0. Furthermore, the corresponding variable y is denoted by $y(t, x_0)$.

The origin is called *stable* if, for each $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that

$$\|x_0\| < \delta(\varepsilon) \Rightarrow \|x(t, x_0)\| < \varepsilon, \quad \forall t \geq 0 \tag{3}$$

holds, that is, we use the term 'stable' in the sense of Lyapunov. The origin is called *attractive* if $\lim_{t\to\infty} x(t, x_0) = 0$ for any initial state x_0. The origin is called *globally asymptotically stable* if it is stable and attractive.

Let us begin with investigation of existence of piecewise quadratic Lyapunov functions with S-procedure [9], [10].

Proposition 1 ([11]): Consider the BPLS (1)–(2). Then there exists a quadratic Lyapunov function with S-procedure for the system, only if both A_1 and A_2 are Hurwitz. $\triangleleft$

Proposition 1 implies that no piecewise quadratic Lyapunov functions exist for systems with unstable dynamics, even if the origin is stable as illustrated in the following example. In other words, the piecewise quadratic Lyapunov method can not lead us to any necessary and sufficient conditions. This motivates us to propose a new approach for stability analysis which is different from the quadratic Lyapunov method.

Example 1: Let us consider a BPLS with

$$A_1 = \begin{bmatrix} -1, & 1 \\ -1, & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1, & 3 \\ -3, & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1, & 0 \end{bmatrix}.$$

Figure 1 shows a trajectory of the system. Clearly the system is asymptotically stable, although $\dot{x} = A_2 x$ is unstable. $\triangleleft$

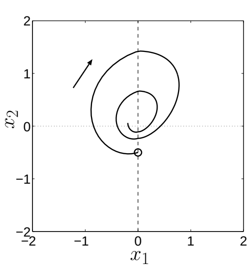

Fig. 1. Trajectory of Example 1.

III. Preliminaries

In this section, we describe some properties which will play important roles for stability analysis in the subsequent sections.

For the sake of briefly, we omit the index i of A_i in the equations (1)–(2), i.e., we consider the system of the form

$$\dot{x} = Ax, \quad y = Cx, \tag{4}$$

where $A \in \mathcal{R}^{n\times n}$, $C \in \mathcal{R}^{1\times n}$ and C is not zero. Let U be the observability matrix of the pair (C, A), i.e.

$$U = \begin{bmatrix} C^\top & (CA)^\top & \ldots & (CA^{n-1})^\top \end{bmatrix}^\top.$$

We first discuss the observability property. We assume matrices A and C are expressed as

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \tag{5}$$

where the pair $(C_1,\ A_{11})$ is observable[2] without loss of generality [7]. We denote the size of A_{11} by $m(\leq n)$, i.e., m denotes the observability index of the pair (C, A). In addition, let $[x_1^\top, x_2^\top]^\top := x$ ($x_1 \in \mathcal{R}^m$, $x_2 \in \mathcal{R}^{n-m}$).

We next describe the property of eigenvalues of A_{11}. Let p denote the number of real and distinct eigenvalues of A_{11}, and let $2q$ denote the number of complex and distinct eigenvalues. There exists a unique linearly dependent eigenvector associating with each distinct eigenvalue, because the pair (C_1, A_{11}) is observable. Let λ_i $(i = 1, \ldots, p)$ be real and distinct eigenvalues of A_{11}. Furthermore, the multiplicity of λ_i is denoted by σ_i. Then, there exist non-zero vectors $v_{ij} \in \mathcal{R}^m$ $(i = 1, \ldots, p,\ j = 2, \ldots, \mu_i)$ for each λ_i such that

$$(\lambda_i I - A_{11})v_{i1} = 0, \tag{6}$$
$$(\lambda_i I - A_{11})v_{ij} = -v_{i(j-1)}, \tag{7}$$

hold. Similarly, we denote complex and distinct eigenvalues and its multiplicity by $\sigma_i \pm \mathrm{j}\omega_i$ $(\sigma_i \in \mathcal{R},\ \omega_i \in \mathcal{R}_+)$ and ν_i $(i = 1, \ldots, q)$, respectively. There exist non-zero vectors $r_{ij} \in \mathcal{R}^m$ and $g_{ij} \in \mathcal{R}^m$ $(i = 1, \ldots, q,\ j = 2, \ldots, \nu_i)$ for each $\sigma_i \pm \mathrm{j}\omega_i$ such that

$$\{(\sigma_i \pm \mathrm{j}\omega_i)I - A_{11}\}(r_{i1} \pm \mathrm{j}g_{i1}) = 0, \tag{8}$$
$$\{(\sigma_i \pm \mathrm{j}\omega_i)I - A_{11}\}(r_{ij} \pm \mathrm{j}g_{ij}) = -(r_{i(j-1)} \pm \mathrm{j}g_{i(j-1)}), \tag{9}$$

hold. Here, v_{ij} and $w_{ij} := [r_{ij}, g_{ij}]$ are called elements of a basis of the generalized eigenspace. Note that

$$C_1 v_{i1} \neq 0, \quad (i = 1, \ldots, p) \tag{10}$$
$$C_1 w_{i1} \neq 0, \quad (i = 1, \ldots, q) \tag{11}$$

hold, because the pair (C_1, A_{11}) is observable.

Finally, we make the following assumption for the matrix A_{11}.

[2] If the pair (C, A) is observable, then $A_{11} = A$ and $C_1 = C$.

Assumption 1: We denote the complex eigenvalues of a given square matrix by $\sigma_i \pm \mathrm{j}\omega_i$ ($\sigma_i \in \mathcal{R},\ \omega_i \in \mathcal{R}_+,\ i = 1,\ldots,q$). Then $\sigma_i \neq \sigma_j$ ($i,j \in K,\ i \neq j$) hold, where $K := \{i \in \{1,\ldots,q\} \mid \sigma_i \geq 0\}$. ◁

Assumption 1 causes no practical restrictions, because it is satisfied for almost all square matrices. Especially, all $n \times n$ real matrices satisfy Assumption 1, when $n \leq 3$.

IV. A Necessary Condition for Asymptotic Stability

Let us begin with the following lemma in order to derive a necessary condition for the asymptotic stability.

Lemma 1: Consider the system (4)–(5). Suppose that the matrix A_{11} satisfies Assumption 1. Then the following statements (a), (b) and (c) are equivalent.

(a) There exists an initial state x_0 such that the following two properties hold:
  - (a-i) $\forall t \geq 0,\ Ux(t,x_0) \succeq (\preceq)\ 0$,
  - (a-ii) $\lim_{t\to\infty} x(t,x_0) \neq 0$.

(b) At least one of the following properties holds:
  - (b-i) The pair (C,A) is not detectable,
  - (b-ii) The matrix A has a non-negative real eigenvalue.

(c) At least one of the following properties holds:
  - (c-i) The matrix A_{22} has an eigenvalue in the closed right half complex plane.
  - (c-ii) The matrix A_{11} has a non-negative real eigenvalue. ◁

Applying Lemma 1 to the system (1)–(2), we can see that the origin is not asymptotically stable if either A_1 or A_2 has a non-negative real eigenvalue. Moreover, from the condition (a-i), there exists an initial state such that no events occur when the condition (b) holds. Conversely, suppose that the condition (b) does not holds. Then, we see that an event always takes place if the trajectory does not converges to 0 as time goes to infinity.

Hence, we obtain the following theorem which provides a necessary condition for asymptotic stability of the BPLS (1)–(2).

Theorem 1: Consider the BPLS (1)–(2). Let assume that A_i ($i = 1,2$) and C have the forms

$$T_i^{-1} A_i T_i = \begin{bmatrix} A_{11}^i & 0 \\ A_{21}^i & A_{22}^i \end{bmatrix},\quad CT_i = \begin{bmatrix} C_1^i & 0 \end{bmatrix},$$

with nonsingular matrices T_1 and T_2, where the pairs (C_1^1, A_{11}^1) and (C_1^2, A_{11}^2) are observable. Suppose that the two matrices A_{11}^1 and A_{11}^2 satisfy Assumption 1. Then the origin is asymptotically stable, only if the following two conditions hold:

(1-i) Neither A_{22}^1 nor A_{22}^2 has any eigenvalue in the closed right half complex plane, that is, both pairs (C,A_1) and (C,A_2) are detectable.

(1-ii) Neither A_{11}^1 nor A_{11}^2 has any non-negative real eigenvalue. In other words, neither the pair (C,A_1) nor the pair (C,A_2) has any non-negative real observable mode. ◁

V. A Sufficient Condition for Attractiveness

We here give the following lemma which lead us to a sufficient condition for stability of the BPLS (1)–(2).

Lemma 2: For the system (4)–(5), the following statements (d) and (e) are equivalent.

(d) Two conditions
  - (d-i) $\forall t \geq 0,\ Ux(t,x_0) \succeq (\text{resp.} \preceq)\ 0$,
  - (d-ii) $\lim_{t\to\infty} x(t,x_0) = 0$,

  hold for any x_0 satisfying

$$Cx_0 = 0 \ \text{ and } \ Ux_0 \succeq (\text{resp.} \preceq)\ 0. \qquad (12)$$

(e) The pair (C,A) is detectable and at least one of following two conditions holds:
  - (e-i) The observable index m is equal to 1 for the pair (C,A).
  - (e-ii) The observable index m is equal to 2 and all observable modes are negative real for the pair (C,A). ◁

Lemma 2 means that any state starting from the switching plane (12) tends to zero as the time goes to infinity with no events. In other words, once the state reaches at the switching plane (12), it converges to zero.

Applying Lemmas 1 and 2 to the BPLS (1)–(2), we obtain the following theorem which provides a sufficient condition for attractiveness.

Theorem 2: Suppose that the system (1)–(2) satisfies the necessary condition of Theorem 1. Then the origin is attractive, if the following two conditions hold for the pair (C,A_1) or (C,A_2):

(2-i) The observable index is less than or equal to 2.

(2-ii) All the observable modes are negative real. ◁

Note that the sufficient condition does not imply that both A_1 and A_2 are Hurwitz, while the piecewise quadratic Lyapunov approach needs the stability of subsystems (see Proposition 1).

VI. Stability Analysis for the Planar Bimodal Piecewise Linear System

Our focus in this section is restricted to the planar system, i.e. $n = 2$. Fortunately, we can obtain a necessary and sufficient condition for this case.

We first investigate conditions corresponding to those of Lemma 1 and Lemma 2 for the planar system.

Corollary 3: Consider the system (1)–(2) with $n = 2$. The following two statements (a) and (b) are equivalent.

(a) There exists an initial state x_0 such that the following two conditions hold:

(a-i) $\forall t \geq 0,\ U_1 x(t,x_0) \succeq 0$ (resp. $U_2 x(t,x_0) \preceq 0$),

(a-ii) $\lim_{t\to\infty} x(t,x_0) \neq 0$.

(b) At least one of the eigenvalues of A_1 (resp. A_2) is non-negative real. ◁

Corollary 4: Consider the system (1)–(2) with $n = 2$. If all the eigenvalues of A_1 (resp. A_2) are negative real, then

(d-i) $\forall t \geq 0,\ U_1 x(t,x_0) \succeq 0$ (resp. $U_2 x(t,x_0) \preceq 0$),

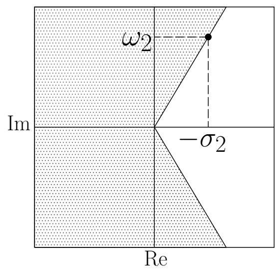

Fig. 2. Stability condition for the eigenvalues of A_1, where $\sigma_2 \pm \mathrm{j}\omega_2$ are the complex eigenvalues of A_2

(d-ii) $\lim_{t\to\infty} x(t, x_0) = 0$,
hold for any x_0 satisfying

$$Cx_0 = 0 \quad \text{and} \quad U_1 x_0 \succeq 0 \quad (\text{resp. } U_2 x_0 \preceq 0). \tag{13}$$

Using Corollaries 3 and 4, we obtain a necessary and sufficient condition for stability of the planar BPLS.

Theorem 5: For the BPLS (1)–(2) with $n = 2$, the following statements are true.

(i) Suppose that either A_1 or A_2 has a real eigenvalue. Then the origin is globally asymptotically stable, if and only if all the real eigenvalues of A_1 and A_2 are negative.

(ii) Suppose that A_1 and A_2 have complex eigenvalues of the forms $\sigma_1 \pm \mathrm{j}\omega_1$ and $\sigma_2 \pm \mathrm{j}\omega_2$, respectively, where $\sigma_1, \sigma_2 \in \mathcal{R}$ and $\omega_1, \omega_2 \in \mathrm{R}_+$. Then the origin is globally asymptotically stable, if and only if

$$\frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} < 0 \tag{14}$$

holds. ◁

Let us now investigate the conditions in Theorem 5. Let A_2 be given. If A_2 has a non-negative real eigenvalue, then the system is not asymptotically stable for any A_1. If A_2 has complex eigenvalues of the form $\sigma_2 \pm \mathrm{j}\omega_2$, then the stability condition for the eigenvalues of A_1 is characterized by the shaded portion in Figure 2. It is seen that the matrix A_1 does not need to be Hurwitz, when $\sigma_2 < 0$.

Note that Theorem 2 with $n = 2$ is equivalent to Theorem 5–(i). In other words, Theorem 2 is a generalization of Theorem 5–(i).

Theorem 5 includes the stability tests for the two special cases.

Corollary 6: For the BPLS (1)–(2) with $n = 2$, the following statements are true.

(i) If both A_1 and A_2 are Hurwitz, then the origin is globally asymptotically stable.

(ii) If neither A_1 nor A_2 is Hurwitz, then the origin is not globally asymptotically stable. ◁

Remark 1: A version of Theorem 5 with continuity of vector fields has been obtained in a recent, independent work by Çamlıbel et al. [1]. They have derived the same stability condition with Theorem 5 for a subclass of planar BPLSs represented by

$$\dot{x} = \begin{cases} Ax & \text{if } cx \geq 0, \\ Ax - bcx & \text{if } cx \leq 0, \end{cases} \tag{15}$$

where $A \in \mathcal{R}^{2\times 2}$, $b \in \mathcal{R}^2$, and $c \in \mathcal{R}^{1\times 2}$. They have investigated the stability for the class of planar bimodal linear complementarity systems represented by

$$\begin{aligned} \dot{x} &= Ax + bz, && (16)\\ w &= cx + z, && (17)\\ z &\geq 0, w \geq 0, zw = 0, && (18) \end{aligned}$$

which can be rewritten as (15). ◁

Example 2: Let us consider a planar BPLS with

$$\begin{aligned} A_1 &= \begin{bmatrix} \zeta, & 1 \\ -1, & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1, & 3 \\ -3, & 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1, & 0 \end{bmatrix}, \end{aligned}$$

where ζ is a constant value. A_2 has complex eigenvalues $1 \pm \mathrm{j}3$. We here examine the relationship between stability of the origin and the value of ζ.

(i) When $\zeta \geq 2$, A_1 has a non-negative real eigenvalue. Thus, the origin is not globally asymptotically stable from Theorem 5–(i). Furthermore, from Corollary 3, there exists a trajectory which does not tend to zero as $t \to \infty$ as illustrated in Figure 3–(i).

(ii) When $\zeta \leq -2$, all eigenvalues of A_1 are negative real. Thus, the origin is globally asymptotically stable from Theorem 1–(i). In addition, from Corollary 4, all trajectories tend to zero as $t \to \infty$, once the state reaches at the switching plane (13). See Figure 3–(ii).

(iii) When $-2/\sqrt{10} < \zeta < 2$, A_1 has complex eigenvalues which does not satisfy the condition (14). Therefore, the origin is not globally asymptotically stable from Theorem 5–(ii) as shown in Figure 3–(iii).

(iv) When $-2 < \zeta < -2/\sqrt{10}$, A_1 has complex eigenvalues satisfying the condition (14). Therefore, the origin is globally asymptotically stable from Theorem 5–(ii). See Figure 3–(iv).

(v) When $\zeta = -2/\sqrt{10}$, A_1 has complex eigenvalues satisfying

$$\frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} = 0. \tag{19}$$

Thus, each trajectory is a closed orbit corresponding to the given initial state as illustrated in Figure 3–(v). ◁

VII. A STABILITY TEST BASED ON COEFFICIENTS OF CHARACTERISTIC POLYNOMIALS

In this section, another stability condition for the planar BPLS is derived. The condition is characterized by coefficients of characteristic polynomials. Let

$$\det(sI - A_i) = s^2 + \alpha_i s + \beta_i, \quad i = 1, 2$$

for A_i with $n = 2$. Then the following theorem provides another necessary and sufficient condition for the stability.

Theorem 7: Consider the BPLS (1)–(2) with $n = 2$. The origin is globally asymptotically stable, if and only if all

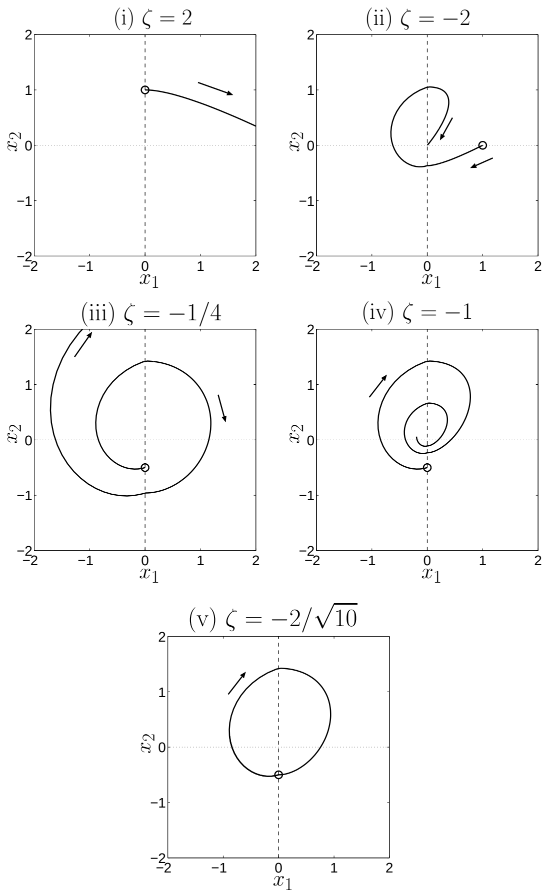

Fig. 3. Trajectories of Example 2. (i) A_1 has a non-negative real eigenvalue. (ii) All eigenvalues of A_1 are negative real. (iii) The condition (14) is not satisfied. (iv) The condition (14) is satisfied. (v) The condition (19) is satisfied.

the following inequalities hold;

$$\beta_i > \max(0, -\frac{|\alpha_i|\alpha_i}{4}), \quad i = 1, 2 \tag{20}$$

$$\frac{|\alpha_1|\alpha_1}{\beta_1} + \frac{|\alpha_2|\alpha_2}{\beta_2} > 0. \tag{21}$$

Let us now investigate the conditions in Theorem 7. The condition (20) is imposed on each subsystem. It is illustrated as the shaded portion in Figure 4. Clearly, $\beta_i > 0$ holds, when the condition (20) holds. The inequality (20) implies that A_i has no non-negative real eigenvalue. Roughly speaking, the inequality (20) corresponds to Theorem 5 (i). On the other hand, the condition (21) is a coupling condition of two subsystems. The inequality (21) corresponds to Theorem 5 (ii). Let A_2, i.e., α_2 and β_2 be given. Then, the shaded portions of Figure 5 show the stability condition for A_1.

VIII. Stabilization for Bimodal Piecewise Linear Systems

In this section, we are interested in the BPLS with control inputs of the form

$$\dot{x} = \begin{cases} A_1 x + B_1 u_1, & \text{if } Cx \geq 0, \\ A_2 x + B_2 u_2, & \text{if } Cx \leq 0, \end{cases} \tag{22}$$

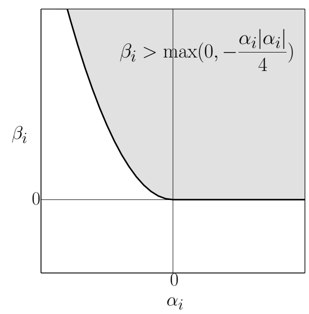

Fig. 4. The shaded portion corresponds to the region represented by the inequality (20).

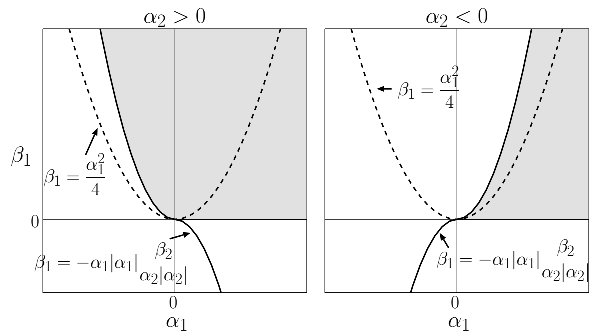

Fig. 5. Stability condition for α_1 and β_1. The shaded portions imply the stability conditions.

where $x \in \mathcal{R}^n$ is the state, and $u_1 \in \mathcal{R}^{m_1}$ and $u_2 \in \mathcal{R}^{m_2}$ are the inputs. The objective here is to find feedback gains K_1 and K_2 that stabilize the system (22). Especially, we consider a controller design based on the stability condition derived in Section V. To this end, we must make the observability index of one of the pairs $(C, A_i + B_i K_i)$, $i = 1, 2$ less than or equal to two. The following proposition provides a sufficient condition for the existence of such a feedback gain.

Proposition 2: Consider the system (22).

(i) There exists a feedback gain K_i such that the triple (C, A_i, B_i) satisfies the conditions of Lemma 2, if the following conditions hold:
  - (i-1) $\min\{\rho \mid CA_i^{\rho-1} B_i \neq 0\} \leq 2$.
  - (i-2) The pair (A_i, B_i) is controllable.
  - (i-3) All invariant zeros[3] of (C, A_i, B_i) are in the open left half complex plane.

(ii) Suppose that K_i satisfies the conditions of (i). The origin is attractive, if there exists a feedback gain K_j $(j \neq i)$ such that the closed loop system is well-posed and the pair $(C, A_j + B_j K_j)$ does not satisfy the condition (b) of Lemma 1.

[3]A complex number s is called an *invariant zero* for the triple (C, A, B) where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{\ell\times n}$, if

$$\text{rank} \begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix} < n + \min(\ell, m)$$

holds.

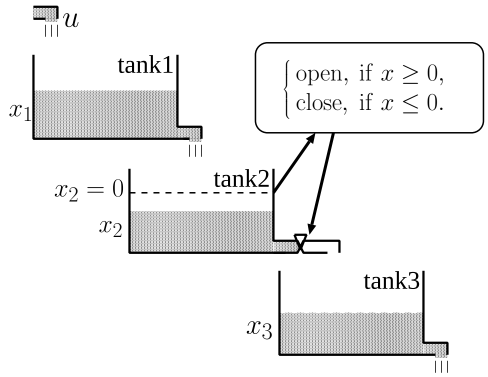

Fig. 6. 3-tank system.

Example 3: Consider an illustrative example of 3-tank system as shown in Figure 6, where x_i is the water level of tank i, $i = 1, 2, 3$, the input u is the volume of water discharged into tank 1. The valve at tank 2 is open if $x_2 \geq 0$, and it is closed if $x_2 \leq 0$ as illustrated in Figure 6. For simplicity, all coefficients are normalized to 1. Then, equations of motion of the system at a neighborhood of the origin are given by

$$\dot{x} = \begin{cases} A_1 + Bu, & \text{if } Cx \geq 0, \\ A_2 + Bu, & \text{if } Cx \leq 0, \end{cases} \tag{23}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Suppose that $u = 0$. Noting that the matrix A_2 has a non-negative real eigenvalue, we see that the origin is not asymptotically stable. Figure 7 (i) shows trajectories of the system (23) with $u = 0$.

Let us now design a controller for the origin to be attractive. The triple of (C, A_1, B) satisfies the condition (ii) of Proposition 2; $\min\{\rho \mid CA_1^{\rho-1}B \neq 0\} = 2$, the pair of (A_1, B) is controllable, and the invariant zero is $s = -1$. Indeed, the triple (C, A_1, B_1) satisfies the conditions of Lemma 2 for $K_1 = [-1, -2, 0]$. Also, we choose $K_2 = [0, -2, 0]$. Then, the closed system is attractive, because the closed loop system is well-posed and the triple of $(C, A_2 + BK_2)$ does not satisfy the condition (b) of Lemma 1. Figure 7 (i) shows trajectories of the system (23) with $K_1 = [-1, -2, 0]$ and $K_2 = [0, -2, 0]$, where the initial state is set $[-1, -1, 1]^\top$. In this case, an event takes place at $t = 1.85$. $\triangleleft$

IX. Conclusion

In this paper, we have investigated the stability problem for a class of bimodal piecewise linear systems (BPLSs). We have first discussed some properties of trajectories of BPLSs and derived a necessary condition and a sufficient condition for the stability. The conditions are given in terms of the eigenvalue loci and the detectability of subsystems. In addition, we have provided two necessary and sufficient conditions for the planar BPLSs to be stable. These two conditions are given in terms of eigenvalue loci of subsystems and coefficients of characteristic polynomials, respectively. Furthermore, we have discussed a stabilizing controller design based on the derived sufficient condition for BPLSs.

An extension to the multi-modal case appears in [5].

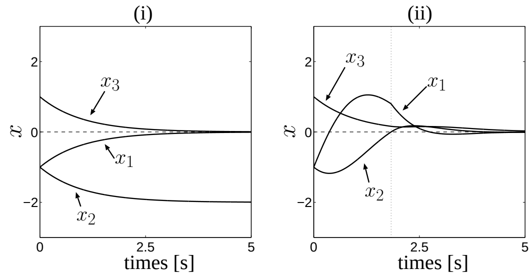

Fig. 7. Trajectories of the tank system (23).

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References

[1] M. K. Çamlıbel, W. P. M. H. Heemels, and J. M. Schumacher; Stability and controllability of planar bimodal linear complementarity systems, In *Proc. of the 42th IEEE Conference on Decision and Control*, 1651-1656, 2003.

[2] R. Decarlo, M. Branicky, S. Petersson, and B. Lennartson; Perspective and results on the stability and stabilizability of hybrid systems, *Proceedings of the IEEE*, **88**-7:1069-1082, 2000.

[3] J. M. Gonçalves, A. Megretski, and A. Dahleh; Global analysis of piecewise linear systems using impact maps and surface Lyapunov functions, *IEEE Trans. on Automatic Control*, **48**-12:2089-2106, 2003.

[4] J. Imura and A. J. van der Schaft; Characterization of well-posedness of piecewise linear systems, *IEEE Trans. on Automatic Control*, **45**-9:1600-1619, 2000.

[5] Y. Iwatani and S. Hara; A necessary and sufficient stability condition for planar multi-modal piecewise linear systems, submitted to *the 43rd IEEE Conference on Decision and Control.*

[6] H. K. Khalil; *Nonlinear systems*, Third Edition, Prentice Hall, 2002.

[7] R. Kalman; Mathematical description of linear dynamical systems, *SIAM J. Control*, **1**:152-192, 1963.

[8] J. Lygeros, K. H. Johansson, S. N. Simic, J. Zhang, and S. S. Sastry; Dynamical Properties of Hybrid Automata, *IEEE Trans. Automatic Control*, **48**-1:2-17, 2003.

[9] S. Pettersson and B. Lennartson; Hybrid system stability and robustness verification using linear matrix inequalities, *Int. J. Control*, **75**:1335-1355, 2002.

[10] A. Rantzer and M. Johansson; Piecewise linear quadratic optimal control, *IEEE Trans. Automatic Control*, **45**-4:629-637, 2000.

[11] A. J. van der Schaft and H. Schmacher; *An Introduction to Hybrid Dynamical Systems*, Springer, 2000.